Computing *V mode* power spectra

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Based on M. Lembo, F. Forastieri et al. (in preparation)

Outline:

- Stokes parameters;
- summary of some circular polarisation sources: Faraday conversion;
- Boltzmann equation with the Faraday conversion like term;
- calculation of the four source terms: $\tilde{S}_I^{+,\times}$, $\tilde{S}_Q^{+,\times}$, $\tilde{S}_U^{+,\times}$ and $\tilde{S}_V^{+,\times}$;
- implementation in the CAMB code;
- some results for the 1^{st} order approximation in $\Delta\alpha$.

Stokes Parameters

For a nearly monochromatic plane electromagnetic wave propagating in the z-direction, the components of the electric field are:

$$E_x = a_x(t)\cos(\omega_0 t - \delta_x) \qquad \text{and} \qquad E_y = a_y(t)\cos(\omega_0 t - \delta_y),$$

where ω_0 is the mean frequency, a_x and a_y are the amplitudes, and δ_x and δ_y the phase angles.

Stokes parameters

$$I \equiv \langle a_x^2 \rangle + \langle a_y^2 \rangle$$

$$Q \equiv \langle a_x^2 \rangle - \langle a_y^2 \rangle$$

$$U \equiv \langle 2 a_x a_y \cos(\delta_x - \delta_y) \rangle$$

$$V \equiv \langle 2 a_x a_y \sin(\delta_x - \delta_y) \rangle$$

- $Q = U = V = 0$
  unpolarized radiation
- Q and/or $U \neq 0$
  linearly polarized radiation
- $V \neq 0$
  circular polarized radiation

Summary of some *circular polarisation* sources

Source	Mechanism for CP	Frequency dependence	B dependence	Predicted CP signal in δV (K) at ν=10 GHz
Primordial	primordial B+ Compton scattering [24]	ν^{-3}	B	10^{-9}
Primordial	Lorentz invariance violations [28]	ν^{-3}	NA	10^{-12}
primordial	Non-commutivity [25, 26]	ν^{-1}	NA	10^{-12}
primordial	B+Thomson scattering [23]	ν^{-3}	B^2	10^{-12}
Cosmic neutrino background (CνB)	Scattering with left handed neutrinos [27]	ν^{-1}	NA	10^{-8}
Pop III stars	FC [20, 31]	ν^{-3}	B^2	few $\times\ 10^{-6}$ ($\ell \sim 1000$, $t_{\rm age}=10^4$ yr, $N_{\rm p}=100$) few $\times\ 10^{-5}$ ($\ell \sim 1000$, $t_{\rm age}=10^4$ yr, $N_{\rm p}=1000$) few $\times\ 10^{-7}$ ($\ell \sim 100$, $t_{\rm age}=10^4$ yr, $N_{\rm p}=100$)
Galaxy clusters	FC	ν^{-3}	B^2	10^{-10} ($\ell \sim 1000$ [39])
Galactic synchrotron	intrinsic emission [21]	$\nu^{(-2-\alpha_{\rm sync}/2)}$	$B^{3/2}$	10^{-8} ($\ell \sim 100$) $<10^{-9}$ ($\ell \sim 500$)

[S. King, P. Lubin (2016)]

Faraday conversion

Linearly polarised light propagates through cold magnetised plasma, it undergoes Faraday rotation.
Light traversing relativistic plasma undergoes both Faraday conversion and Faraday rotation.

Faraday rotation: rotation of the linear polarisation between Q and U parameters (it does not lead to circular polarisation!)

$$\dot{Q} = -2U\frac{d\Delta\phi_{FR}}{dt} \qquad and \qquad \dot{U} = 2Q\frac{d\Delta\phi_{FR}}{dt} \qquad \text{with} \qquad \Delta\phi_{FR} \propto B$$

Faraday conversion: in presence of relativistic electrons, linearly polarised radiation can be *converted* into circularly polarised radiation

$$\dot{V} = 2U\frac{d\Delta\phi_{FC}}{dt} \qquad \text{with} \qquad \Delta\phi_{FC} \propto B^2$$

[A. Cooray, A. Melchiorri, J. Silk (2002)]

Boltzmann equation for tensor perturbation

In linear perturbation theory, *scalar* and *tensor* perturbations evolve independently.

