α_s determination from static potential:

OPE with renormalon subtraction and lattice QCD

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Based on collab. with

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Plan

1. Introduction

   Problem in OPE, Solution

2. Analysis of $V_{\mathrm{QCD}}(r)$

   Strategy, Validity tests of OPE

3. Results and Conclusions

   α_s determination, New aspects in OPE

Problem in OPE

If we compute Wilson coeff. naively in pert. QCD

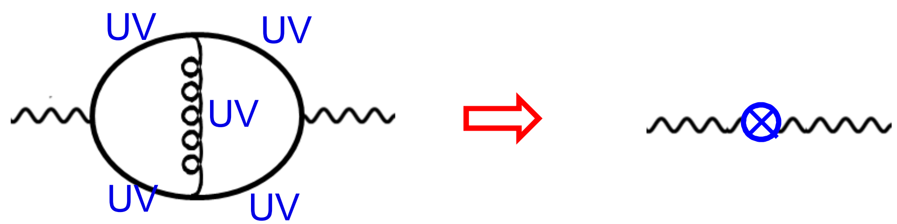
⇨ Renormalon uncertainty ~ non-pert. matrix elem.

Müller

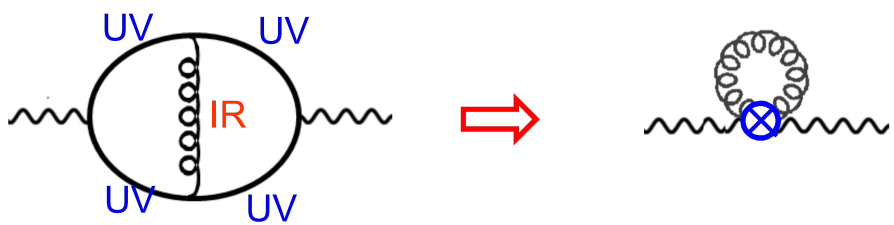
$$X = \underline{C_1^X(Q^2)} \cdot \langle \mathbf{1} \rangle + C_{G^2}^X(Q^2) \cdot \underline{\frac{\langle G_{\mu\nu}^a G_{\mu\nu}^a \rangle}{Q^4}} + \cdots$$

<u>Correspondence by expansion-by-regions method in pert. QCD</u>

Mishima,YS,Takaura



$\sim \langle \mathbf{1} \rangle$



$\sim \dfrac{\langle (G_{\mu\nu}^a)^2 \rangle}{Q^4}$

Solution

Subtract IR renormalons from Wilson coeffs. in a form consistent with OPE.

UV quantity

"Contour-deformation prescription"

$$V_{\mathrm{QCD}}(r) = \underline{V_S(r)} \cdot \langle \mathbf{1} \rangle + \text{const.} + \delta E_{US}(r) + \cdots$$

Brambilla,Pineda,Soto,Vairo

$$\delta E_{US}(r) = g^2 \int_0^\infty dt\, e^{-it\Delta V} \left\langle \vec{r} \cdot \vec{E}^a(t)\, \varphi_{ab}(t)\, \vec{r} \cdot \vec{E}^b(0) \right\rangle$$

Expand in r: $V_C(r) + C_0^V \cdot \Lambda_{\mathrm{QCD}} + C_1^V \cdot \Lambda_{\mathrm{QCD}}^2 r + C_2^V \cdot \Lambda_{\mathrm{QCD}}^3 r^2 + \cdots$

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$$= V_S^{RF}(r) \cdot \langle \mathbf{1} \rangle + \text{const.} + \delta E_{US}^{RF}(r) + \cdots$$

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$\updownarrow$ compare

$$V_{\rm latt}(r)$$

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$$= V_S^{RF}(r) \cdot \langle \mathbf{1} \rangle + \text{const.} + \boxed{\delta E_{US}^{RF}(r)} + \cdots$$

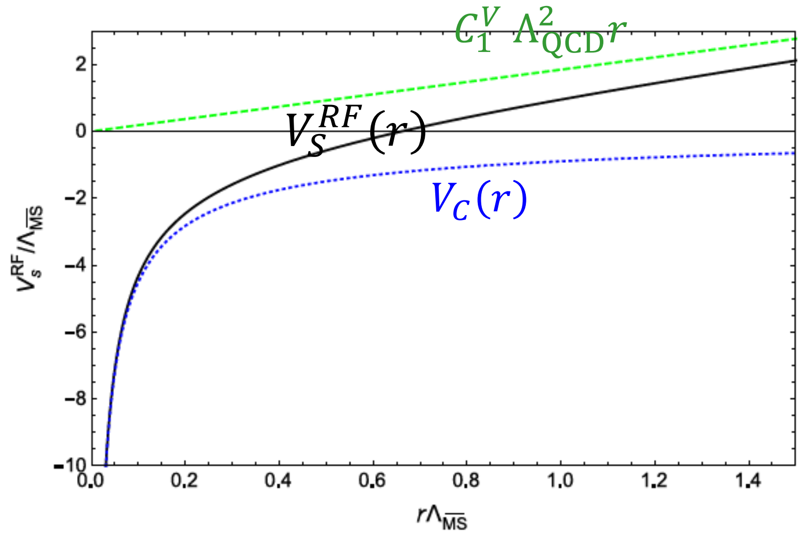
$A_2 r^2$ or $A_2 r^2 (1 + c \log r)$

fitting param.

compare

$V_{\mathrm{latt}}(r)$

$V_S^{RF}(r)$



- $\mathrm{N^3LL}$ accuracy

- Unlike usual RG improvement, free from unphysical singularity at $r \sim \Lambda_{\rm QCD}^{-1}$ caused by running coupling.

- Free from renormalon uncertainties of $O(\Lambda_{\rm QCD})$ and $O(\Lambda_{\rm QCD}^3\, r^2)$.

⇨ Wider validity range of prediction expected.

