Higher derivative quantum theories: classical unstability & quantum problems

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1542

INFORMATION GEOMETRY, QUANTUM MECHANICS AND APPLICATIONS

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Einstein Great Question

What really interest me most is whether **God** had any freedom in the creation of the **World**

Einstein Approach to Gravitation

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- General Relativity:

$$S_E(g) = \varkappa \int \sqrt{g} R$$

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$$S_{E_\lambda}(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g}$$

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Good agreement with LCDM
and Cosmological Data

Lovelock Theorem

2^{nd}-order EoM

Symmetric 2-tensors:

$G_{\mu\nu}$ $\qquad g_{\mu\nu}$

Bianchi identities

$$S_{ES}(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g} + \beta \int \sqrt{g} \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right)$$

Horndeski scalar-tensor theories

$$
\begin{aligned}
S_H \; = & \int \sqrt{g}\,[K(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R] \\
+ & \int \sqrt{g}\,\big[G_{4,X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi \\
- & \int \sqrt{g}\,\big[\tfrac{1}{6}G_{5,X}[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\big] \\
+ & \int \sqrt{g}\,\big[2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)\big]
\end{aligned}
$$

$$X = -\tfrac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$$

Why to look at beyond GR?

- GR theory is not PT-renormalizable: UV problem
- Modified gravity at large distances: IR problem
- GR could be nonPT-renormalizable:
  $\exists$ non-trivial renormalization group IR fixed point

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$$S_{ES}(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g} + \mu \int \sqrt{g} R^2$$

Higher derivative theories

- Smoothing singularities
- Renormalizable theories of Gravity

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- Smoothing singularities
- Renormalizable theories of Gravity
- Classical Instabilities Ostrogradsky
- Quantum Ghosts: Unitarity loss

Ostrogradski instabilities
[1850]

Ostrogradski instabilities

Theories with high time derivatives are unstable

Classical Mechanics:
A Lagrangian with higher derivatives:

$$L(x, \dot{x}, \ddot{x})$$

canonical variables

$$q_1 = x \qquad p_1 = \partial_{\dot{x}} L - \frac{d}{dt} \partial_{\ddot{x}} L$$

$$q_2 = \dot{x} \qquad p_2 = \partial_{\ddot{x}} L$$

Ostrogradski instabilities

If L is regular there exist a function $\Lambda(q_1, q_2, p_2)$ such that

$$\partial_{\ddot{x}} L \Big|_{\substack{x=q_1, \dot{x}=q_2 \\ \ddot{x}=\Lambda(q_1,q_2,p_2)}} = p_2$$

$$H = p_1 q_2 + p_2 \Lambda(q_1, q_2, p_2) - L(q_1, q_2, \Lambda(q_1, q_2, p_2))$$

H is unbounded below in p_1.

Similar arguments apply for higher time derivative Lagrangians.

Theories with high time derivatives are unstable

Do quantum theories solve this problem?

Smoothing UV Singularities.

Super-renormalizable theories of gravity

$$S_{SR}(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g}$$

$$+ \sum_{n=1}^{N} \int \sqrt{g} \, \left(\alpha_n \, R_{\mu\nu\alpha\beta} \, \Box^n R^{\mu\nu\alpha\beta} - \beta_n \, R_{\mu\nu} \, \Box^n R^{\mu\nu} + \gamma_n \, R \, \Box^n R \right)$$

only one loop divergences

[M. A., JL. Lopez and I. Shapiro]

QFT Principles

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Implications:

- Analytic continuation to Euclidean time

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Implications:

- Analytic continuation to Euclidean time
- Absence of unphysical ghosts

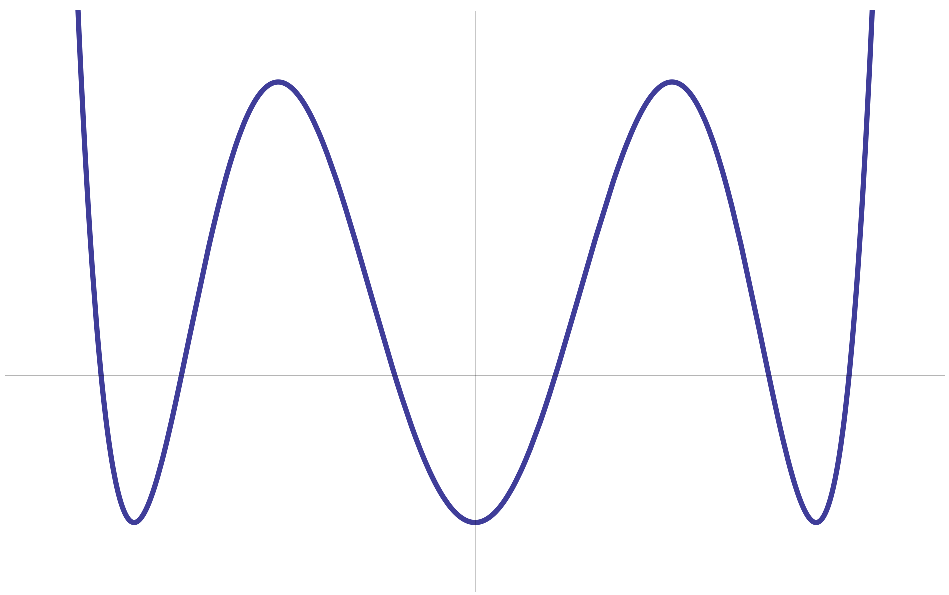
Real Ghosts

Real Ghosts

$$G(k) = \left(\sum_{n=1}^{N} c_n \, k^{2n+2} \right)^{-1} = \sum_{n=1}^{N} \frac{a_n}{k^2 + m_n^2}$$

