(Bound) Entanglement Detection with Mutually Unbiased Bases

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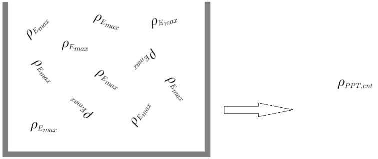
Outline

Motivation

Goal: detect bound entangled states

What is bound entanglement?



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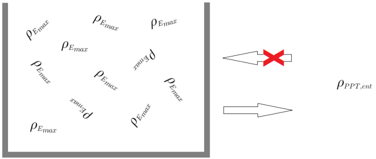
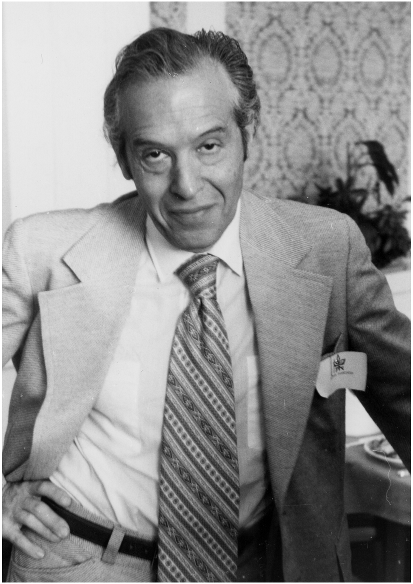


Peres-Horodecki Criterion:
$(\mathbb{I} \otimes T)\rho_{sep} \geq 0$

- for $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ with $d_1 \times d_2 \geq 6$ necessary but not sufficient!



Two unitary operators A and B are complementary if

[1] J. Schwinger, "UNITARY OPERATOR BASES", , Proceedings of the National Academy of Sciences **46**, 570–579 (1960).

Two unitary operators A and B are complementary if

- both generators of complete, orthonormal operator bases, having d non-degenerate eigenvalues
    - Orthonormality: $\langle v_{a,i} | v_{a,j} \rangle = \delta_{ij}$
    - Completeness Relation: $\sum_{i=1}^{d} |v_{a,i}\rangle\langle v_{a,i}| = \sum_{i=1}^{d} |v_{b,i}\rangle\langle v_{b,i}| = 1$
    - Nondegeneracy of eigenvalues: $\lambda_{a,i} \neq \lambda_{a,j}$

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- all normalized vectors $|v_{a,i}\rangle, |v_{b,i}\rangle$ are maximally incompatible, such that the $p(v_{a,i}, v_{b,j})$ for a≠b becomes
  - Mutual Unbiasedness:

$$p(v_{a,i}, v_{b,j}) \overset{a \neq b}{=} |\langle v_{a,i} | v_{b,j} \rangle|^2 \quad (1)$$

$$\overset{a \neq b}{=} \delta_{a,b}\delta_{i,j} + \frac{1}{d}(1 - \delta_{a,b}) \quad (2)$$

$$\overset{a \neq b}{=} \frac{1}{d} \quad (3)$$

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What are MUBs?
J.Schwinger - Complementarity of Unitary Operators[1]

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MUBs feature the following properties:

- outcomes of systems prepared in an eigenstate of A are equally likely when the prepared system is measured in any eigenbasis of operator B
- Lower bound of entropic uncertainty relations maximized

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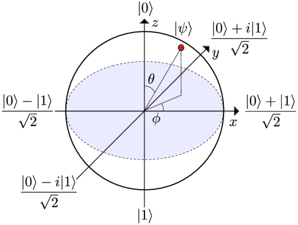

Bloch sphere representation of qubit[a]

$B_1 = \{|0\rangle, |1\rangle\}$ Eigenvectors of σ_x

$B_2 = \{\frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \frac{|0\rangle - i|1\rangle}{\sqrt{2}}\}$ Eigenvectors of σ_y

$B_3 = \{\frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle - |1\rangle}{\sqrt{2}}\}$ Eigenvectors of σ_z

[a] A. Frisk Kockum, *Quantum optics with artificial atoms*, (2014).