Lattice simulation

JLQCD T. Kaneko

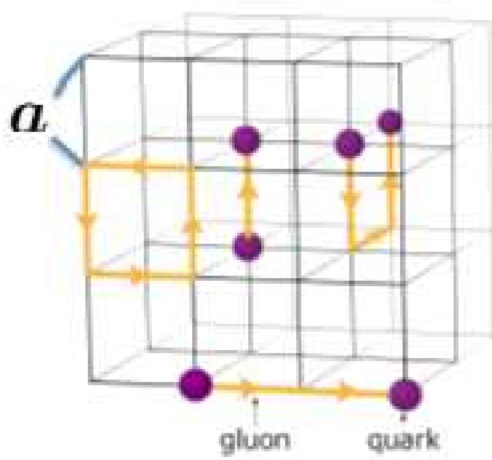
Size 32^3x64,48^3x96,64^3x128

Fermion 2(u,d)+1(s) Domain-wall fermion

Action O(a)-improved action : Discretization error is O(a^2)

$$M_\pi \sim 300\text{MeV}$$
$$M_K \sim 520\text{MeV}$$

$$a^{-1} = 2.453(4), 3.610(9), 4.496(9)\text{GeV}$$



Analysis strategy

Analysis (I)

(i) Continuum extrapolation of lattice data
(ii) Matching of OPE and continuum limit (i) ⇨ determine $\alpha_s(M_Z)$

while checking:

- smooth continuum limit
- $V_{\rm latt}(r) - V_S^{RF}(r)$ consistent with $O(r^2)$
- wider validity range

Analysis (II)

- Perform (i) and (ii) simultaneously by a global fit

First-principle analysis and smaller error   our final result

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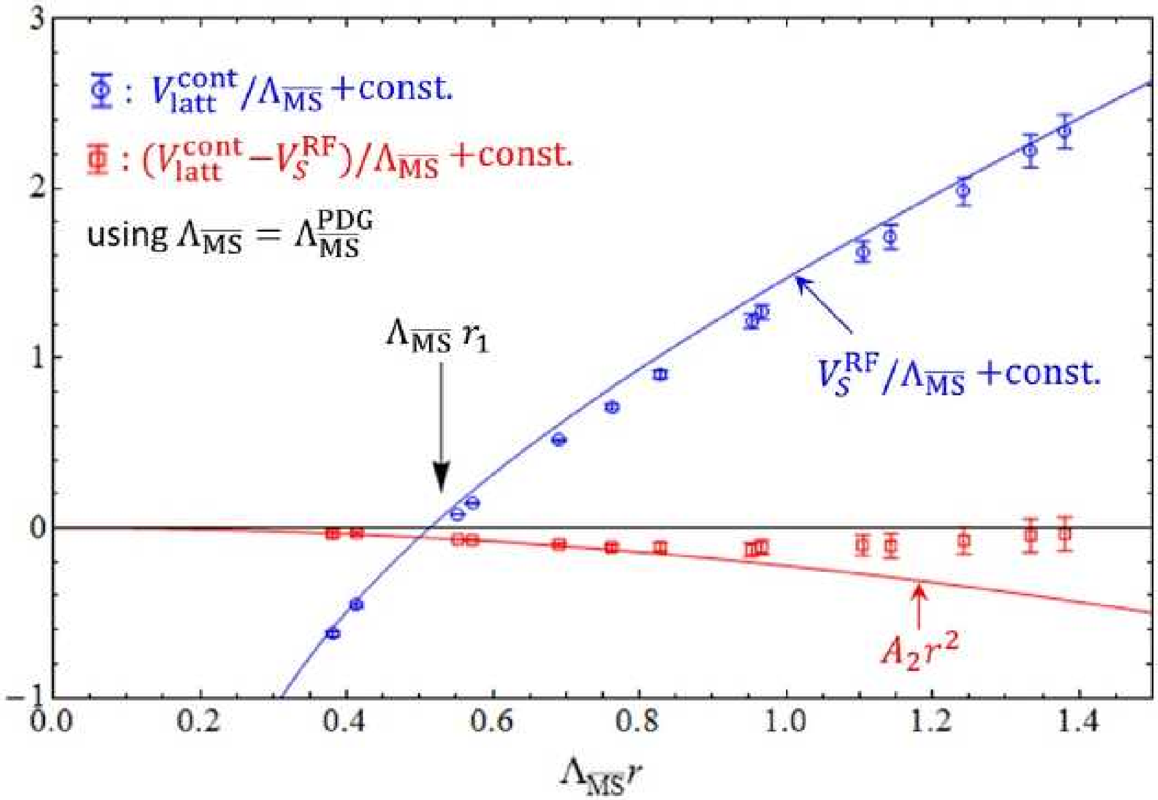
Validity range of OPE

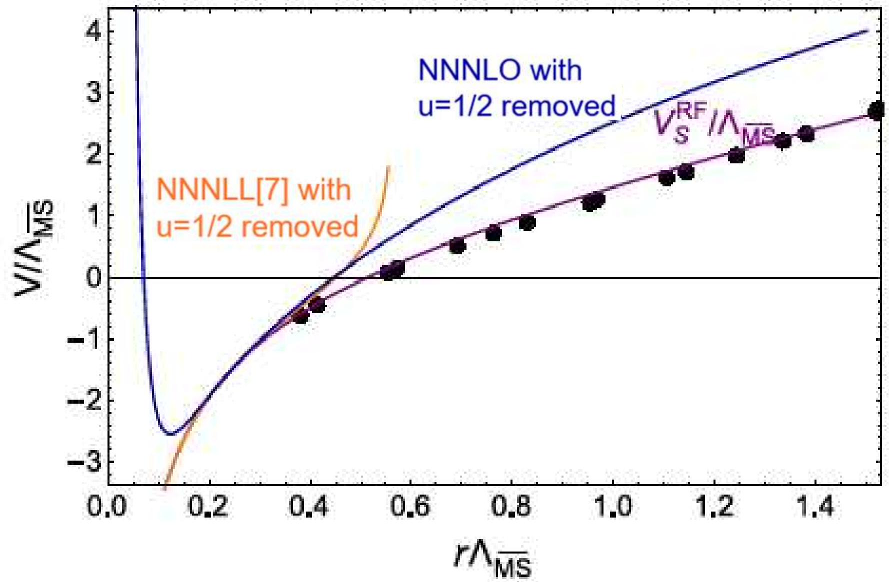
$V_{\mathrm{latt}}(r) - V_S^{RF}(r)$ consistency with $O(r^2)$ using $\Lambda_{\overline{\mathrm{MS}}} = \Lambda_{\overline{\mathrm{MS}}}^{\mathrm{PDG}}$

