DIPARTIMENTO INTERATENEO DI FISICA "M. MERLIN"
INFN
Istituto Nazionale di Fisica Nucleare

Continuous and Pulsed Quantum Control

Giovanni GRAMEGNA

Joint work with: P. FACCHI, S. PASCAZIO and D. BURGARTH

INFORMATION GEOMETRY, QUANTUM MECHANICS AND APPLICATIONS

26 June 2018 POLICETA - SAN RUFO (SALERNO)

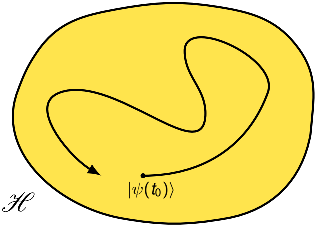
Outline

1. Two techniques to control the evolution of a quantum system
   - Strong Continuous Coupling
   - Bang-bang evolution
2. Comparing the two paradigms

Strong continuous coupling

$\mathscr{H}$ finite dimensional Hilbert space.

H be the Hamiltonian of the system



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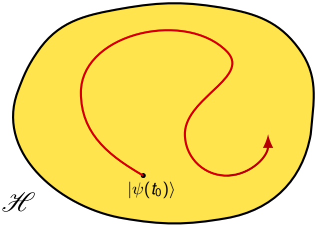
Evolution operator: $U_K(t) = e^{-it(H+KV)}$

Control potential

$$V = \sum_{\mu} \lambda_{\mu} P_{\mu}$$



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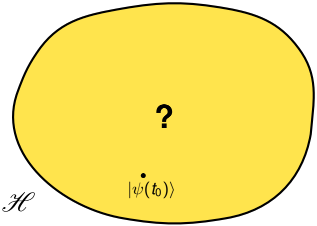
Strong Continuous Coupling ($K \to \infty$)



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$$U_K(t) = e^{-iKVt} e^{-iH_Z t} + O\left(\frac{1}{K}\right)$$

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Zeno Hamiltonian:

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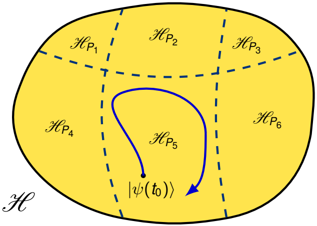

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The total Hilbert space is partitioned into superselection sectors:

$$\mathscr{H} = \bigoplus_\mu \mathscr{H}_{P_\mu}, \quad \mathscr{H}_{P_\mu} = P_\mu \mathscr{H}$$

Control potential

$$V = \sum_\mu \lambda_\mu P_\mu$$

Zeno Hamiltonian:

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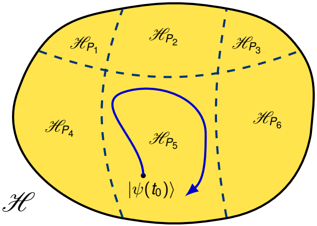

A brief detour: the Adiabatic Theorem

Time-dependent Schrödinger equation:

$$\begin{cases} i\dfrac{d\mathcal{U}}{dt} = \mathcal{H}(t)\mathcal{U}(t), \\ \mathcal{U}(0) = \mathbb{I} \end{cases} \qquad t \in [0, T]$$

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Rescaled Schrödinger equation

$$i\frac{dU_T}{ds} = TH(s)U_T(s), \quad U_T(0) = 1$$

Where:

$$H(s) \equiv \mathcal{H}(sT)$$
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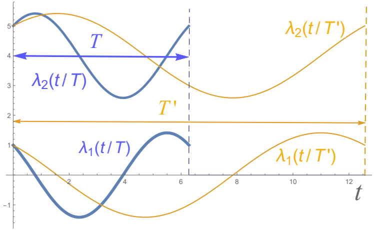

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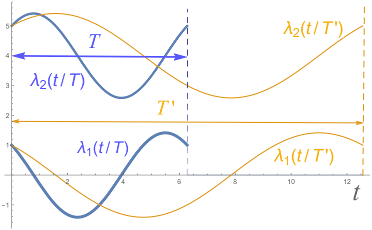

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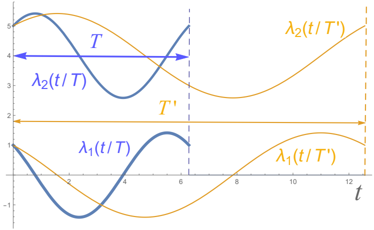