How can MUBs be constructed?

Various constructions via Fourier transformations (pair), Latin squares, Galois fields, generalized Pauli groups but only for d=p or d=p^n !

[2] T. Durt, et al., "On mutually unbiased bases", International Journal of Quantum Information **08**, 535–640 (2010).

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- cyclic unitary operators X and Z (analogously to Pauli matrices)
  - $X|\hat{j}\rangle = |\hat{j}\rangle \omega^j$, $X^d = 1$
  - $Z|k\rangle = |k\rangle \omega^k$, $Z^d = 1$

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- in the computational basis
  - $X = \sum_{k=0}^{d-1} |(k+1 \mod d)\rangle\langle k|$
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- for d=p: eigenvectors of X,Z,XZ^2,...,XZ^{d-1}
- for d=p^n: redefine shifts as n shifts mod p, l=(0...d-1)
  - $V_l^0 = X^l = \sum_{k=0}^{d-1} |k \oplus l\rangle\langle k|$
  - $V_0^l = Z^l = \sum_{k=0}^{d-1} \omega^{(k \odot l)}|k\rangle\langle k|$
  - $V_i^j = V_0^j V_i^0 = \sum_{k=0}^{d-1} \omega^{(k\oplus i)\odot j}|k \oplus i\rangle\langle k|$
  - d+1 commuting sets of d elements from these d^2 unitary operators

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The MUB Criterion[3]

Consider two observables a and b on system A and B

Mutual Predictability C_{ab}

$$C_{ab} = \sum_{i=0}^{d-1} P_{ab}(i, i) \tag{4}$$

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$$C_{ab} = \sum_{i=0}^{d-1} \langle i_a| \otimes \langle i_b|\rho|i_a\rangle \otimes |i_b\rangle \tag{5}$$

If $C_{ab} = 1 \rightarrow$ fully correlated
Else if $C_{ab} = \frac{1}{d} \rightarrow$ completely uncorrelated

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Solution: Conduct measurements in multiple MUBs

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The MUB Criterion

For an arbitrary product state $\rho = |a\rangle\langle a| \otimes |b\rangle\langle b|$

$$?? \leq I_m(\rho_{sep}) = \sum_{k=1}^{m} C_{kk} = \sum_{k=1}^{m} \sum_{i} |\langle i_a|a\rangle|^2 |\langle i_b|b\rangle|^2 \leq 1 + \frac{m-1}{d} \qquad (6)$$

For a complete set of $d+1$ MUBs

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Proof: Use inequality of arithmetic and geometric means:

$$\frac{\sum_i^n x_i}{n} \geq \sqrt[n]{\prod_i x_i} \qquad (8)$$

Then

$$I_m(\rho_{sep}) = \sum_{k=1}^{m} C_{kk} = \sum_{k=1}^{m} \sum_{i} |\langle i_a|a\rangle|^2 |\langle i_b|b\rangle|^2 \leq \frac{1}{2}\left(\sum_{k=1}^{m} \sum_{i} |\langle i_a|a\rangle|^4 + |\langle i_b|b\rangle|^4\right) \qquad (9)$$

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Because for any pure state $|a\rangle$ it holds $\sum_{k=1}^{m} \sum_{i=0}^{d-1} |\langle i_k|a\rangle|^4 \leq 1 + \frac{m-1}{d}$, we have

$$I_m \leq 1 + \frac{m-1}{d} \tag{10}$$

Entanglement Witnesses[4][5]

[4] R. A. Bertlmann, and P. Krammer, "Entanglement witnesses and geometry of entanglement of two-qutrit states", Annals of Physics **324**, 1388 –1407 (2009).

[5] M. Lewenstein, et al., "Optimization of entanglement witnesses", Physical Review A **62**, 052310 (2000).