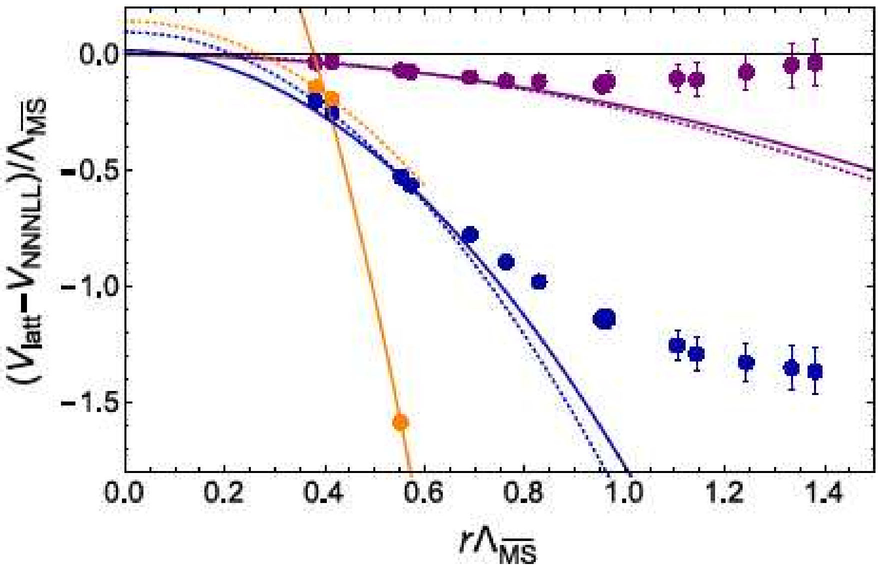


Validity range: $\Lambda_{\overline{\mathrm{MS}}}\, r \lesssim 0.8 \quad (r \lesssim 0.5\ \mathrm{fm})$

[Conventional analyses' validity range: $\Lambda_{\overline{\mathrm{MS}}}\, r \lesssim 0.3 \quad (r \lesssim 0.2\ \mathrm{fm})$]

Comparison with other methods

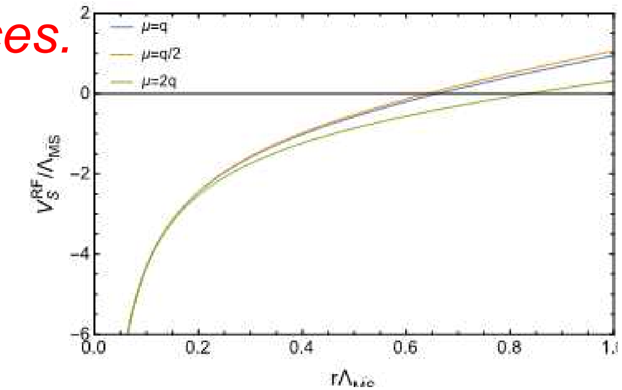


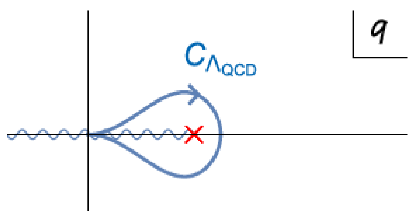


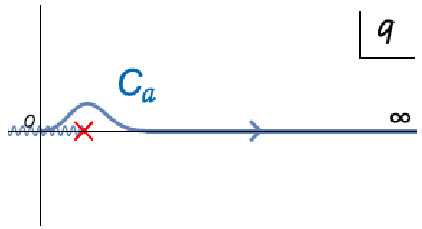
[7] Bazavov, Brambilla, Garcia i Tormo, Petreczky, Soto, Vairo

Scale dep. is much smaller compared to other methods.

r^2 behavior stably observed up to larger distances.







$\alpha_s(M_Z)$ determination

Analysis (II): Global fit

Basic idea: *Lattice data after continuum limit matches OPE at short dist.*

$$V_{\text{latt}}^{\text{cont}}(r) = V_{\text{latt},d,i}(\vec{r}) - \underbrace{\kappa_{d,i}\left(\frac{1}{r} - \left[\frac{1}{r}\right]_{d,i}\right)}_{\text{tree-level corr.}} + f_d\,\frac{a_i^2}{r^3} - c_{0,d,i}$$

