Correlated photon emission by two excited atoms in a waveguide

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Information Geometry, Quantum Mechanics and Applications

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Outline

- Waveguide QED
  - Non-relativistic Hamiltonian & Dipolar interaction
  - Excitation sectors in the Friedrichs-Lee model
- One-excitation sector
  - Diagrams & Self-energy
  - Entanglement by relaxation
- Two-excitation sector
  - Perturbative expansion
  - Non-perturbative weak coupling limit
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Waveguide QED

Multi-level quantum systems decay can be controlled by means of boundary conditions and dimensional reduction (e.g. embedding in a waveguide).

- Rectangular cross section \((L_y < L_z)\);

- Cut-off frequencies:
  \[ M_{n,m} = \pi \sqrt{\frac{n^2}{L_z^2} + \frac{m^2}{L_y^2}}; \]

- Dispersion relation:
  \[ \omega(k) = \sqrt{k^2 + M_{n,m}^2}. \]

Electromagnetic field quantization:
\[ [a(k), a^\dagger(k')] = \delta(k - k'). \]

The model concerns photons in the lowest-cutoff energy \(TE_{1,0}\) mode.
Building block: the coupled qubit

Artificial atoms engineering

Hilbert space: $\mathcal{H} \otimes \mathcal{H}_B$

\[\begin{align*}
\mathcal{H} &= \text{span}\{|e\rangle, |g\rangle\} \\
\mathcal{H}_B &= \text{span}\{|k\rangle\}, \text{ with } k \in X
\end{align*}\]
Building block: the coupled qubit

Artificial atoms engineering

Quantum Dot

SQUID

Structured bath

dispersion relation $\omega : X \rightarrow \mathbb{R}$

spectrum of a multiplication operator: $\Omega |k\rangle = \omega(k) |k\rangle$

Examples:

$\omega_0 \in \sigma_{ac}(\Omega)$: decay is energetically allowed (in most cases);

$\omega_0 \notin \sigma_{ac}(\Omega)$: decay inhibition

Hilbert space: $\mathcal{H} \otimes \mathcal{H}_B$

$\begin{cases} 
\mathcal{H} = \text{span}\{|e\>, |g\rangle\} \\
\mathcal{H}_B = \text{span}\{|k\rangle\}, \text{ with } k \in X
\end{cases}$
Non-relativistic Hamiltonian & Dipolar interaction

Hamiltonian of an atom trapped in a potential $V(\mathbf{r})$:

$$H_{\text{at}} = \frac{1}{2m_e} (\mathbf{p} - e\mathbf{A}(\mathbf{r}))^2 - V(\mathbf{r}) = H_0 - \frac{e}{m_e} \mathbf{p} \cdot \mathbf{A}(\mathbf{r}) + \mathcal{O}(e^2),$$

- nondynamical atoms center of mass position $\mathbf{r}_0$;
- **dipolar approximation**: atoms are assumed pointlike dipoles,

$$\langle e | H_{\text{int}}^{(\text{dip})} | g \rangle = - \frac{e}{m_e} A_z^{(1,0)}(\mathbf{r}_0) \langle e | p_z | g \rangle$$

$$= - i e \omega_0 A_z^{(1,0)}(\mathbf{r}_0) z_{eg}.$$  

Dipole moment

$$d_{eg} = e |z_{eg}|$$
Non-relativistic Hamiltonian & Dipolar interaction

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- nondynamical atoms center of mass position $\mathbf{r}_0$;
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$$\langle e | H_{\text{int}}^{(\text{dip})} | g \rangle = -\frac{e}{m_e} A_z^{(1,0)}(\mathbf{r}_0) \langle e | p_z | g \rangle = -ie\omega_0 A_z^{(1,0)}(\mathbf{r}_0) z_{eg}. $$

Dipole moment

$$d_{eg} = e|z_{eg}|$$

$|g\rangle = |n = 1, l = 0, m = 0\rangle$

$|e\rangle = |n = 2, l = 1, m = 0\rangle$
Friedrichs-Lee model

Vacuum instability description

Hamiltonian $H_g = H_0 + \lambda V_g$

in dipolar and rotating wave approximations:

$$H_0 = \omega_0 |e\rangle \langle e| \otimes \mathbb{1} + \mathbb{1} \otimes \int dk \; \omega(k) a^\dagger(k) a(k),$$

$$V_g = \int dk \left[ \sigma^+ \otimes g(k) a(k) + \sigma^- \otimes g^*(k) a^\dagger(k) \right],$$

with form factor $g$, coupling constant $\lambda$, identity $\mathbb{1}$, ladder operators $\sigma^+$, $\sigma^-$. 