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- only witnesses related to indecomposable PNCP maps may detect bound entanglement in the PPT region

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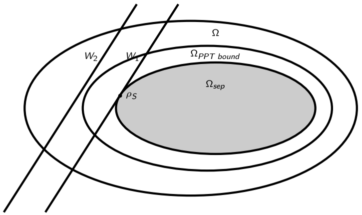

Entanglement witnesses and geometrical optimality

[4] R. A. Bertlmann, and P. Krammer, "Entanglement witnesses and geometry of entanglement of two-qutrit states", Annals of Physics **324**, 1388 –1407 (2009).

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Convex subset $\mathcal{S}$ in an affine space

- a collection of points where for any two points x_1, x_2 it holds that a mixture x also belongs to it:

$$x = \lambda_1 x_1 + \lambda_2 x_2 \in \mathcal{S}, \quad \lambda_1 + \lambda_2 = 1, \quad \lambda_i \geq 0 \tag{11}$$

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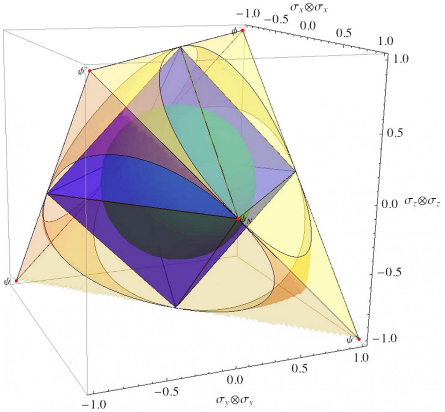
$$x = \sum_{i=0}^{p} \lambda_i x_i, \quad \sum_{i=0}^{p} \lambda_i = 1, \quad 0 \leq \lambda_i \leq 1 \tag{12}$$

p	0	1	2	3
geometrical object	point	line segment	triangle	tetraeder

[6] I. Bengtsson, and K. Zyczkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement*, (Cambridge University Press, 2006).



- 4 Bell states as corner points
- yellow: positivity
- blue: PPT (separable states)
- green: Kuś-Życzkowski ball
- dark yellow (local states satisfying Bell inequality)

[7] W. Thirring, et al., "Entanglement or separability: The choice of how to factorize the algebra of a density matrix", The European Physical Journal D **64**, arXiv: 1106.3047, 181–196 (2011).

Detecting (Bound) Entanglement with MUBs within the Magic Simplex

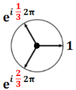
Special Simplex in $\mathcal{H}_d \otimes \mathcal{H}_d$[8]

For bipartite systems in $\mathcal{H}_d \otimes \mathcal{H}_d$, a simplex can be constructed via

$$\mathcal{W} = \left\{ \sum_{kl} c_{kl} P_{kl} \,\middle|\, c_{kl} \geq 0, \sum_{kl} c_{kl} = 1 \right\} \tag{13}$$

where

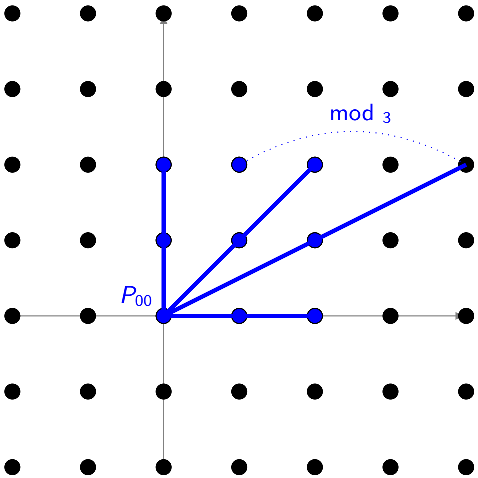
- $P_{kl} = |\Omega_{kl}\rangle\langle\Omega_{kl}|$
- $|\Omega_{kl}\rangle = (W_{kl} \otimes \mathbb{I})|\Omega_{00}\rangle$ with $W_{kl} = \sum_{j=0}^{d-1} e^{\frac{2\pi jk}{d}} |j\rangle\langle j+l|$ and $|\Omega_{00}\rangle = \frac{1}{\sqrt{d}} \sum_s |s\rangle \otimes |s\rangle$
- $P_{kl} = (W_{kl} \otimes \mathbb{I}) P_{00} (W_{kl}^{\dagger} \otimes \mathbb{I})$



n = 3

$$1 + e^{i\frac{1}{3}2\pi} + e^{i\frac{2}{3}2\pi} = 0$$

[8] B. Baumgartner, et al., "A special simplex in the state space for entangled qudits", Journal of Physics A: Mathematical and Theoretical **40**, 7919 (2007).