$$a_n = -a_{n+1} \, , m_n \leq m_{n+1}$$

alternating signs



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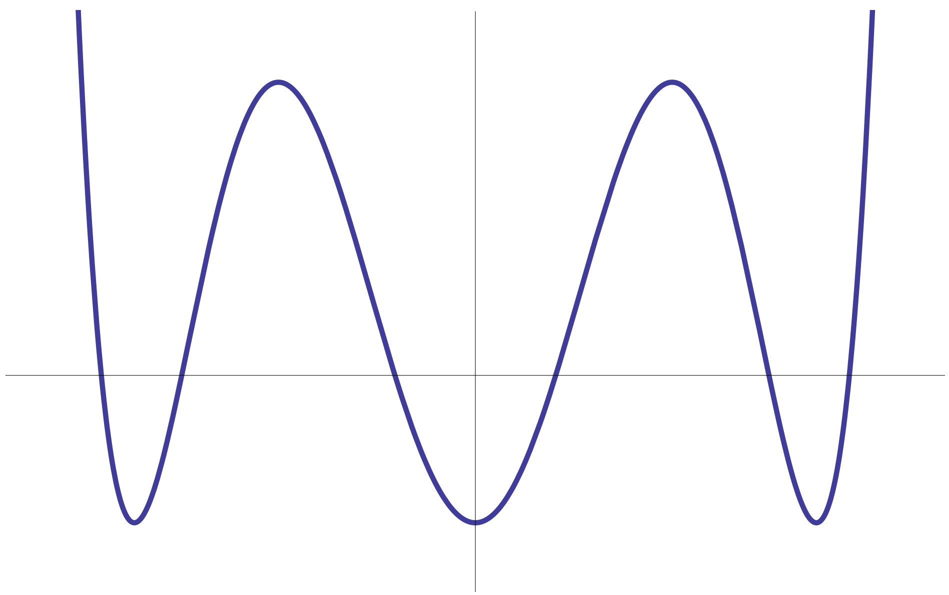
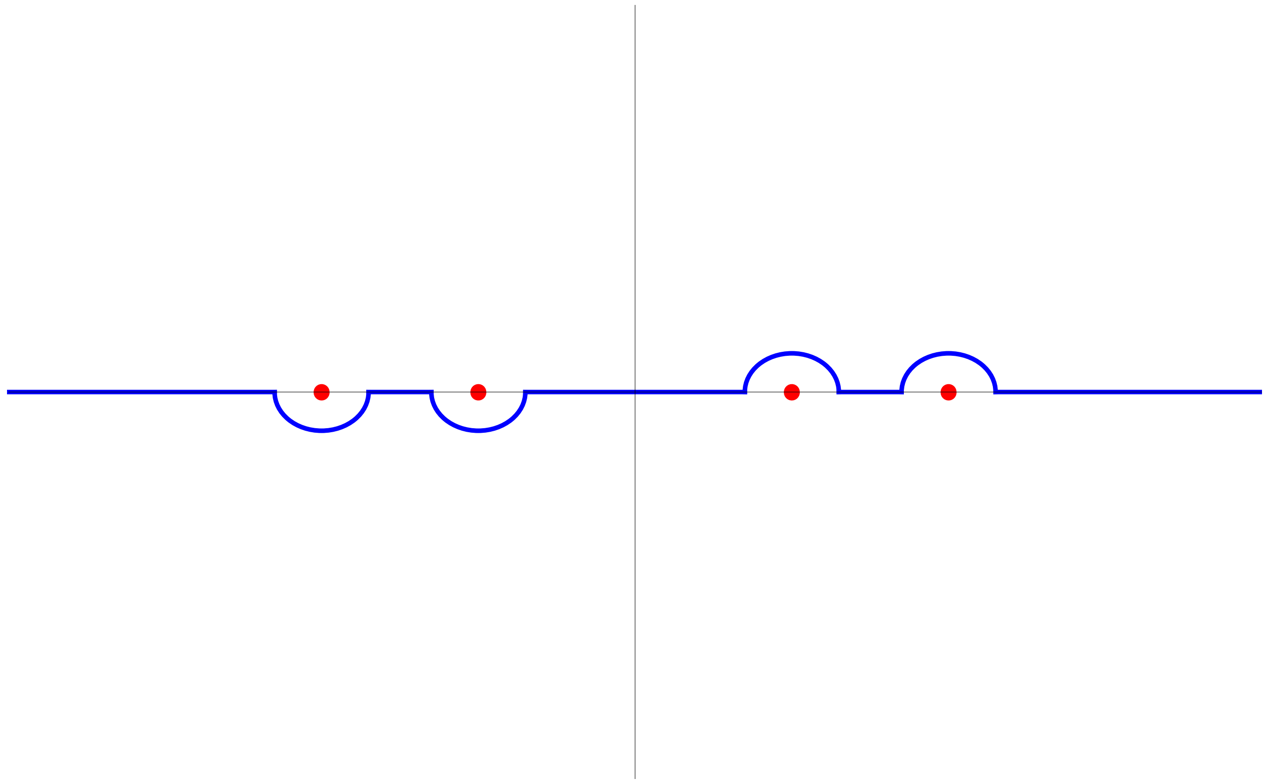