$i = 1,2,3$: different lattices

$d = 1,2$: direction of $\vec{r}$

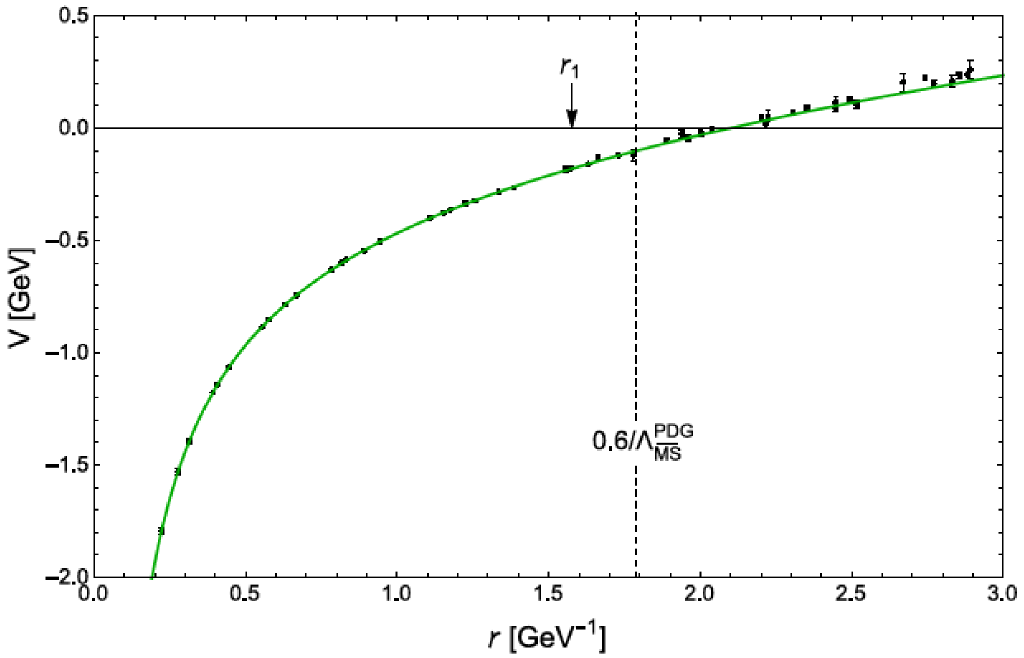
$$||$$

$$V_{\text{OPE}}(r) = z\,[V_S^{RF}(zr)/\Lambda_{\overline{\text{MS}}}] + A_2 r^2 \qquad (\ z = \Lambda_{\overline{\text{MS}}}\ [\text{GeV}]\ )$$

Fitting param.: $\{z = \Lambda_{\overline{\text{MS}}}[\text{GeV}], A_2\,, \kappa_{d,i}\,, f_d\,, c_{0,d,i}\}$

Fit range: $a \approx 0.05\ \text{fm}\ \leq r < 0.35\ \text{fm}$ $\quad (0.07 \leq \Lambda_{\overline{\text{MS}}}\, r < 0.6)$

Fit result



$\chi^2/\mathrm{d.o.f.} = 8.7/(30-16)$

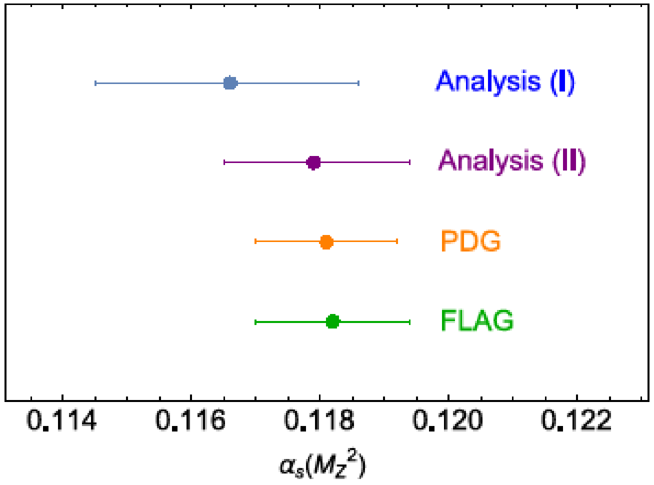
$\Lambda_{\overline{\rm MS}} = 334 \pm 10(stat.)\ \mathrm{MeV}$ ⇨ $\alpha_s(M_Z) = 0.1179 \pm 0.0007(stat.)$

Including syst. errors: $\alpha_s(M_Z) = 0.1179 \pm 0.0007(stat.)^{+0.0014}_{-0.0012}(sys.)$

Table of syst. errors: $10^4 \times \Delta\alpha_s(M_Z)$

	finite a	h.o.	US	Mass	range	fact. scheme	latt. spacing
error	± 2	$^{+12}_{-10}$	± 2	± 0	± 4	± 3	± 4

Results and Conclusions



- Final result:

  $\alpha_s(M_Z) = 0.1179^{+0.0015}_{-0.0014}$ [from Analysis (II)]

- Current error of $\alpha_s(M_Z)$ dominated by higher-order pert. corr. (beyond $\mathrm{N^3LL}$)

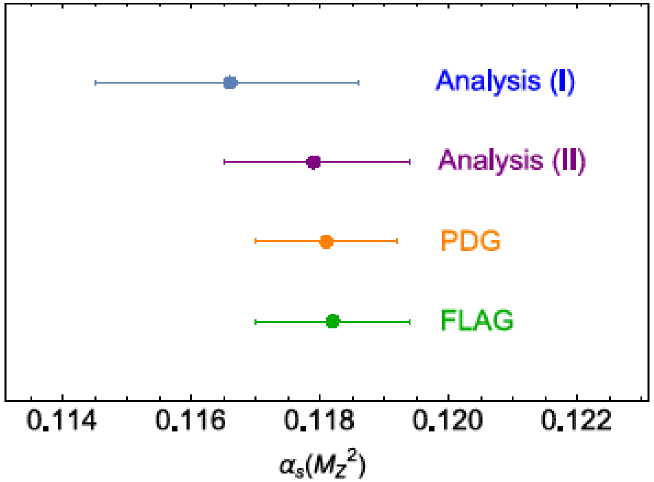
  ⇨ Expected to reduce if we use finer lattices.

- High quality agreement with OPE

  By subtracting $O(\Lambda_{\mathrm{QCD}}), O(\Lambda_{\mathrm{QCD}}^3 r^2)$ renormalons, accuracy and stability of prediction improves. ⇨ *wide matching range*

  Validity range extends to larger dist. $r \lesssim 0.5\ \mathrm{fm}$

Results and Conclusions



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Conclusions

- We use OPE with renormalon subtraction as a solution to the window problem.