Non-parallel phase space lines in d=3 originating from (0,0)

[9] B. C. Hiesmayr, and W. Löffler, "Mutually unbiased bases and bound entanglement", Physica Scripta **2014**, 014017 (2014).

Investigate family of $U \otimes U^*$ symmetric states[9]

- $\rho_M[d] = \left(1 - \frac{q[1]}{(d^2-(d+1))} - \frac{q[2]}{(d+1)}\right) \frac{1}{d^2} \mathbb{I}_{d^2}[d] + \frac{q[1]}{(d^2-(d+1))} P_{00}[d] + \frac{q[2]}{(d+1)(d-1)} \sum_{i=1}^{d-1} P_{i0}[d] + \sum_{j=3}^{d+1} \sum_{i=0}^{d-1} \left(\frac{(-q[j])}{d^2} \mathbb{I}_{d^2}[d] + \frac{q[j]}{d} P_{i,j-2}[d] \right)$

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- $\rho_M[d] = \left(1 - \frac{q[1]}{(d^2-(d+1))} - \frac{q[2]}{(d+1)}\right) \frac{1}{d^2} \mathbb{I}_{d^2}[d] + \frac{q[1]}{(d^2-(d+1))} P_{00}[d] + \frac{q[2]}{(d+1)(d-1)} \sum_{i=1}^{d-1} P_{i0}[d] + \sum_{j=3}^{d+1} \sum_{i=0}^{d-1} \left(\frac{(-q[j])}{d^2} \mathbb{I}_{d^2}[d] + \frac{q[j]}{d} P_{i,j-2}[d] \right)$
- $I_d^j = Tr\Big(\sum_i |b_i^j\rangle\langle b_i^j| \otimes (W_{kl}|b_i^j\rangle)^* (\langle b_i^j|W_{kl}^\dagger)^* \rho_M \Big)$

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- $\min_{\substack{\forall k_j, l_j:\, 0 \le k_j, l_j \le d-1 \\ \forall q[i]:\, 1 \le i \le d+1}} \{\langle\mathfrak{W}(k_j, l_j, q[i])\rangle_{\rho_M} | \, \rho_M(q[i]) \geq 0, \, \rho_M^\Gamma(q[i]) \geq 0\}$

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- Ensure that $\mathrm{Tr}\Big(\mathfrak{W}\rho_{sep}\Big) \geq 0\ \forall \rho_{sep}$

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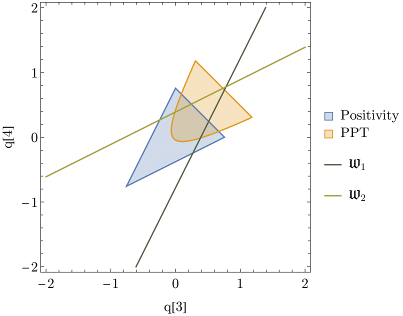
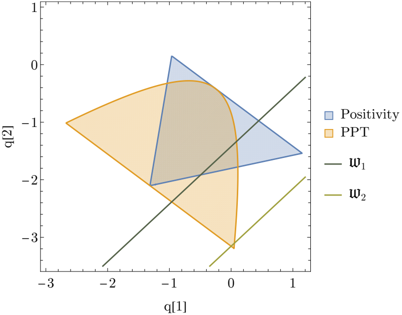
Special Simplex in $\mathcal{H}_d \otimes \mathcal{H}_d$