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Friedrichs-Lee model

Vacuum instability description

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$$H_0 = \omega_0 |e\rangle \langle e| \otimes 1 + 1 \otimes \int dk \ \omega(k) a^\dagger(k) a(k),$$

$$V_g = \int dk \left[ \sigma^+ \otimes g(k) a(k) + \sigma^- \otimes g^*(k) a^\dagger(k) \right],$$

with form factor $g$, coupling constant $\lambda$, identity $1$, ladder operators $\sigma^+$, $\sigma^-$. Excitation number: $\mathcal{N} = |e\rangle \langle e| + \int dk \ a^\dagger(k) a(k) \implies [H_g, \mathcal{N}] = 0$

One-excitation sector:

$$\mathcal{H} \otimes \mathcal{H}_B = \mathbb{C} \oplus L^2(\mathbb{R})$$

Vacuum state: $|\text{vac}\rangle = |e\rangle \otimes |0\rangle$

$|\psi\rangle = c |e\rangle \otimes |0\rangle + |g\rangle \otimes |\varphi\rangle$

with $c \in \mathbb{C}$, $\varphi \in L^2(\mathbb{R})$

▶ Excited state probability $|c|^2$;

▶ One photon wavefunction $\varphi$. 

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Infinite waveguide

\[ H_d = H_0 + \lambda V_d \]

\[ H_0 = \omega_0 (|e_A\rangle \langle e_A| + |e_B\rangle \langle e_B|) \]

\[ + \int dk \ \omega(k)a^\dagger(k)a(k), \]

\[ V_d = \int \frac{dk}{\sqrt{\omega(k)}} \left[ (\sigma_A^+ + \sigma_B^+e^{ikd})a(k) + h.c. \right], \]

\[ g_x(k) = \frac{e^{ikx}}{\sqrt{\omega(k)}} \]
Infinite waveguide

\[ H_d = H_0 + \lambda V_d \]

\[ H_0 = \omega_0 (|e_A \rangle \langle e_A| + |e_B \rangle \langle e_B|) + \int dk \, \omega(k) a^\dagger(k) a(k), \]

\[ V_d = \int \frac{dk}{\sqrt{\omega(k)}} \left[ \left( \sigma_A^+ + \sigma_B^+ e^{i k d} \right) a(k) + h.c. \right], \quad g_x(k) = \frac{e^{i k x}}{\sqrt{\omega(k)}} \]

Excitation number

\[ \mathcal{N} = |e_A \rangle \langle e_A| + |e_B \rangle \langle e_B| + \int dk \, a^\dagger(k) a(k) \]

\[ \mathcal{N} = 1 : \quad |\psi\rangle = (c_A |e_A, g_B\rangle + c_B |g_A, e_B\rangle) \otimes |0\rangle + |g_A, g_B\rangle \otimes |\varphi\rangle \]

with \( \varphi \in L^2(\mathbb{R}) \) s.t.

\[ |\varphi\rangle = \int dk \, \varphi(k) a^\dagger(k) |0\rangle \in \mathcal{H}_B. \]
Bound states

\[ H |\psi\rangle = E |\psi\rangle \quad \text{with} \quad |\psi\rangle \propto \frac{|e_A, g_B\rangle \pm |g_A, e_B\rangle}{\sqrt{2}} \otimes |0\rangle + \ldots \]

- **E < M**: Evanescent mode interaction
  - for \( \omega_0 \) below the threshold there are both symmetric and antisymmetric eigenstates;
  - the electromagnetic field contribution in these states is not negligible and \( E = \omega_0 + O(\lambda^2) \).
Bound states

\[ H |\psi\rangle = E |\psi\rangle \quad \text{with} \quad |\psi\rangle \propto \frac{|e_A, g_B\rangle \pm |g_A, e_B\rangle}{\sqrt{2}} \otimes |0\rangle + \ldots \]

**E < M**: Evanescent mode interaction

- for \(\omega_0\) below the threshold there are both symmetric and antisymmetric eigenstates;
- the electromagnetic field contribution in these states is not negligible and \(E = \omega_0 + O(\lambda^2)\).