- In the SP framework, U and V have no source terms;
- In the TP framework, U has source term, V does not (at least in the ΛCDM).

$$\begin{aligned}
\frac{\partial \tilde{\Delta}_I^{+,\times}}{\partial \eta} + \imath k \mu \tilde{\Delta}_I^{+,\times} - 2\frac{\partial h^{+,\times}}{\partial \eta} &= \frac{d\tau}{d\eta}\left(\tilde{\Delta}_I^{+,\times} + \tilde{\Lambda}^{+,\times}\right) \\
\frac{\partial \tilde{\Delta}_Q^{+,\times}}{\partial \eta} + \imath k \mu \tilde{\Delta}_Q^{+,\times} &= \frac{d\tau}{d\eta}\left(\tilde{\Delta}_Q^{+,\times} - \tilde{\Lambda}^{+,\times}\right) \\
\frac{\partial \tilde{\Delta}_U^{+,\times}}{\partial \eta} + \imath k \mu \tilde{\Delta}_U^{+,\times} &= \frac{d\tau}{d\eta}\left(\tilde{\Delta}_U^{+,\times} - \tilde{\Lambda}^{+,\times}\right) \\
\frac{\partial \tilde{\Delta}_V^{+,\times}}{\partial \eta} + \imath k \mu \tilde{\Delta}_V^{+,\times} &= \frac{d\tau}{d\eta}\tilde{\Delta}_V^{+,\times}
\end{aligned}$$

where the source term is

$$\tilde{\Lambda}^{+,\times} \equiv -\frac{3}{70}\tilde{\Delta}_{I4}^{+,\times} + \frac{1}{7}\tilde{\Delta}_{I2}^{+,\times} - \frac{1}{10}\tilde{\Delta}_{I0}^{+,\times} + \frac{3}{70}\tilde{\Delta}_{Q4}^{+,\times} + \frac{6}{7}\tilde{\Delta}_{Q2}^{+,\times} + \frac{3}{5}\tilde{\Delta}_{Q0}^{+,\times} .$$

[A. Kosowsky (1994)]

Boltzmann equation for tensor perturbation plus a Faraday conversion like term

$$
\begin{aligned}
\frac{\partial \tilde{\Delta}_I^{+,\times}}{\partial \eta} + \imath k \mu \tilde{\Delta}_I^{+,\times} - 2\frac{\partial h^{+,\times}}{\partial \eta} &= \frac{d\tau}{d\eta}\left(\tilde{\Delta}_I^{+,\times} + \tilde{\Lambda}^{+,\times}\right) \\
\frac{\partial \tilde{\Delta}_Q^{+,\times}}{\partial \eta} + \imath k \mu \tilde{\Delta}_Q^{+,\times} &= \frac{d\tau}{d\eta}\left(\tilde{\Delta}_Q^{+,\times} - \tilde{\Lambda}^{+,\times}\right) \\
\frac{\partial \tilde{\Delta}_U^{+,\times}}{\partial \eta} + \imath k \mu \tilde{\Delta}_U^{+,\times} &= \frac{d\tau}{d\eta}\left(\tilde{\Delta}_U^{+,\times} - \tilde{\Lambda}^{+,\times}\right) -2\,\tilde{\Delta}_V^{+,\times}\frac{d(\Delta\alpha)}{d\eta} \\
\frac{\partial \tilde{\Delta}_V^{+,\times}}{\partial \eta} + \imath k \mu \tilde{\Delta}_V^{+,\times} &= \frac{d\tau}{d\eta}\tilde{\Delta}_V^{+,\times} +2\,\tilde{\Delta}_U^{+,\times}\frac{d(\Delta\alpha)}{d\eta}
\end{aligned}
$$

where the source term is

$$
\tilde{\Lambda}^{+,\times} \equiv -\frac{3}{70}\tilde{\Delta}_{I4}^{+,\times} + \frac{1}{7}\tilde{\Delta}_{I2}^{+,\times} - \frac{1}{10}\tilde{\Delta}_{I0}^{+,\times} + \frac{3}{70}\tilde{\Delta}_{Q4}^{+,\times} + \frac{6}{7}\tilde{\Delta}_{Q2}^{+,\times} + \frac{3}{5}\tilde{\Delta}_{Q0}^{+,\times}\,.
$$