The Case of d=3



	$\langle \mathfrak{W}_1 \rangle_{\rho_M}$	$\langle \mathfrak{W}_2 \rangle_{\rho_M}$
$q[1]$	$\frac{5}{3}(-1+\sqrt{3})$	$\frac{5}{3}(-1+\sqrt{3})$
$q[2]$	0	0
$q[3]$	$\frac{2}{3}$	$\frac{2}{3}-\frac{1}{\sqrt{3}}$
$q[4]$	$\frac{2}{3}-\frac{1}{\sqrt{3}}$	$\frac{2}{3}$
$2-I_d$	$1-\frac{2}{\sqrt{3}}$	$1-\frac{2}{\sqrt{3}}$

Table of optimized parameters $q[i]$ for 2 witnesses

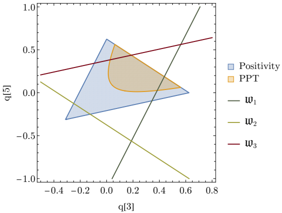


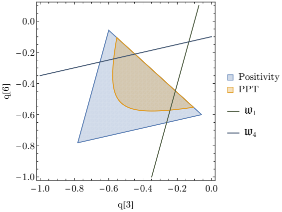
	$\langle \mathfrak{W}_1 \rangle_{\rho_M}$	$\langle \mathfrak{W}_2 \rangle_{\rho_M}$
$q[1]$	$\frac{5}{9}(-5+3\sqrt{3})$	$\frac{5}{3}(-5+4\sqrt{3})$
$q[2]$	$-\frac{16}{9}$	$\frac{8}{9}(-2+\sqrt{3})$
$q[3]$	0	0
$q[4]$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
$2-I_d$	$1-\frac{2}{\sqrt{3}}$	$1-\frac{2}{\sqrt{3}}$

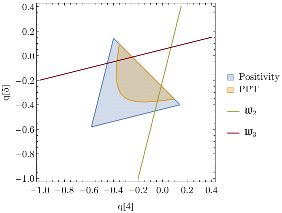
Table of optimized parameters $q[i]$ for 2 witnesses with $q[3]=0$

Special Simplex in $\mathcal{H}_d \otimes \mathcal{H}_d$

Bound Entanglement Detection for d=4-5







	d=4	d=5
2-I_d	$-\frac{1}{8}$	$-1+\frac{2}{\sqrt{5}}$

- witnesses can be rewritten as $\mathfrak{W}_i = (2-d)\,P_{00} + 2\sum_{m=1}^{D-1} P_{m0} + \sum_{m=0}^{d-1}\sum_{n=1}^{d-1}(1-\delta_{ni})\,P_{mn}$
- for even dimensions $d \leq 8$ not all Weyl-shifted witnesses (optimally) detect bound entanglement

$\mathfrak{W}_i$	d=6	d=7	d=8
$\mathfrak{W}_1$	-0.091752	-0.08136	-0.07322
$\mathfrak{W}_2$	-0.091752	-0.08136	-0.07322
$\mathfrak{W}_3$	$-1.14826 \cdot 10^{-6}$	-0.08136	-0.07322
$\mathfrak{W}_4$	-0.091752	-0.08136	$-2.13422 \cdot 10^{-7}$
$\mathfrak{W}_5$	-0.091752	-0.08136	-0.07322
$\mathfrak{W}_6$	-	-0.08136	-0.07322
$\mathfrak{W}_7$	-	-	-0.07322

Table of optimal values for bound entanglement for $6 \leq d \leq 8$

Summary and Outlook

Summary:

- $\exists\ \rho_{PPT,ent} \in \mathcal{H}_d \otimes \mathcal{H}_d$ with $d \geq 3$
- characterizing ρ with respect to its separability $\rightarrow$ NP hard!
- introduce reduced state space (simplex)
- turn to MUB witnesses to detect classes of such states

Future goals:

- new bounds on bound entangled and separable states within the magic simplex
- investigate lower bound of MUB criterion

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