

# Manifolds of Quantum States

Relative Entropies

Monotonic Quantum Metrics

To the memory of George Sudarshan (1931 - 2018)

# Manifolds of quantum states

Covariance with respect to all coordinate transformations

From  $\langle \psi | \psi \rangle$  on  $\mathcal{H}$  to  $\langle d\psi | d\psi \rangle$  on  $T\mathcal{H}$

To avoid technical difficulties  $\dim \mathcal{H} = n$

Complex coordinates

$$\langle \psi | \psi \rangle = \bar{z}^k z_k$$

$$z = x + iy$$

$$\langle d\psi | d\psi \rangle = d\bar{z}^k \otimes dz_k$$

$$dx^k \otimes dx_k + dy^k \otimes dy_k \quad \mathfrak{g}$$

$$+ i (dx^j \wedge dy_j) \quad i\omega$$

$$J = dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x}$$

Complex projective space  $\mathbb{C}_0 \rightarrow \mathbb{H}_0 \rightarrow \mathbb{C}P(\mathbb{H})$

Pure states

$$\begin{aligned} h &= \frac{\langle d\psi | d\psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle d\psi | \psi \rangle \langle \psi | d\psi \rangle}{\langle \psi | \psi \rangle^2} \\ &= \frac{d\bar{z}^j \otimes dz_j}{\bar{z}^k z_k} - \frac{(d\bar{z}^j z_j)(dz_k \bar{z}^k)}{(\bar{z}^l z_l)^2} \end{aligned}$$

nonlinear change of coordinates  $z_j = \sqrt{p_j} e^{i\varphi_j}$   
 $p_1, p_2, \dots, p_n \geq 0$   $\sum_j p_j = 1$   $\langle \cdot \rangle_p$  expectation value with respect to  $p$

$$\begin{aligned} h &= \frac{1}{4} \left[ \langle d \log p \otimes d \log p \rangle_p - \langle d \log p \rangle_p \otimes \langle d \log p \rangle_p \right. \\ &\quad \left. + \langle d\varphi \otimes d\varphi \rangle_p - \langle d\varphi \rangle_p \otimes \langle d\varphi \rangle_p + \right. \\ &\quad \left. \frac{i}{2} \left[ \langle d \log p \wedge d\varphi \rangle_p - \langle d \log p \rangle_p \wedge \langle d\varphi \rangle_p \right] \right] \end{aligned}$$

## Continuum case

$$\mathcal{H} = L^2(X) \quad p(x) = \psi^*(x)\psi(x)$$

$$\langle \psi | \psi \rangle = \int_X \psi(x)^* \psi(x) dx = 1$$

Immersion of a parameter space

into  $\mathbb{C}P(\mathcal{H})$

$$\psi(x; \theta) = p(x; \theta)^{\frac{1}{2}} e^{i\alpha(x; \theta)}$$

$p$  a probability density

$$\mathbb{E}_p(f) = \int_X f p dx$$

$$\mathbb{E}_p(df) = \int_X df p dx$$

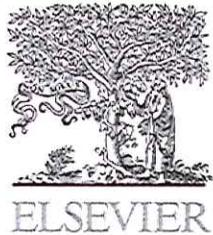
$$df = \frac{\partial f}{\partial \theta_\kappa} d\theta_\kappa$$

$$h_x = \frac{1}{4} \int_X (d \log p \otimes d \log p) p dx + \int_X (d\alpha \otimes d\alpha) p dx - \left( \int_X d\alpha p dx \right)^2$$

$$- i \int_X (d \log p \otimes d\alpha - d\alpha \otimes d \log p) p dx$$

notice:  $\int d p dx = \int (d \log p) p dx = 0$  from  $\int p dx = 1$

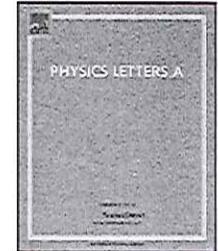
Fisher classical information metric hidden in the quantum metric



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## Classical and quantum Fisher information in the geometrical formulation of quantum mechanics

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### ABSTRACT

The tomographic picture of quantum mechanics has brought the description of quantum states closer to that of classical probability and statistics. On the other hand, the geometrical formulation of quantum mechanics introduces a metric tensor and a symplectic tensor (Hermitian tensor) on the space of pure states. By putting these two aspects together, we show that the Fisher information metric, both classical and quantum, can be described by means of the Hermitian tensor on the manifold of pure states.