$\boxed{\text{Window problem}}$ $\underbrace{1-2\ \mathrm{GeV}}_{\text{perturbation}} \lesssim E \lesssim \underbrace{1-2\ \mathrm{GeV}}_{\text{Lattice QCD}}$

$\boxed{\text{Our determination}}$ $\underbrace{0.6\ \mathrm{GeV}}_{\text{OPE+renormalon sub.}} \lesssim E \lesssim \underbrace{2\ \mathrm{GeV}}_{\text{Lattice QCD}}$

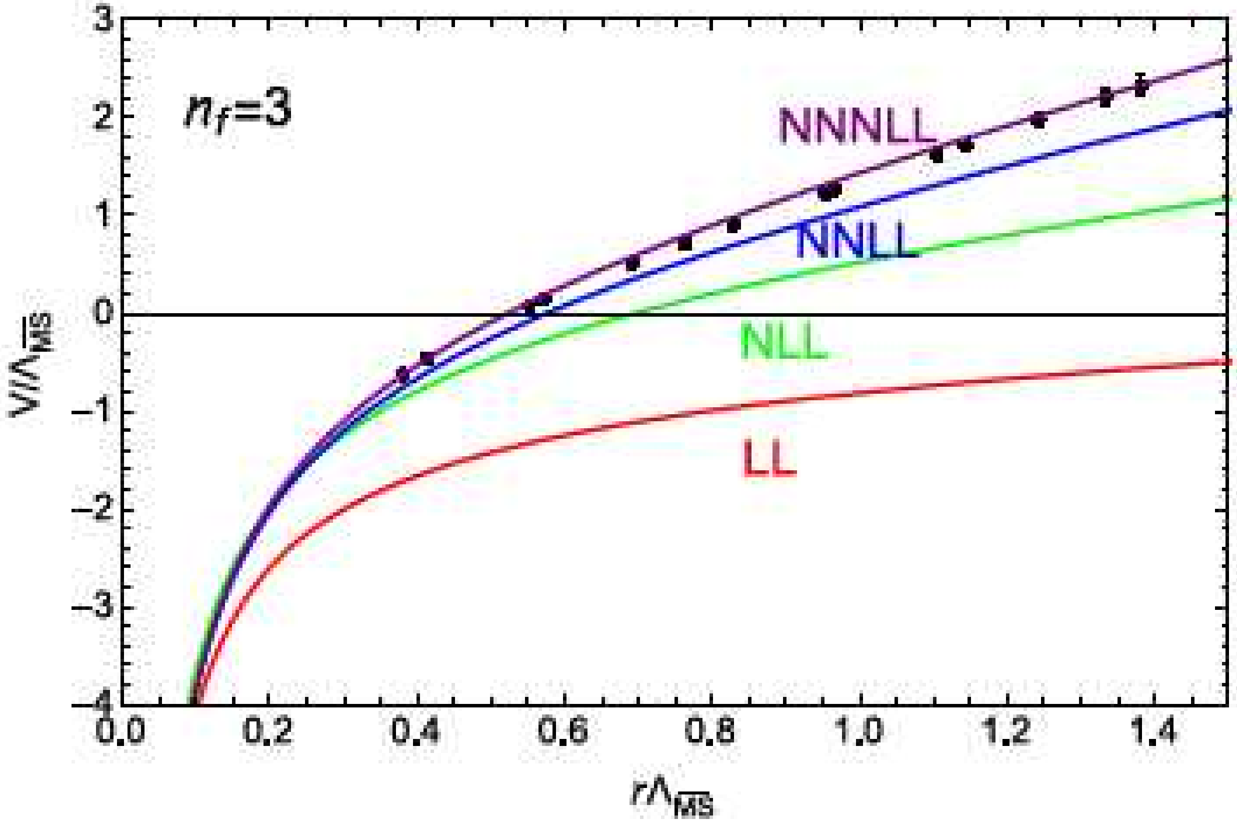
Reasonable fit for the wide range gives a reliable value of α_s .

- Dominant error comes from higher order uncertainty, which can be reduced by finer lattice simulation.

Rough estimate: In $a/2$, theoretical error would become ~1/3.

Consistency check

$\Lambda_{\overline{\mathrm{MS}}}^{\mathrm{PDG}} = 336\mathrm{MeV}$



Systematic errors

(a) Finite a effect (± 0.0002)

Previous analysis: $r \geq a \longrightarrow r \geq 2a$

(b) Higher order corrections $(^{+0.0012}_{-0.0010})$

$$V_S^{\mathrm{N^3LL}}(r) \to V_S^{\mathrm{N^3LL}}(r) \pm \delta V_S(r)$$

with $\delta V_S(r) = V_S^{\mathrm{N^3LL}}(r) - V_S^{\mathrm{N^2LL}}(r)$

(c) Matching range with OPE (± 0.0004)

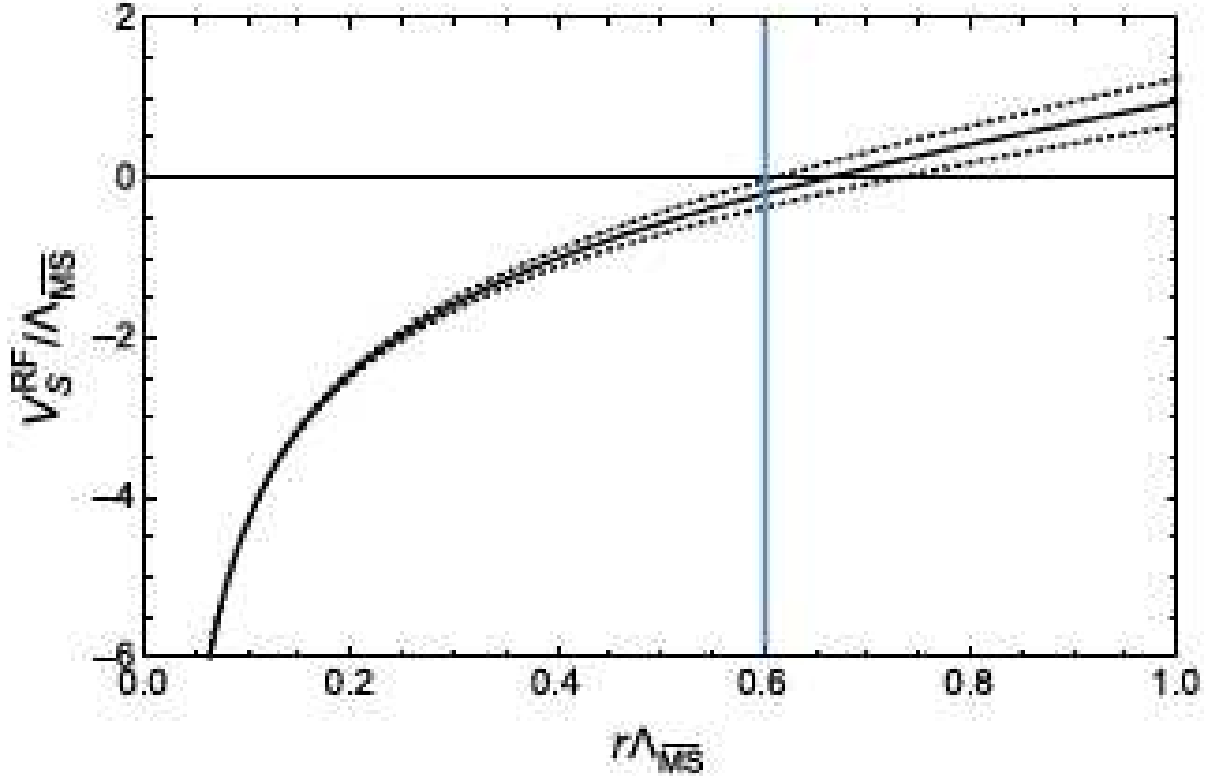
$$\Lambda_{\overline{MS}}^{\mathrm{PDG}} r < 0.6 \longrightarrow \Lambda_{\overline{MS}}^{\mathrm{PDG}} r < 0.5 \text{ or } 0.8$$