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Adiabatic limit ($T \to \infty$)

$$U_T(s)P(0) = e^{-iT\int_0^s \lambda(\sigma)\,\mathrm{d}\sigma}U(s)P(0) + O\left(\frac{1}{T}\right)$$

Intertwining property: $P(s)U(s) = U(s)P(0)$

The strong coupling limit: a sketch of the proof

The evolution $U_K(t)$ generated by the continuous coupling satisfies the equation:

$$i\frac{dU_K}{dt} = (H + KV)U_K(t)$$

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Continuous Coupling
Bang-Bang control
Comparison

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Going to the interaction picture: $U_K^I(t) = e^{itH}U_K(t), \qquad V^I(t) = e^{itH}Ve^{-itH}$

Interaction Picture

$$i\frac{dU_K^I}{dt} = KV^I(t)U_K^I(t)$$

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Adiabatic limit in interaction picture ($K \to \infty$)

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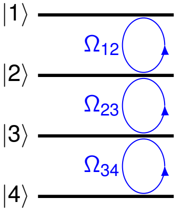
Going back to the Schrodinger picture we obtain the result

Example: four level system

$$H = \begin{pmatrix} 0 & \Omega_{12} & 0 & 0 \\ \Omega_{12} & 0 & \Omega_{23} & 0 \\ 0 & \Omega_{23} & 0 & \Omega_{34} \\ 0 & 0 & \Omega_{34} & 0 \end{pmatrix}$$



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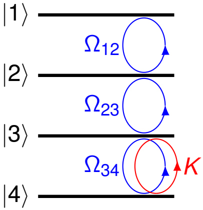

$$KV = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K \\ 0 & 0 & K & 0 \end{pmatrix}$$

V eigenspaces

$$\mathscr{H}_{P_1} = \text{Span}\{|1\rangle, |2\rangle\}$$
$$\mathscr{H}_{P_+} = \text{Span}\{|3\rangle + |4\rangle\}$$
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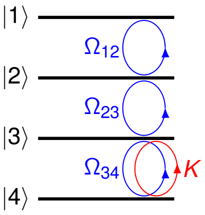

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$K \to \infty$?

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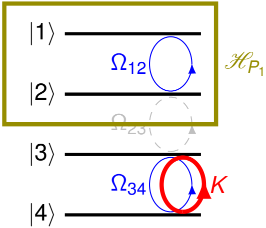

$$H_Z = \begin{pmatrix} 0 & \Omega_{12} & 0 & 0 \\ \Omega_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_{34} \\ 0 & 0 & \Omega_{34} & 0 \end{pmatrix}$$

Zeno subspaces

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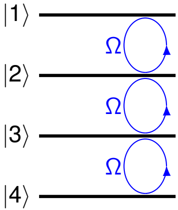
Example: transition probabilities

$$P_{1\to j}(t) = \left|\left\langle j \left| e^{-i(H+KV)t} \right| 1 \right\rangle\right|^2$$

- $K = 0$



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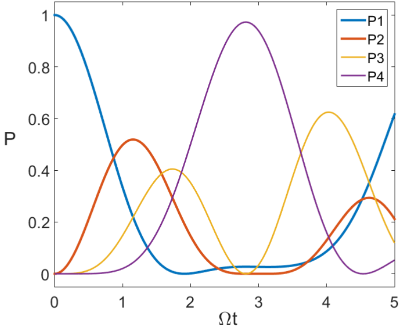

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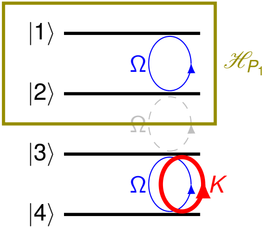

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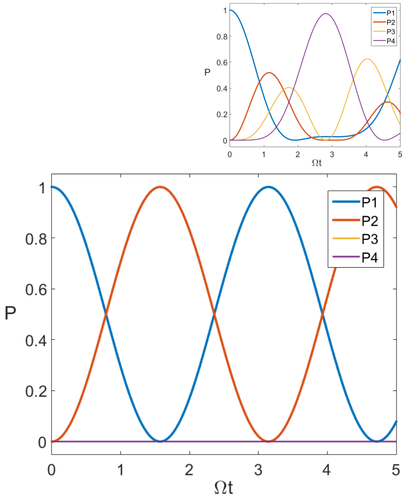
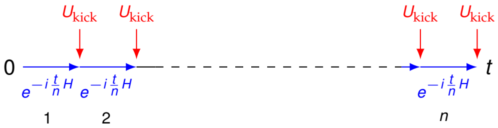