Negative signs corresponds to ghosts

Real Ghosts

Analytic continuation to Euclidean time



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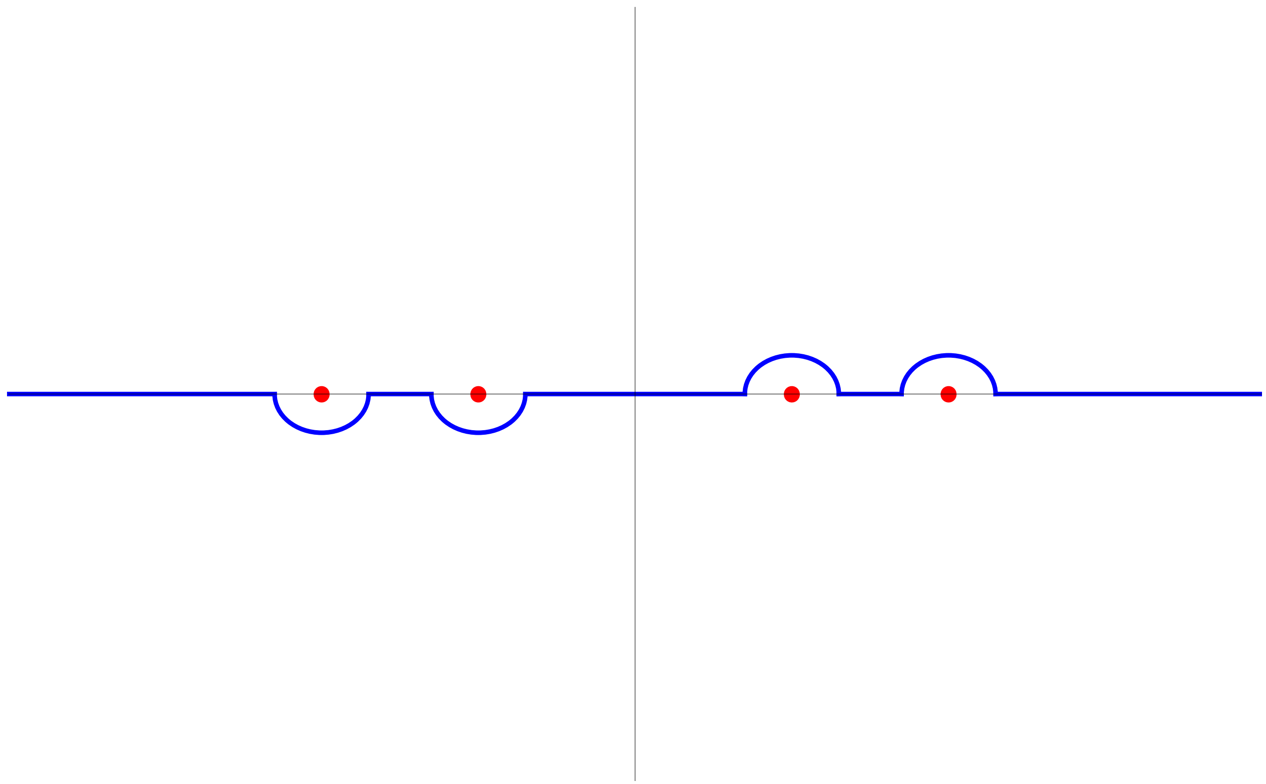