(d) IR divergence at 3-loop (± 0.0002)

Way to regularize IR divergence in 3-loop perturbative coefficient

Higher order uncertainty

$$V_S^{\rm RF} \pm \delta V_S^{\rm RF}(r) \quad \text{with} \quad \delta V_S^{\rm RF}(r) = V_S^{\rm RF}|_{\rm N^3LL} - V_S^{\rm RF}|_{\rm N^2LL}$$



Systematic errors

(e) Mass correction (± 0.0000)

Input masses in lattice simulation differ from physical values.

$$V_{\rm latt}(r;\{m^{\rm latt}\}_{u,d,s}) \to$$

$$V_{\rm latt}(r,\{\overline{m}\}_{u,d,s}) = V_{\rm latt}(r;\{m^{\rm latt}\}_{u,d,s}) + [V_{\rm pt}(r;\{\bar{m}\}_{u,d,s}) - V_{\rm pt}(r;\{m^{\rm latt}\}_{u,d,s})]$$

$\overline{m}$:MSbar mass ⟶ Also use constituent quark mass of 300MeV

(f) Scheme to extract μ_f independent part (± 0.0003)

O(r^3) term is added to V_s

(g) Lattice spacing (± 0.0004)

Shift of lattice spacing within its uncertainty (error of relative latt. spacing and the Wilson-flow scale)

Ultrasoft

Soft contribution

$$\alpha_V(q) = \sum_{n=0}^{3} [P_n(\log(\mu/q)) + \delta P_n(\log(\mu/q)]\alpha_s(\mu)\left(\frac{\alpha_s(\mu)}{4\pi}\right)^n$$

with $\delta P_3 = 72\pi^2\left(\frac{1}{\epsilon} + 6\log(\mu/q)\right)$

Ultrasoft correction

$$\left[-\delta P_3 + 72\pi^2\left(2\log(C_A\alpha_s(\mu)) + 2\gamma_E - \frac{5}{3}\right)\right]\alpha_s(\mu)\left(\frac{\alpha_s(\mu)}{4\pi}\right)^3$$

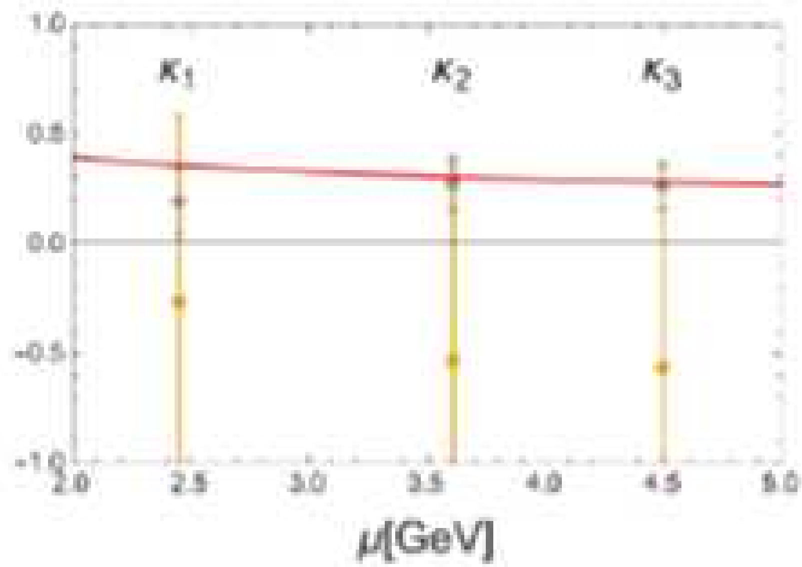
How to remove divergence

(a) Ultrasoft correction added

(b) Remove divergence by IR cutoff

Fitting parameters in global fit

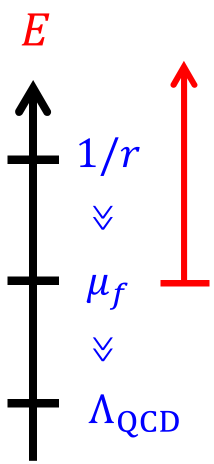
i (size)	$i=1$ ($32^3\times64$)		$i=2$ ($48^3\times96$)		$i=3$ ($64^3\times128$)	
d ($N_{i,d}$)	$d=1$ (4)	$d=2$ (3)	$d=1$ (6)	$d=2$ (4)	$d=1$ (8)	$d=2$ (5)
κ	0.19(15)	−0.26(85)	0.27(12)	−0.53(88)	0.27(11)	−0.57(91)
c_0 [GeV]	2.245(11)	2.300(87)	3.012(11)	3.099(89)	3.546(10)	3.631(86)
χ^2	$\chi^2/\mathrm{d.o.f.}=8.7/(30-16)$ (global fit)					
f_d	$f_1=0.0004(18)$, $f_2=-0.025(32)$ (common to all i)					
A_2	$A_2=-0.0091(54)$ GeV^3 (common to all i,d)					



UV contribution

$$V_{\rm UV}(r;\mu_f) \equiv -\int_{q>\mu_f} \frac{d^3\vec{q}}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}}\, C_F \frac{4\pi\alpha_{\rm 1L}(q)}{q^2}$$

$$\alpha_{\rm 1L}(q) = \frac{2\pi}{\beta_0 \log(q/\Lambda_{\rm QCD})}$$: LL resum.