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## Kähler structures and potential functions

Using the complex structure  $J = dx_j \otimes \frac{\partial}{\partial y_j} - dy_j \otimes \frac{\partial}{\partial x_j}$

$$f = \langle \Psi | \Psi \rangle = (x^j x_j + Y^j Y_j)$$

$$\tilde{f} = \log \langle \Psi | \Psi \rangle = \log (x^j x_j + Y^j Y_j)$$

$$dd_J f = 2 dy^k \wedge dx_k$$

an associated metric

$$g_f(X, Y) = (dd_J f)(X, JY) \Rightarrow 2(dy^k \otimes dy_k + dx^k \otimes dx_k) = g_f$$

As the quantum metric is derived from a potential it is reasonable to expect that also the Fisher-Rao metric may be derivable from a potential even if the parametric statistical manifold does not have even dimensions

Potential functions for the Riemannian metric of information geometry

Shun-ichi Amari

Information Geometry and its applications

Springer - Japan 2016

Differential-Geometrical Methods in Statistics  
Lecture Notes in Statistics, 28 - Springer, 1985

with Hiroshi Nagaoka

Methods of Information Geometry

Mathematical Monographs, volume 191

AMS, 2007

Potential functions had been christened

"divergence functions", "contrast functions"

"directed distances", "distinguishability functions"

# Riemannian metric and potential functions

The potential function from  $QM$  requires  $J$  twice,

to define  $dd_J f$  and  $g_f(x, y) = dd_J f(x, J(y))$

We notice that  $T^*M$  is an "almost" linearization of  $M \times M$

$\phi_t : \mathbb{R} \times T^*M \rightarrow T^*M$  a flow associated with some completely integrable Hamiltonian system

Given  $(m, p) \in T_m^*M \rightarrow \pi(\phi_{t=1}(m, p)) \in M$

Thus we may replace  $T^*M$  with  $M \times M$

Our potential function should be a 2-point function on  $M \times M$

One such function is known to us from H. J. theory

$H(q, \frac{\partial S}{\partial q}(q, Q)) = E$ , a complete solution  $S(q, Q)$

$\omega = \frac{\partial^2 S}{\partial q \partial Q} dq \wedge dQ$ , we may use  $J = dq \otimes \frac{\partial}{\partial Q} - dQ \otimes \frac{\partial}{\partial q}$

to get

$$g = \frac{\partial^2 S}{\partial q \partial Q} dq \otimes dQ + \frac{\partial^2 S}{\partial Q \partial q} dQ \otimes dq$$

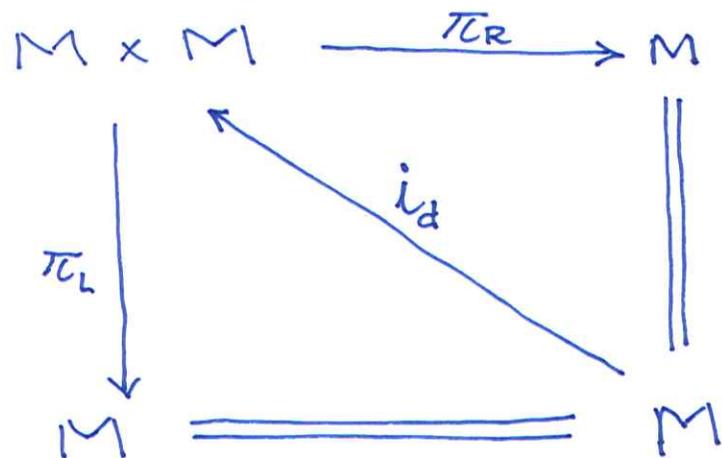
This may be restricted to the diagonal in  $M \times M$  and

gives a metric on  $M$

## Formalizing a bit

To use our machinery we have to develop an exterior differential calculus on the product manifold  $M \times M$

We have



Vector fields  $\mathfrak{X}(M)$  may be lifted in various manners  
left, right, diagonal

Forms on  $M$  may be pulled-back to  $M \times M$ , where we may  
define differentials  $d_L, d_R, d$  and  $d \otimes \mathbb{1}, \mathbb{1} \otimes d$

Various submodules in  $\mathfrak{X}(M \times M)$  with coefficients in  
 $\pi_L^*(\mathfrak{F}(M)), \pi_R^*(\mathfrak{F}(M)), \mathfrak{F}(M \times M)$

In particular, it is possible to consider differential forms at  $x$  with values differential forms at  $y$  (double forms, bi-forms, ...)

A direct way to arrive at the metric out of a potential function  $F$  would be to define bivectors  $X_L \otimes Y_R$ , lift of vector fields  $X$  and  $Y$  on  $M$ , we may define

$$g_F(X, Y) = i_d^* (L_{X_L} L_{Y_R} F)$$

If we define a basis  $X_1, X_2, \dots, X_n$ , and a dual basis  $\theta^1, \dots, \theta^n$   $\theta^j(X_k) = \delta_k^j$ , we have

$$g_F = i_d^* (L_{X_k^e} L_{X_j^r} F) \theta^k \otimes \theta^j$$

An almost-complex structure

$$J = \theta_L^j \otimes X_j^r - \theta_R^j \otimes X_j^e$$

symplectic structure

$$i_d^* (L_{X_L} L_{Y_R} F) \theta_L^x \wedge \theta_R^y$$

## Quantum case :

- Quantum states replace probability distributions
- .. The manifold of quantum states is a stratified manifold (strata are orbits of the action of  $SL(\mathbb{H})$  by congruence  $g^+ \rho g / \text{Tr}(g^+ \rho g)$ )
- ... Potential functions provided by relative entropies