Dynamical decoupling

Pulsed evolution:



Kicking Unitary

$$U_{\text{kick}} = \sum_\mu e^{-i\lambda_\mu} P_\mu$$

Dynamical decoupling

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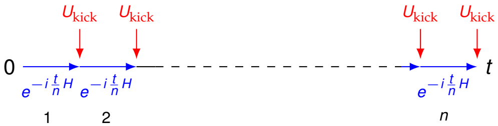

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Idea behind the method: suppose that $U_{\text{kick}}^m = \mathbb{I}$ for some $m \in \mathbb{N}$

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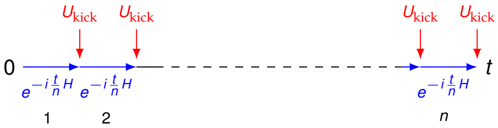

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Then, for $n = km$:

$$U_n(t) = \underbrace{\left(U_{\text{kick}} e^{-i\frac{t}{n}H}\right)\left(U_{\text{kick}} e^{-i\frac{t}{n}H}\right)\cdots\left(U_{\text{kick}} e^{-i\frac{t}{n}H}\right)}_{n\text{ times}}$$

$$= U_{\text{kick}}^n U_{\text{kick}}^{\dagger n-1} e^{-i\frac{t}{n}H} U_{\text{kick}}^{n-1} \cdots U_{\text{kick}}^\dagger e^{-i\frac{t}{n}H} U_{\text{kick}} e^{-i\frac{t}{n}H}$$

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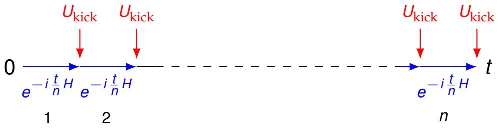

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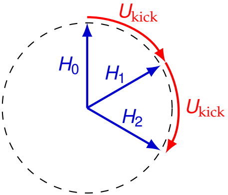
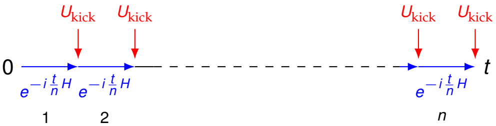


$$H_\ell = U_{\text{kick}}^{\dagger\ell} H U_{\text{kick}}^\ell$$

Dynamical decoupling

Pulsed evolution:



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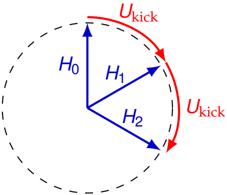

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An “effective” average is taking place:

$$\overline{H} = \frac{1}{n}\sum_{\ell=0}^{n-1} H_\ell = \frac{1}{n}\sum_{\ell=0}^{n-1} U_{\text{kick}}^{\dagger\ell} H U_{\text{kick}}^\ell = \cdots = \sum_\mu P_\mu H P_\mu = H_Z \qquad (n = km)$$

$[H_Z, U_{\text{kick}}] = 0$, only the diagonal part survives the limit while the off-diagonal part $H_{od} = H - H_Z$ (satisfying $[H_{od}, U_{\text{kick}}] \neq 0$) is averaged to zero.

Trotter product formula

A and B operators on some finite dimensional Hilbert space $\mathscr{H}$.

$$[A, B] \neq 0 \implies e^{A+B} \neq e^A e^B$$

“Remedy”:

Trotter product formula

$$e^{A+B} = \lim_{n\to\infty} \left(e^{A/n} e^{B/n}\right)^n$$

n large but finite:

$$e^{A+B} = \left(e^{A/n} e^{B/n}\right)^n + O\left(\frac{1}{n}\right)$$

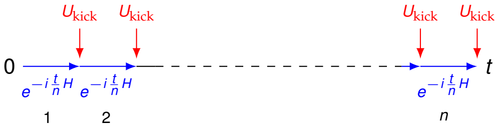
This still works with the sum of a ***finite*** number of operators $A_1, \ldots, A_m$:

$$e^{A_1+\cdots+A_m} = \left(e^{A_1/n} e^{A_2/n} \cdots e^{A_m/n}\right)^n + O\left(\frac{1}{n}\right)$$

We cannot go further if we want to retain this rate of convergence.