The $\Delta\alpha(\eta)$ function is the phase shift of the *Faraday conversion*.

Disentangling the U and V equations

In order to disentangle the U and V equations, we write them in a vectorial form, introducing

$$\vec{\Delta}^{+,\times} \equiv \begin{pmatrix} \tilde{\Delta}_V^{+,\times} \\ \tilde{\Delta}_U^{+,\times} \end{pmatrix} \qquad \text{and} \qquad \vec{\Lambda}^{+,\times} \equiv \begin{pmatrix} 0 \\ \tilde{\Lambda}^{+,\times} \end{pmatrix} .$$

Then, diagonalising the *Faraday conversion like term* through the rotation matrix

$$R \equiv \begin{pmatrix} 1 & -\imath \\ 1 & \imath \end{pmatrix} ,$$

we obtain

$$\begin{aligned} \tilde{\Delta}_{V,\,\mathrm{new}}^{+,\times\,\prime} + \imath k\mu \tilde{\Delta}_{V,\,\mathrm{new}}^{+,\times} &= \tau' \tilde{\Delta}_{V,\,\mathrm{new}}^{+,\times} + 2\imath(\Delta\alpha)' \tilde{\Delta}_{V,\,\mathrm{new}}^{+,\times} + \imath\tau' \tilde{\Lambda}^{+,\times} \\ \tilde{\Delta}_{U,\,\mathrm{new}}^{+,\times\,\prime} + \imath k\mu \tilde{\Delta}_{U,\,\mathrm{new}}^{+,\times} &= \tau' \tilde{\Delta}_{U,\,\mathrm{new}}^{+,\times} - 2\imath(\Delta\alpha)' \tilde{\Delta}_{U,\,\mathrm{new}}^{+,\times} - \imath\tau' \tilde{\Lambda}^{+,\times} , \end{aligned}$$

where $\vec{\Delta}_{\mathrm{new}}^{+,\times} \equiv R\vec{\Delta}^{+,\times}$.

Line of sight approach

We formally integrate the Boltzmann equation with the FC like term along the photon past light cone, and, then, we write an equation for each multipole ℓ.

For example for $\tilde{\Delta}_U^{+,\times}$ and $\tilde{\Delta}_V^{+,\times}$, we obtain:

$$\tilde{\Delta}_{U;\ell}^{+,\times}(k,\eta_0) = \int_{\eta_i}^{\eta_0} d\eta\, \jmath_\ell\left[k(\eta_0-\eta)\right]\, \tilde{S}_U^{+,\times}(\eta)$$

$$\tilde{\Delta}_{V;\ell}^{+,\times}(k,\eta_0) = \int_{\eta_i}^{\eta_0} d\eta\, \jmath_\ell\left[k(\eta_0-\eta)\right]\, \tilde{S}_V^{+,\times}(\eta)$$

where the source terms are

$$\tilde{S}_U^{+,\times} \equiv g\, \cos\left[2\left(\Delta\alpha(\eta) - \Delta\alpha_0\right)\right]\tilde{\Lambda}^{+,\times} \text{and } \tilde{S}_V^{+,\times} \equiv -g\, \sin\left[2\left(\Delta\alpha(\eta) - \Delta\alpha_0\right)\right]\tilde{\Lambda}^{+,\times}.$$