Example : von Neumann-Umegaki relative entropy, a noncommutative version of Kullback-Leibler entropy

$$\text{Tr}(\rho \log \rho - \rho \log \sigma)$$

$$\sum_j p_j (\log p_j - \log q_j)$$

By using a tomographic map

$$\{\text{States}\} \rightarrow \{\text{Probability vectors}\}$$

we get a simplex  $\Delta_n$

We can parametrize states by means of unitary matrices and probability vectors

$$\rho = U^+ \begin{vmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{vmatrix} U$$

To deal with a smooth manifold we restrict to states of maximal rank, the open interior of the simplex  $\Delta_n$ , 10

$$SU(n) \times \overset{\circ}{\Delta}_n$$

Using

$$F(\rho, \sigma) = \text{Tr} \rho (\log \rho - \log \sigma)$$

We may set  $\rho = U \rho_0 U^{-1}$ ,  $\sigma = V \sigma_0 V^{-1}$  with  $\rho_0$  and  $\sigma_0$  diagonal

We would get

$$\mathcal{G}_F = -i_d^* (d\tilde{d}F) = +i_d^* \text{Tr} d\rho (\log \rho - \log \sigma) \otimes \tilde{d}\rho \log \sigma$$

by using the chosen parametrization

$$\mathcal{G} = \text{Tr} \rho_0^{-1} d\rho_0 \otimes d\rho_0 + \text{Tr} [U^{-1}dU, \log \rho_0] \otimes [U^{-1}dU, \rho_0]$$

For the q-bit  $\rho_0 = \frac{1}{2}(\sigma_0 + w\sigma_3)$ ;  $U^{-1}dU = i \sigma_j \theta^j$   $w^2 \leq 1$

$$\mathcal{G} = \frac{1}{1-w^2} dw \otimes dw + 2w \log \frac{1+w}{1-w} (\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2)$$

A natural question

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A state is uniquely reconstructed from a "quorum" of tomograms, fair probability distributions.

Fisher-Rao metric, on fair probability distributions, is uniquely singled out by monotonicity requirements (Chentsov theorem).

How is it possible to recover the infinity of monotonic quantum metrics from the uniquely defined Fisher-Rao?

Metrics from relative entropies define geodesic motions, what information we gain on separability and entanglement?

Is the manifold of separable states a totally geodesic submanifold?

What is the meaning of the dynamics defined by the metric obtained from a relative entropy?

### **Aspects of Geodesical Motion with Fisher-Rao Metric: Classical and Quantum**

By: Ciaglia, Florio M.; Di Cosmo, Fabio; Felice, Domenico; et al.

OPEN SYSTEMS & INFORMATION DYNAMICS Volume: 25 Issue: 1 Article Number: 1850005 Published: MAR 2018

### **Tomographic reconstruction of quantum metrics**

By: Laudato, Marco; Marmo, Giuseppe; Mele, Fabio M.; et al.

JOURNAL OF PHYSICS A-MATHEMATICAL AND THEORETICAL Volume: 51 Issue: 5 Article Number: 055302

Published: FEB 2 2018

### **Geometrical structures for classical and quantum probability spaces**

Ciaglia, Florio Maria; Ibort, Alberto; Marmo, Giuseppe

INTERNATIONAL JOURNAL OF QUANTUM INFORMATION Volume: 15 Issue: 8 Article Number: 1740007 Published: DEC 2017

### **Tensorial dynamics on the space of quantum states**

By: Carinena, J. F.; Clemente-Gallardo, J.; Jover-Galtier, J. A.; et al.

JOURNAL OF PHYSICS A-MATHEMATICAL AND THEORETICAL Volume: 50 Issue: 36 Article Number: 365301 Published: SEP 8 2017

### **Metric on the space of quantum states from relative entropy. Tomographic reconstruction**

Man'ko, Vladimir I.; Marmo, Giuseppe; Ventriglia, Franco; et al.

JOURNAL OF PHYSICS A-MATHEMATICAL AND THEORETICAL Volume: 50 Issue: 33 Article Number: 335302

Published: AUG 18 2017

### **Hamilton-Jacobi approach to potential functions in information geometry**

By: Ciaglia, Florio M.; Di Cosmo, Fabio; Felice, Domenico; et al.

JOURNAL OF MATHEMATICAL PHYSICS Volume: 58 Issue: 6 Article Number: 063506 Published: JUN 2017

### **A pedagogical intrinsic approach to relative entropies as potential functions of quantum metrics: The q-z family**

F.M. Ciaglia, F. Di Cosmo, M. Laudato, G. Marmo, F.M. Mele, F. Ventriglia, P. Vitale

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