We can separate μ_f -dependent and μ_f -independent parts as follows.

$$V_{\rm UV}(r;\mu_f) \overset{\int d\Omega_{\vec{q}}}{=} -\frac{2C_F}{\pi}\int_{\mu_f}^{\infty} dq\, \frac{\sin(qr)}{qr}\,\alpha_{\rm 1L}(q) = -\frac{2C_F}{\pi}{\rm Im}\int_{\mu_f}^{\infty} dq\, \frac{e^{iqr}}{qr}\,\alpha_{\rm 1L}(q)$$

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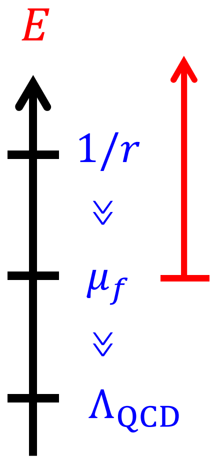

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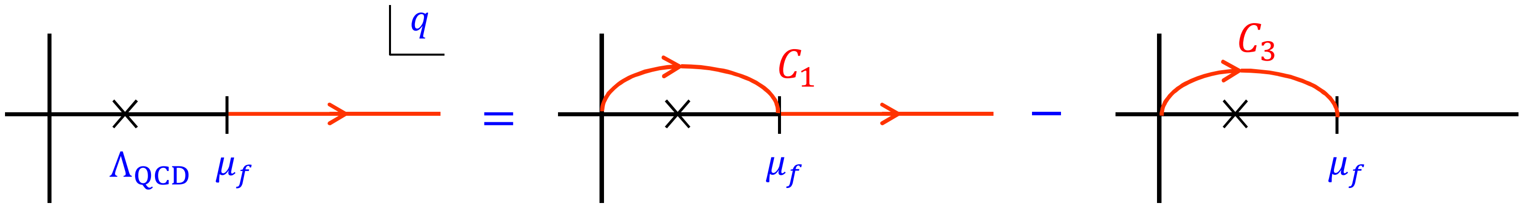

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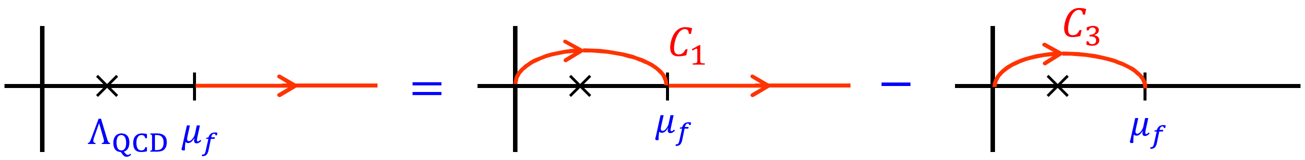
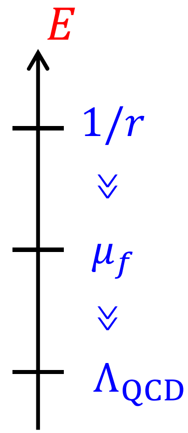


$$V_{UV}(r;\mu_f) = -\frac{2C_F}{\pi}\,\mathrm{Im}\int_{C_1} dq\,\frac{e^{iqr}}{qr}\,\alpha_{1L}(q) + \frac{2C_F}{\pi}\,\mathrm{Im}\int_{C_3} dq\,\frac{e^{iqr}}{qr}\,\alpha_{1L}(q)$$

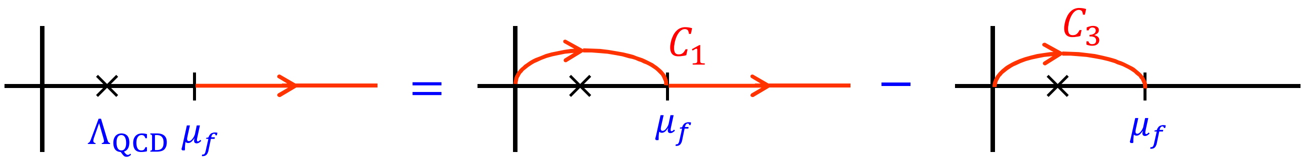


- Along C_3, justified to expand e^{iqr} in iqr, since $\mu_f r \ll 1$:

$$\frac{2C_F}{\pi}\mathrm{Im}\int_{C_3} dq\,\frac{1 + iqr + \frac{1}{2}(iqr)^2 + \cdots}{qr}\,\alpha_{1\mathrm{L}}(q) = \frac{A}{r} + B + \sigma r + Dr^2 + \mathcal{O}(r^3)$$

A and σ can be computed analytically.