We have already gone back to the unrotated expression, through the relation:

$$\begin{pmatrix} \tilde{\Delta}_{V;\ell}^{+,\times} \\ \tilde{\Delta}_{U;\ell}^{+,\times} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \tilde{\Delta}_{V,\,\text{new};\,\ell}^{+,\times} + \tilde{\Delta}_{U,\,\text{new};\,\ell}^{+,\times} \\ \imath\tilde{\Delta}_{V,\,\text{new};\,\ell}^{+,\times} - \imath\tilde{\Delta}_{U,\,\text{new};\,\ell}^{+,\times} \end{pmatrix}.$$

The 1^{st} order approximation in $\Delta\alpha$

At the first order in $(\Delta\alpha(\eta) - \Delta\alpha_0)$, the formal solutions of the Boltzmann equation plus the FC like term are:

$$\begin{aligned}
\tilde{\Delta}^{+,\times}_{I;\,\ell}(k,\eta_0) &= \int_{\eta_i}^{\eta_0} d\eta\, \jmath_\ell\,[k(\eta_0-\eta)]\ \tilde{S}^{+,\times}_{I}(\eta) \\
\tilde{\Delta}^{+,\times}_{Q;\,\ell}(k,\eta_0) &= \int_{\eta_i}^{\eta_0} d\eta\, \jmath_\ell\,[k(\eta_0-\eta)]\ \tilde{S}^{+,\times}_{Q}(\eta) \\
\tilde{\Delta}^{+,\times}_{U;\,\ell}(k,\eta_0) &= \int_{\eta_i}^{\eta_0} d\eta\, \jmath_\ell\,[k(\eta_0-\eta)]\ \tilde{S}^{+,\times}_{U}(\eta) \\
\tilde{\Delta}^{+,\times}_{V;\,\ell}(k,\eta_0) &= \int_{\eta_i}^{\eta_0} d\eta\, \jmath_\ell\,[k(\eta_0-\eta)]\ \tilde{S}^{+,\times}_{V}(\eta)\,,
\end{aligned}$$

where the source term are

$$\tilde{S}^{+,\times}_{I} \equiv g\,\tilde{\Lambda}^{+,\times} + 2h' e^{-\tau} \qquad \tilde{S}^{+,\times}_{Q} \equiv g\,\tilde{\Lambda}^{+,\times}$$

$$\tilde{S}^{+,\times}_{U} \equiv g\,\tilde{\Lambda}^{+,\times} \qquad \tilde{S}^{+,\times}_{V} \equiv -2\,g\,\left[\Delta\alpha(\eta) - \Delta\alpha_0\right]\,\tilde{\Lambda}^{+,\times}\,.$$

Power spectrum of tensor modes

The power spectra are given by

$$\mathrm{C}_{X\ell}^{(T)} \propto \int k^2 \mathrm{d}k \, P_h(k) \left[\Delta_{X\ell}^{(T)}(k)\right]^2$$

$$\mathrm{C}_{C\ell}^{(T)} \propto \int k^2 \mathrm{d}k \, P_h(k) \, \Delta_{X\ell}^{(T)}(k) \, \Delta_{Y\ell}^{(T)}(k) \,,$$

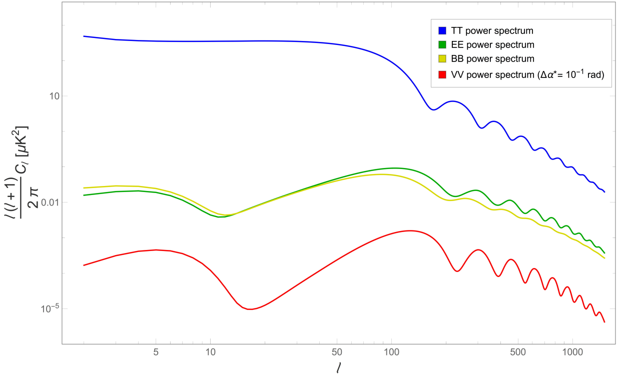
where X and Y stand for T, E, B or V.

While $P_h(k)$ is the primordial power spectrum.