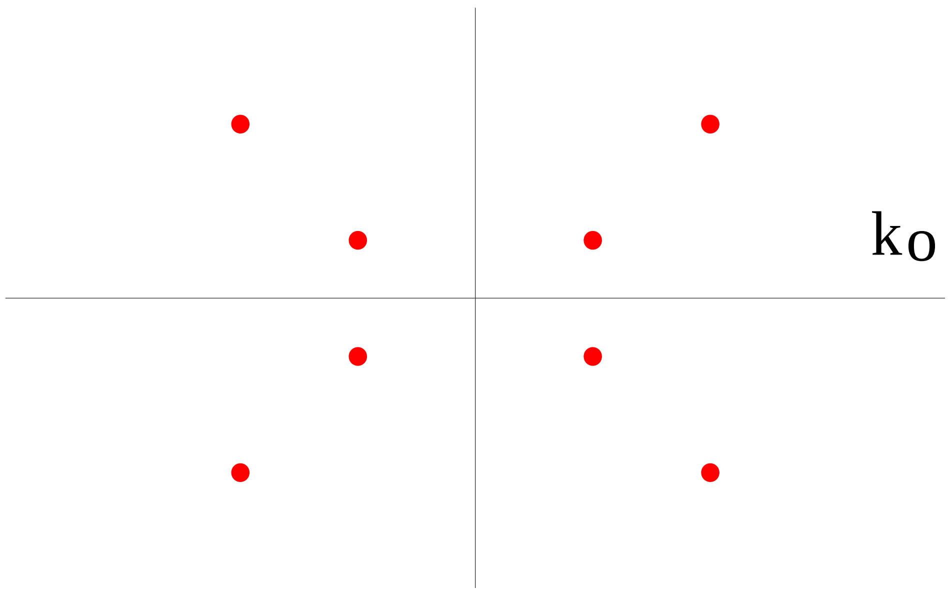
no reflection positivity

Complex Ghosts

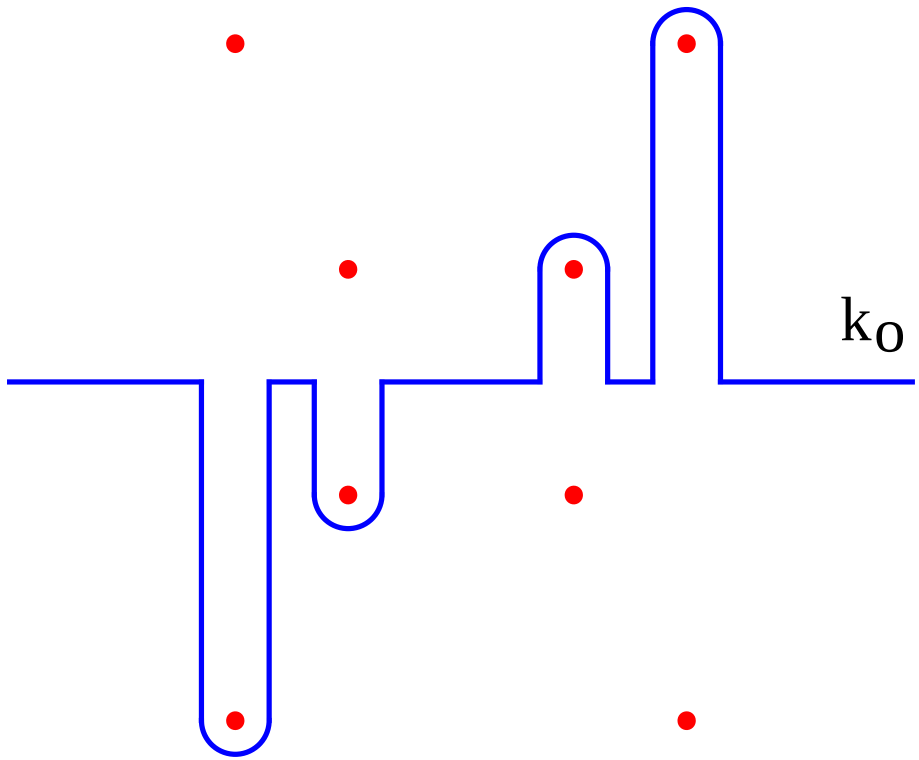
Complex Ghosts

$$G(k) = \left(\sum_{n=1}^{N} c_n \, k^{2n+2} \right)^{-1} = \sum_{n=1}^{N} \frac{a_n}{k^2 + m_n^2} + \sum_{n=-M}^{M} \frac{b_n}{k^2 + \mu_n}$$

$$b_{-n} = b_n = b_n^*, \quad \mu_n^* = \mu_n$$



Complex Ghosts



Violation of Poincaré invariance

Transcendent Theories

Ghosts Free

Transcendent Theories

Ghosts Free

$$S_e(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g}$$

$$+ \int \sqrt{g} \left(\alpha R_{\mu\nu\alpha\beta} \, e^{-\varepsilon \Box^n} R^{\mu\nu\alpha\beta} - \beta R_{\mu\nu} \, e^{-\varepsilon \Box^n} R^{\mu\nu} + \gamma R \, e^{-\varepsilon \Box^n} R \right)$$

Transcendent Theories
Ghosts Free

$$S_e(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g}$$
$$+ \int \sqrt{g} \left(\alpha \, R_{\mu\nu\alpha\beta} \, e^{-\varepsilon \Box^n} R^{\mu\nu\alpha\beta} - \beta \, R_{\mu\nu} \, e^{-\varepsilon \Box^n} R^{\mu\nu} + \gamma \, R \, e^{-\varepsilon \Box^n} R \right)$$

- $n = 1$ has UV divergences in Minkowski space-time
- $n = 2$ leads to finite results both in Euclidean and Minkowski space-times. But ...

Transcendent Theories

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- Existence of analytic continuation to Euclidean space-time
- Osterwalder-Schräder Reflection positivity

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Implications

- Existence of analytic continuation to Euclidean space-time
- Osterwalder-Schräder Reflection positivity
- Existence of a Källén-Lehmann representation of the 2-point function

OS Reflection positivity

OS Reflection positivity

Reflection positivity in quantum mechanics

Quantum mechanics as a 0+1 dimensional QFT

$$S_0(x) = \frac{1}{2}\int \left(\dot{x}^2 - m^2 x^2\right) dt$$

Quantum states in covariant formalism

$$|1\rangle = x|0\rangle \Rightarrow \quad |f\rangle = \int_{-\infty}^{\infty} f(t)x(t)|0\rangle; \quad f \in \mathcal{S}(\mathbb{R})$$

Heisenberg picture : $\quad x(t) = e^{iHt} x e^{-iHt}$

$$\| f \|^2 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds\, f^*(t)\langle 0|x(t)x(s)|0\rangle f(s) = \left| \int_{-\infty}^{\infty} ds\, f(s)\frac{e^{ims}}{2m}\right|^2$$