UV contribution



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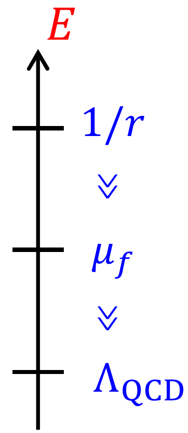

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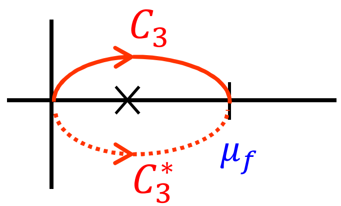
A and σ can be computed analytically:

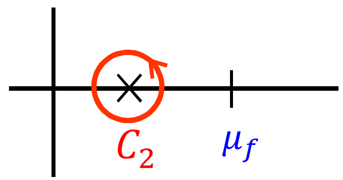


$$A = \frac{2C_F}{\pi}\mathrm{Im}\int_{C_3} dq\,\frac{\alpha_{1\mathrm{L}}(q)}{q} = \frac{C_F}{\pi i}\int_{C_3-C_3^*} dq\,\frac{\alpha_{1\mathrm{L}}(q)}{q}$$

$$= -\frac{C_F}{\pi i}\int_{C_2} dq\,\frac{\alpha_{1\mathrm{L}}(q)}{q} \underset{\uparrow}{=} -\frac{4\pi C_F}{\beta_0}.$$

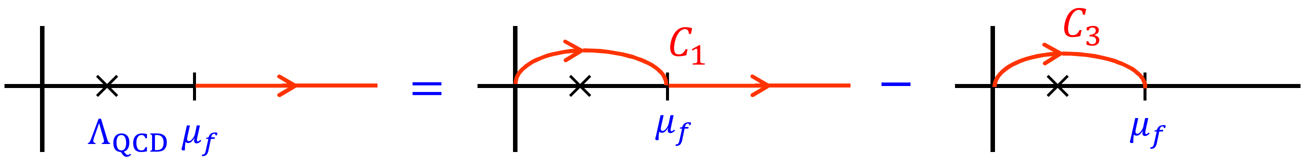
Cauchy thm

μ_f independent !



$$\sigma = \frac{2C_F}{\pi}\mathrm{Im}\int_{C_3} dq\left(-\frac{1}{2}q\right)\alpha_{1\mathrm{L}}(q) = \frac{C_F}{2\pi i}\int_{C_2} dq\,q\,\alpha_{1\mathrm{L}}(q) = \frac{2\pi C_F}{\beta_0}\Lambda_{\rm QCD}^2$$

UV contribution



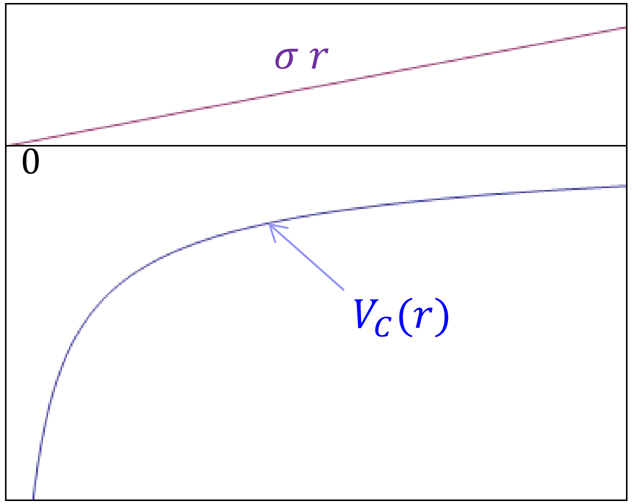
$$V_{UV}(r;\mu_f) = -\frac{2C_F}{\pi}\,\mathrm{Im}\int_{C_1} dq\,\frac{e^{iqr}}{qr}\,\alpha_{1L}(q) + \frac{2C_F}{\pi}\,\mathrm{Im}\int_{C_3} dq\,\frac{e^{iqr}}{qr}\,\alpha_{1L}(q)$$

$$= V_C(r) + B + \sigma r + Dr^2 + O(r^3)$$

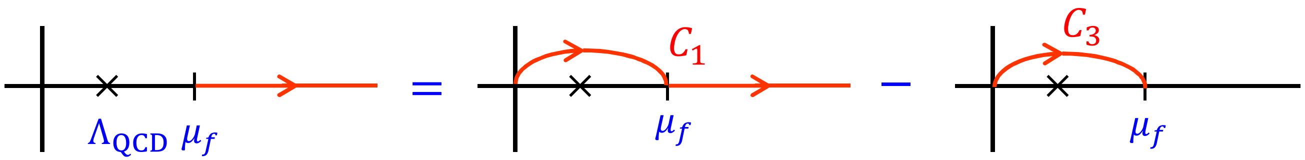
μ_f independent

$$V_C(r) = \frac{A}{r} - \frac{2C_F}{\pi}\,\mathrm{Im}\int_{C_1} dq\,\frac{e^{iqr}}{qr}\,\alpha_{1\mathrm{L}}(q).$$

$$V_C(r) \to \begin{cases} -\dfrac{2\pi C_F}{\beta_0}\,\dfrac{1}{r\,\left|\log(\Lambda_{\rm QCD}\,r)\right|}, & r\to 0, \\ -\dfrac{4\pi C_F}{\beta_0 r}, & r\to\infty. \end{cases}$$



UV contribution



$$V_{UV}(r;\mu_f) = -\frac{2C_F}{\pi}\,\mathrm{Im}\int_{C_1} dq\,\frac{e^{iqr}}{qr}\,\alpha_{1L}(q) + \frac{2C_F}{\pi}\,\mathrm{Im}\int_{C_3} dq\,\frac{e^{iqr}}{qr}\,\alpha_{1L}(q)$$

$$= V_C(r) + B + \sigma r + Dr^2 + O(r^3)$$

$\Leftarrow$ A short-distance expansion in r with correct RG log in Coulomb term (as $r \to 0$)

μ_f independent

$$V_C(r) = \frac{A}{r} - \frac{2C_F}{\pi}\,\mathrm{Im}\int_{C_1} dq\,\frac{e^{iqr}}{qr}\,\alpha_{1\mathrm{L}}(q).$$

$$V_C(r) \to \begin{cases} -\dfrac{2\pi C_F}{\beta_0}\,\dfrac{1}{r\,\left|\log(\Lambda_{\mathrm{QCD}}\,r)\right|}, & r \to 0, \\ -\dfrac{4\pi C_F}{\beta_0 r}, & r \to \infty. \end{cases}$$