[M. Zaldarriaga, U. Seljak (1997)]

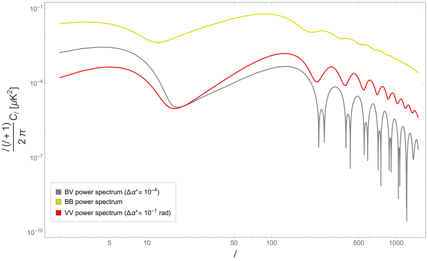
Choosing the $\Delta\alpha(\eta)$ function - *preliminary results*

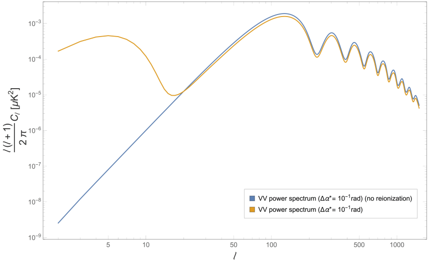
For the rest of the analysis, it is chosen $\Delta\alpha(\eta) - \Delta\alpha_0 = \Delta\alpha^* = 10^{-1} rad$.



These are **only** for tensor perturbations.

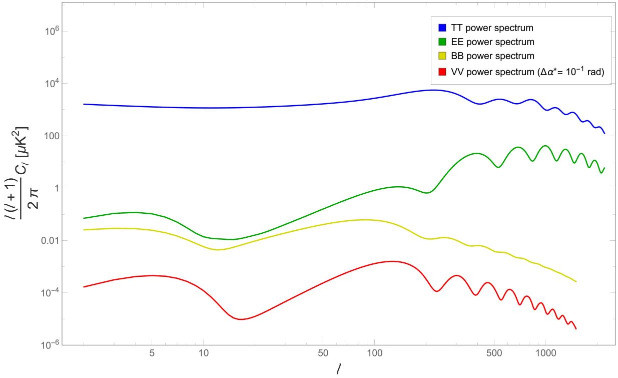
Choosing the $\Delta\alpha(\eta)$ function - *preliminary results*





These are **only** for tensor perturbations.

Choosing the $\Delta\alpha(\eta)$ function - *preliminary results*



These are **both** for scalar and tensor perturbations.

Conclusion and future perspectives:

- we did the groundwork for a new tool allowed to compute the *primordial circular polarisation* no matter is the physics beyond;
- we are working on the implementation of this *tool* at higher orders: the idea is to have a *plug-in* for CAMB;
- we are going to put in the $\Delta\alpha$ function the informations of different physical models;
- the goal is to achieve an upper limit on the primordial circular polarisation for the future experiments (CLASS, SPIDER, PIPER).

Thanks for your attention!

Coordinate trasformation [A. G. Polnarev, 1985]

As concerns tensor perturbations, it is customary to introduce the following coordinate transformation:

$$
\begin{aligned}
\Delta^{+}_{I,\,V} &= (1-\mu^2)\cos(2\phi)\tilde{\Delta}^{+}_{I,\,V}\,, \quad & \Delta^{\times}_{I,\,V} &= (1-\mu^2)\sin(2\phi)\tilde{\Delta}^{\times}_{I,\,V}\,,\\
\Delta^{+}_{Q} &= (1+\mu^2)\cos(2\phi)\tilde{\Delta}^{+}_{Q}\,, \quad & \Delta^{\times}_{Q} &= (1+\mu^2)\sin(2\phi)\tilde{\Delta}^{\times}_{Q}\,,\\
\Delta^{+}_{U} &= -2\mu\sin(2\phi)\tilde{\Delta}^{+}_{U}\,, \quad & \Delta^{\times}_{U} &= 2\mu\cos(2\phi)\tilde{\Delta}^{\times}_{U}\,.
\end{aligned}
$$