Dynamical decoupling

Pulsed evolution:



Kicking Unitary

$$U_{\text{kick}} = \sum_{\mu} e^{-i\lambda_{\mu}} P_{\mu}$$

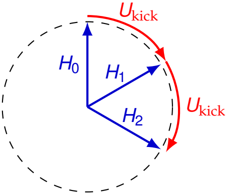
Idea behind the method: suppose that $U_{\text{kick}}^m = \mathbb{I}$ for some $m \in \mathbb{N}$

Then, for $n = km$:

$$U_n(t) = \underbrace{\left(U_{\text{kick}} e^{-i\frac{t}{n}H}\right)\left(U_{\text{kick}} e^{-i\frac{t}{n}H}\right)\cdots\left(U_{\text{kick}} e^{-i\frac{t}{n}H}\right)}_{n \text{ times}}$$

$$= U_{\text{kick}}^n U_{\text{kick}}^{\dagger n-1} e^{-i\frac{t}{n}H} U_{\text{kick}}^{n-1} \cdots U_{\text{kick}}^{\dagger} e^{-i\frac{t}{n}H} U_{\text{kick}} e^{-i\frac{t}{n}H}$$

$$= e^{-i\frac{t}{n}H_{n-1}} \cdots e^{-i\frac{t}{n}H_1} e^{-i\frac{t}{n}H_0}$$



$$H_\ell = U_{\text{kick}}^{\dagger\ell} H U_{\text{kick}}^{\ell}$$

An “effective” average is taking place:

$$\overline{H} = \frac{1}{n}\sum_{\ell=0}^{n-1} H_\ell = \frac{1}{n}\sum_{\ell=0}^{n-1} U_{\text{kick}}^{\dagger\ell} H U_{\text{kick}}^{\ell} = \cdots = \sum_{\mu} P_\mu H P_\mu = H_Z \qquad (n = km)$$

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Dynamical decoupling

For a ***finite*** number of operators $A_1, A_2, \ldots, A_m$:

$$\left(e^{A_1/n} e^{A_2/n} \cdots e^{A_m/n}\right)^n = e^{A_1+\cdots+A_m} + O\left(\frac{1}{n}\right)$$

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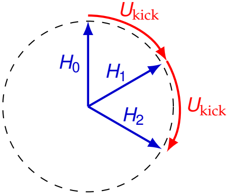
$$= U_{\text{kick}}^n U_{\text{kick}}^{\dagger n-1} e^{-i\frac{t}{n}H} U_{\text{kick}}^{n-1} \cdots U_{\text{kick}}^\dagger e^{-i\frac{t}{n}H} U_{\text{kick}} e^{-i\frac{t}{n}H}$$

$$= e^{-i\frac{t}{n}H_{n-1}} \cdots e^{-i\frac{t}{n}H_1} e^{-i\frac{t}{n}H_0}$$



$$H_\ell = U_{\text{kick}}^{\dagger\ell} H U_{\text{kick}}^\ell$$

An “effective” average is taking place:

$$\overline{H} = \frac{1}{n}\sum_{\ell=0}^{n-1} H_\ell = \frac{1}{n}\sum_{\ell=0}^{n-1} U_{\text{kick}}^{\dagger\ell} H U_{\text{kick}}^\ell = \cdots = \sum_\mu P_\mu H P_\mu = H_Z \qquad (n = km)$$

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Dynamical decoupling

For a ***finite*** number of operators $A_1, A_2, \ldots, A_m$:

$$\left(e^{A_1/n} e^{A_2/n} \cdots e^{A_m/n}\right)^n = e^{A_1 + \cdots + A_m} + O\left(\frac{1}{n}\right)$$

Kicking Unitary

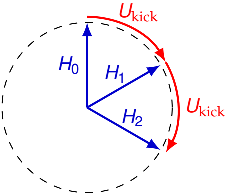
$$U_{\text{kick}} = \sum_\mu e^{-i\lambda_\mu} P_\mu$$