Reflection positivity in quantum mechanics

Euclidean time τ picture:

$$x(\tau) = e^{H\tau} x \, e^{-H\tau}$$

Quantum states in Euclidean covariant formalism

$$|1\rangle = x|0\rangle \Rightarrow \quad |f\rangle = \int_0^\infty f(\tau)x(\tau)|0\rangle; \quad f \in \mathcal{S}(\mathbb{R}_+)$$

$$\| f \|_\vartheta^2 = \int_0^\infty d\tau \int_0^\infty d\tau' f^*(\tau)\langle 0|x(-\tau)x(\tau')|0\rangle f(\tau') = \left| \int_0^\infty d\tau' f(\tau) \frac{e^{-m\tau}}{2m} \right|^2$$

Reflection Positivity

$$\vartheta(\mathbf{x},\, \tau) = \vartheta(\mathbf{x}, -\tau)$$

$$\vartheta f(x) = f^*(\vartheta x) \qquad \mathbf{f} \in \mathcal{S}(\mathbb{R}^4_+)$$

$$\mathbb{R}^4_+ = \{(x,\, \tau) \in \mathbb{R}^4,\, \tau \geq 0\}$$

$$\vartheta\, \mathbf{f_n}(x_1, x_2, \ldots, x_n) = \mathbf{f_n}^*(\vartheta x_1, \vartheta x_2, \ldots, \vartheta x_n) \qquad \mathbf{f}_n \in S(\mathbb{R}^{4n}_+)$$

$$\boxed{\sum_{i,j=0}^{n} \mathcal{S}_{n_i+n_j}(\vartheta\, \mathbf{f_{n_i}}^* . \mathbf{f_{n_j}}) \geq 0}$$

Recovering the Hilbert space

One particle states

Recovering the Hilbert space

One particle states

$$\mathcal{H}_1 = \overline{\{\mathbf{f} \in \mathcal{S}(\mathbb{R}^4_+)\}}_{\vartheta}$$

with the Hermitian product

$$(f, g)_{\vartheta} = (\vartheta f, g) = \int_0^{\infty} dx \int_0^{\infty} dy \, \vartheta f^*(x) \langle 0 | \phi(x) \phi(y) | 0 \rangle g(y)$$

Null states

$$\mathcal{N}_1 = \{\mathbf{f} \in \mathcal{H}_1 ; \| f \|^2_{\vartheta} = 0\}$$

Recovering the Hilbert space

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Recovering the Hilbert space

One particle states

$$\widetilde{\mathcal{H}}_1 = \frac{\mathcal{H}_1}{\mathcal{N}_1}$$

Multiparticle Hilbert space

$$\mathcal{F} = \{\mathbf{f} = (f_0, f_1, \cdots, f_n, 0, \cdots);\ \| \mathbf{f} \|_{\vartheta}^2 \leq \infty\}$$

Null states

$$\mathcal{N}_{\mathcal{F}} = \{\mathbf{f} \in \mathcal{F};\ \| \mathbf{f} \|_{\vartheta}^2 = 0\}$$

Recovering the Hilbert space

$$\widetilde{\mathcal{H}} = \frac{\mathcal{F}}{\mathcal{N}_{\mathcal{F}}}$$

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Euclidean time evolution is just time translation

$$T_t x(\tau) T_t = x(\tau + t) = e^{H(\tau+t)} \, x \, e^{-H(\tau+t)}$$

$$T_t |f\rangle = \int_0^\infty f(\tau) x(\tau + t) |0\rangle = \int_t^\infty f(\tau - t) x(\tau) |0\rangle$$

$$T_t \mathbf{f} = (T_t f_0, T_t f_1, \cdots, T_t f_n, 0, \cdots)$$

$$T_t^\dagger = T_t$$

Reflection positivity

Scalar field theories

Reflection positivity

Scalar field theories

$$S_\phi \;=\; \int \phi^\dagger \left(\Box - m^2\right) \phi$$

Schwinger 2-p functions

D=0+1 Massive case $m \neq 0$

$$S_2(x,y) = \langle \phi^\dagger(x)\phi(y)\rangle = \frac{e^{-m|x-y|}}{2m}$$

Reflection positivity

D=0+1 Massless case $m = 0$

$$S_2(x, y) = \langle \phi^\dagger(x)\phi(y) \rangle = |x - y|$$

Not reflection positive: no normalizable ground state

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$$S_2(x, y) = \langle \phi^\dagger(x)\phi(y) \rangle = -\frac{1}{2\pi} \log |x - y|$$