Stokes Parameters and Boltzmann equation

Stokes parameter in QM description

$$\hat{I} \equiv |\epsilon_1\rangle\langle\epsilon_1| + |\epsilon_2\rangle\langle\epsilon_2|$$
$$\hat{Q} \equiv |\epsilon_1\rangle\langle\epsilon_1| - |\epsilon_2\rangle\langle\epsilon_2|$$
$$\hat{U} \equiv |\epsilon_1\rangle\langle\epsilon_2| + |\epsilon_2\rangle\langle\epsilon_1|$$
$$\hat{V} \equiv i\,|\epsilon_2\rangle\langle\epsilon_1| - i\,|\epsilon_1\rangle\langle\epsilon_2|$$

An arbitrary state for a photon propagating in the z-direction
$|\epsilon\rangle = a_1 e^{i\theta_1}|\epsilon_1\rangle + a_2 e^{i\theta_2}|\epsilon_2\rangle$

Density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} I+Q & U-iV \\ U+iV & I-Q \end{pmatrix} = \frac{1}{2}\left(I\mathbb{1} + Q\sigma_3 + U\sigma_1 + V\sigma_2\right)$$

The time evolution of the number operator, $\hat{\mathscr{D}}_{ij}(\mathbf{k}) = \hat{a}_i^\dagger(\mathbf{k})\hat{a}_j(\mathbf{k})$, in terms of $\rho \Rightarrow$

Boltzmann equation for the density matrix

$$(2\pi)^3\delta^3(0)2k^0\frac{d}{dt}\rho_{ij}(0,\mathbf{k}) = i\langle[\hat{\mathcal{H}}_{\rm int}(0), \hat{\mathcal{D}}_{ij}(\mathbf{k})]\rangle - \frac{1}{2}\int_{-\infty}^{\infty} dt'\ \langle[\hat{\mathcal{H}}_{\rm int}(t'), [\hat{\mathcal{H}}_{\rm int}(0), \hat{\mathcal{D}}_{ij}(\mathbf{k})]]\rangle,$$

Line of signt approach

$$\tilde{\Delta}^{+,\times}_{U,\,\text{new};\,\ell}(k,\eta_0) = \imath \int_{\eta_i}^{\eta_0} d\eta\, \jmath_\ell\,[k(\eta_0-\eta)]\ \tilde{S}^{+,\times}_{U,\,new}(\eta)$$

$$\tilde{\Delta}^{+,\times}_{V,\,\text{new};\,\ell}(k,\eta_0) = \imath \int_{\eta_i}^{\eta_0} d\eta\, \jmath_\ell\,[k(\eta_0-\eta)]\ \tilde{S}^{+,\times}_{V,\,new}(\eta)$$

where the source term are

$$\tilde{S}^{+,\times}_{U,new} \equiv g\, e^{2\imath[\Delta\alpha(\eta)\,-\Delta\alpha(\eta_0)]}\tilde{\Lambda}^{+,\times} \qquad \text{and} \qquad \tilde{S}^{+,\times}_{V,new} \equiv -g\, e^{-2\imath[\Delta\alpha(\eta)\,-\Delta\alpha(\eta_0)]}\tilde{\Lambda}^{+,\times}\,.$$

Going back to the unrotated expression

$$\begin{aligned}\tilde{\Delta}^{+,\times}_{U;\,\ell}(k,\eta_0) &= \int_{\eta_i}^{\eta_0} d\eta\, \jmath_\ell\,[k(\eta_0-\eta)]\ \tilde{S}^{+,\times}_{U}(\eta)\\ \tilde{\Delta}^{+,\times}_{V;\,\ell}(k,\eta_0) &= \int_{\eta_i}^{\eta_0} d\eta\, \jmath_\ell\,[k(\eta_0-\eta)]\ \tilde{S}^{+,\times}_{V}(\eta)\end{aligned}$$

where the source terms are

$$\tilde{S}^{+,\times}_{U} \equiv g\ \cos\left[2\left(\Delta\alpha(\eta)\ -\ \Delta\alpha_0\right)\right]\tilde{\Lambda}^{+,\times} \text{and}\ \ \tilde{S}^{+,\times}_{V} \equiv -g\ \sin\left[2\left(\Delta\alpha(\eta)\ -\ \Delta\alpha_0\right)\right]\tilde{\Lambda}^{+,\times}.$$