Idea behind the method: suppose that $U_{\text{kick}}^m = \mathbb{I}$ for some $m \in \mathbb{N}$
Then, for $n = km$:

$$U_n(t) = \Big[\underbrace{\left(U_{\text{kick}} e^{-i\frac{t}{n}H}\right)\left(U_{\text{kick}} e^{-i\frac{t}{n}H}\right) \cdots \left(U_{\text{kick}} e^{-i\frac{t}{n}H}\right)}_{m \text{ times}} \Big]^k$$

$$= \left[U_{\text{kick}}^m U_{\text{kick}}^{\dagger m-1} e^{-i\frac{t}{n}H} U_{\text{kick}}^{n-1} \cdots U_{\text{kick}}^\dagger e^{-i\frac{t}{n}H} U_{\text{kick}} e^{-i\frac{t}{n}H} \right]^k$$

$$= \left[e^{-i\frac{t}{km}H_{m-1}} \cdots e^{-i\frac{t}{km}H_1} e^{-i\frac{t}{km}H_0} \right]^k = e^{-it\overline{H}} + O\left(\frac{1}{k}\right)$$



$$H_\ell = U_{\text{kick}}^{\dagger \ell} H U_{\text{kick}}^\ell$$

An "effective" average is taking place:

$$\overline{H} = \frac{1}{m} \sum_{\ell=0}^{m-1} H_\ell = \frac{1}{m} \sum_{\ell=0}^{m-1} U_{\text{kick}}^{\dagger \ell} H U_{\text{kick}}^\ell = \cdots = \sum_\mu P_\mu H P_\mu = H_Z$$

$[H_Z, U_{\text{kick}}] = 0$, only the diagonal part survives the limit while the off-diagonal part $H_{od} = H - H_Z$ (satisfying $[H_{od}, U_{\text{kick}}] \neq 0$) is averaged to zero.

Bang-bang control

Pulsed evolution:

Evolution operator:

$$U_n(t) = \left(U_{\text{kick}} e^{-i\frac{t}{n}H}\right)^n$$

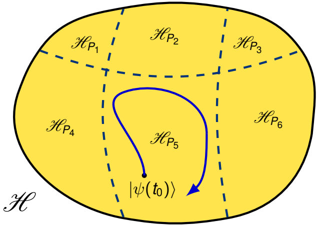
Kicking Unitary

$$U_{\text{kick}} = \sum_\mu e^{-i\lambda_\mu} P_\mu$$

Very frequent kicks: $n \to \infty$

$$\left(U_{\text{kick}} e^{-iHt/n}\right)^n - U_{\text{kick}}^n e^{-iH_Z t} = O\left(\frac{1}{n}\right)$$

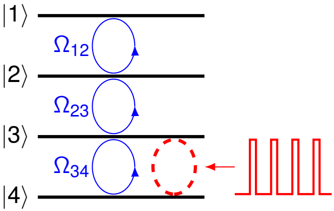
$$H_Z = \sum_\mu P_\mu H P_\mu$$

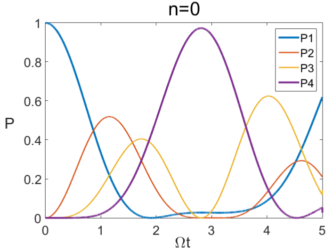


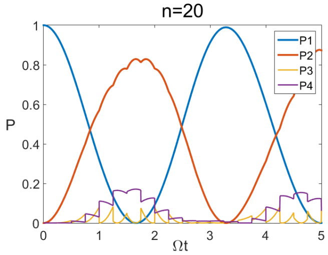
Example: bang-bang control

$$H = \begin{pmatrix} 0 & \Omega_{12} & 0 & 0 \\ \Omega_{12} & 0 & \Omega_{23} & 0 \\ 0 & \Omega_{23} & 0 & \Omega_{34} \\ 0 & 0 & \Omega_{34} & 0 \end{pmatrix}$$

$$U_{\text{kick}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\lambda) & -i\sin(\lambda) \\ 0 & 0 & -i\sin(\lambda) & \cos(\lambda) \end{pmatrix}$$



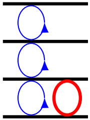




Comparison

Continuous Coupling

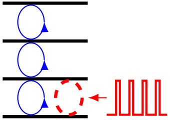
$$U_K(t) = e^{-it(KV+H)}$$



$\longleftrightarrow$?

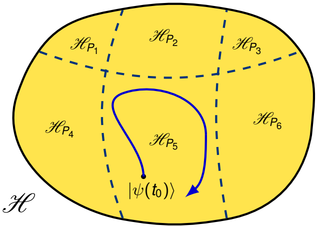
Pulsed evolution

$$U_n(t) = \left(e^{-itV} e^{-i\frac{t}{n}H}\right)^n$$



Both evolutions yield a dynamics generated by

$$H_Z = \sum_\mu P_\mu H P_\mu$$



Thank you for your attention.