Not reflection positive: no normalizable vacuum

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Not reflection positive: no normalizable vacuum

Coleman theorem: No continuous symmetry breaking in 2D

Reflection positivity

Scalar field theories with higher time derivatives

Reflection positivity

Scalar field theories with higher time derivatives

$$
\begin{aligned}
S_4(g) &= \int \phi^\dagger \left(\Box^2 - m^2\right) \phi \\
S_e(g) &= \int \phi^\dagger \, e^{-\varepsilon \Box^{2s}} \phi \\
S_{ep}(g) &= \int \phi^\dagger \, e^{-\varepsilon \Box^{2s}} \left(\Box - m^2\right) \phi
\end{aligned}
$$

Schwinger 2-p functions

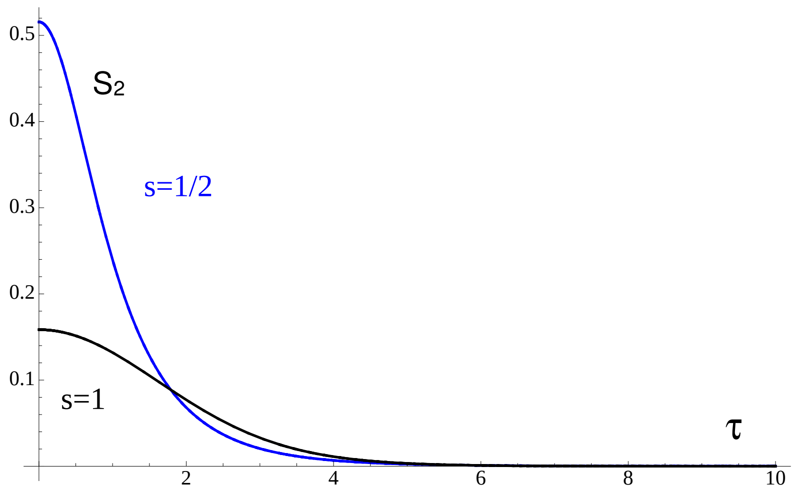
$$S_2(x, y) = \langle \phi^\dagger(x) \phi(y) \rangle$$

Reflection positivity

$$S_{2\vartheta}(x,y) = \langle \vartheta\phi(x)\phi(y)\rangle = \langle \phi^\dagger(\vartheta x)\phi(y)\rangle$$

Reflection positivity implies that

$$S_{2\vartheta}(x,x) \geq 0$$



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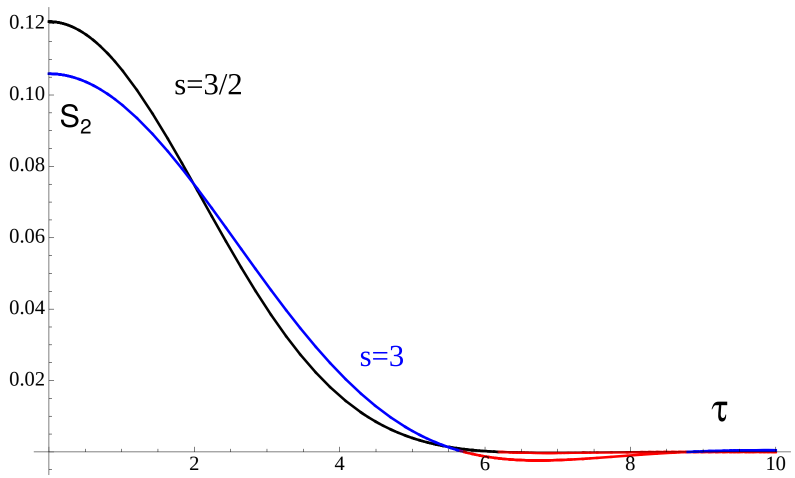

Theorem for Exponential Kernels

$$\Delta(\tau) = S_{2\vartheta}(\mathbf{x},\, \tau; \mathbf{x},\, \tau) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-(p_0^2+\mathbf{p}^2)^s}}{p_0^2 + \mathbf{p}^2 + m^2}$$

Theorem:

- i) For $s \leq 1$ (packing n-sphere problem)

$$\Delta(\tau) = S_{2\vartheta}(x, x) \geq 0$$

- ii) For $s > 1$ there exist $\tau > 0$ such that

$$\Delta(\tau) = S_{2\vartheta}(x, x) < 0$$

Reflection positivity

$$S_{2\vartheta}(x,y) = \langle \vartheta\phi(x)\phi(y)\rangle = \langle \phi^{\dagger}(\vartheta x)\phi(y)\rangle$$

Reflection positivity implies that

$$S_{2\vartheta}(x,x) \geq 0$$

$$K = \frac{e^{-\varepsilon\Box^n}}{\Box}, \qquad K = e^{-\varepsilon\Box^n}$$

$$K = \frac{1}{(\Box + m^2)(\Box + M^2)} = \frac{1}{M^2 - m^2}\left(\frac{1}{(\Box + m^2)} - \frac{1}{\Box + M^2}\right)$$

[M. Christodoulou- L. Modesto, 2018]

Källén-Lehmann

GUNNAR KÄLLÉN

Källén-Lehmann

Theorem: $S_2(x,y)$ is reflection positive iff the Fourier transform $S_2(k)$ has a Källén-Lehmann representation

$$S_2(k) = \int_0^\infty d\mu \frac{\rho(\mu)}{k^2+\mu^2} \quad \text{with } \rho(\mu) \geq 0$$

$S_2(k)$ is strongly positive:

$$\frac{d}{dk^2} k^2 S_2(k) \geq 0$$

$$\frac{(-k^2)^{n-1}}{n!(n-2)!} \left(\frac{d}{dk^2}\right)^{2n-1} k^{2n} S_2(k) \geq 0 \quad n > 1$$

[Widder, 1934]

Källén-Lehmann

None of the transcendent kernels

$$K(x,y) = \frac{e^{-\varepsilon \Box^n}}{\Box + m^2}(x,y)$$

$$K(x,y) = \frac{e^{-\varepsilon \Box^n}}{\Box}(x,y)$$

$$K(x,y) = e^{-\varepsilon \Box^n}(x,y)$$

has a Källén-Lehmann representation, because their Fourier transforms $S_2(k)$ are not strongly positive

[M. A., L. Rachwał and I. Shapiro, 2018]

Super-renormalizable QG

Super-renormalizable QG

$$S(g) = \lambda\varkappa\int\sqrt{g} + \varkappa\int\sqrt{g}R + \alpha\varkappa\int\sqrt{g}\,R_{\mu\nu\alpha\beta}\,e^{(\Box/\Lambda^2)^s}R^{\mu\nu\alpha\beta}$$
$$+\ \beta\varkappa\int\sqrt{g}\,R_{\mu\nu}\,e^{(\Box/\Lambda^2)^s}R^{\mu\nu} + \gamma\varkappa\int\sqrt{g}\,R\,e^{(\Box/\Lambda^2)^s}R\,..$$

2-point function

$$S^{(2)}_{ijij}(p) = \frac{2}{3}\int\frac{d^4p}{(2\pi)^4}\frac{(1+p_i^2p_j^2)}{p^2-\beta/2\,e^{p^{2s}/\Lambda^{2s}}p^4-2\,\gamma\,e^{p^{2s}/\Lambda^{2s}}p^4}$$
$$-\ \frac{1}{3}\int\frac{d^4p}{(2\pi)^4}\frac{(1+p_i^2p_j^2)}{p^2+(\beta/2+6\alpha)\,e^{p^{2s}/\Lambda^{2s}}p^4+8\,\gamma\,e^{p^{2s}/\Lambda^{2s}}p^4}$$

does not admit a Källén-Lehmann representation

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- Ghost free theories are not necessarily unitary
- Higher polynomial and transcendent Euclidean theories are not reflection positive
- Källén-Lehmann representation is not possible for higher polynomial and transcendent theories
- Hořava-Lifshitz and $f(R)$ theories are not affected by these problems