**E > M**: Propagating photon interaction

- a single atom decays if its excitation energy is immersed in the continuum;
- nontrivial bound states occur for particular values of inter-atomic distance.
Resonance condition

\[ |\psi\rangle = (c_A |e_A, g_B\rangle + c_B |g_A, e_B\rangle) \otimes |0\rangle + |g_A, g_B\rangle \otimes |\varphi\rangle \]

This is a stationary state associated with the eigenvalue \( E = \sqrt{\kappa^2 + M^2} \) if:

- atoms distance satisfy a resonance constraint

\[ c_A + c_B e^{\pm i\kappa d} = 0 \implies \kappa d = n\pi, \ n \in \mathbb{N}; \]

- there are real solutions for

\[ E = \omega_0 + \lambda^2 \int dk \frac{1 - (-1)^n e^{-i\kappa d}}{\omega(k)(E - \omega(k))}. \]
Resonance condition

\[ |\psi\rangle = (c_A |e_A, g_B\rangle + c_B |g_A, e_B\rangle) \otimes |0\rangle + |g_A, g_B\rangle \otimes |\varphi\rangle \]

This is a stationary state associated with the eigenvalue \( E = \sqrt{k^2 + M^2} \) if:

- atoms distance satisfy a resonance constraint

\[ c_A + c_B e^{\pm ikd} = 0 \implies kd = n\pi, \quad n \in \mathbb{N}; \]

- there are real solutions for

\[ E = \omega_0 + \lambda^2 \int dk \frac{1 - (-1)^n e^{-ikd}}{\omega(k)(E - \omega(k))}. \]

Triplet and singlet are entangled bound states:

\[ |\psi_n\rangle = \sqrt{p_n} \frac{|e_A, g_B\rangle + (-1)^{n+1} |g_A, e_B\rangle}{\sqrt{2}} \otimes |0\rangle + \mathcal{O}(\lambda), \]

where \( p_n \simeq \left(1 + n \frac{2\pi^2 \lambda^2 M}{k^3}\right)^{-1} \) is a decreasing function with distance.
Diagrams & Self-energy

\[ \frac{1}{z - H_0} \]

\[ \langle k | \ldots | k \rangle \]

\[ \langle \Psi^\pm | \ldots | \Psi^\pm \rangle \]

Propagator renormalization:

\[ = \quad + \quad + \quad + \ldots \]

Interaction vertex: \( \lambda V_d \)

Self-energy:

\[ \Sigma^\pm(z) = \langle \Psi^\pm | V_d | \psi^\pm \rangle \]
Diagrams & Self-energy

\[
\frac{1}{z - H_0} \langle k \vert ... \vert k \rangle
\]
\[
\langle \Psi^\pm \vert ... \vert \Psi^\pm \rangle
\]
\[
(\pm)
\]

Interaction vertex: \( \lambda V_d \)

Propagator rinormalization:

\[
\langle \Psi^\pm \vert \frac{1}{z - H_d} \vert \Psi^\pm \rangle = \frac{1}{z - \omega_0} \sum_{n=0}^{\infty} \left( \frac{\lambda^2 \Sigma^\pm(z)}{z - \omega_0} \right)^n = \frac{1}{z - \omega_0 - \lambda^2 \Sigma^\pm(z)} = G^\pm(z)
\]

Self-energy:

\[
\Sigma^\pm(z) = \langle \Psi^\pm \vert V_d \frac{1}{z - H_0} V_d \vert \Psi^\pm \rangle
\]
Resolvent formalism

The propagator elements for the atomic excitation is:

\[ G_{ij}(z) = \langle i \vert \frac{1}{z - H_d} \vert j \rangle \text{ with } i, j = 1, 2 = (e_A, g_B), (g_A, e_B). \]

Free case \((\lambda = 0)\):

propagator in the basis \(\{|e_A, g_B\}, \{|g_A, e_B\}\}\)

\[ G_0(z) = \frac{1}{z - \omega_0} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

unique pole \(z = \omega_0\)
The propagator elements for the atomic excitation is:

\[ G_{ij}(z) = \langle i | \frac{1}{z - H_d} | j \rangle \] with \( i, j = 1, 2 = (e_A, g_B), (g_A, e_B) \).

**Free case \((\lambda = 0)\):**

propagator in the basis \( \{|e_A, g_B\}, |g_A, e_B\}\) \( G_0(z) = \frac{1}{z - \omega_0} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) unique pole \( z = \omega_0 \)

**Interacting case \((\lambda > 0)\):**

\[ G(z) = [G_0(z)^{-1} - \lambda^2 \Sigma(z)]^{-1} \] the self-energy \( \Sigma(z) \) lifts the degeneracy

the propagator is diagonalized in the Bell basis \( |\Psi^\pm\rangle = \frac{|e_A, g_B\rangle \pm |g_A, e_B\rangle}{\sqrt{2}} \):

\[ G(z) = \frac{|\Psi^+\rangle \langle \Psi^+|}{z - \omega_0 - \lambda^2 \Sigma_+ (z)} + \frac{|\Psi^-\rangle \langle \Psi^-|}{z - \omega_0 - \lambda^2 \Sigma_+ (z)}. \]
Entanglement by relaxation

\[ |e_A, g_B\rangle = \frac{|e_A, g_B\rangle + |g_A, e_B\rangle}{2} + \frac{|e_A, g_B\rangle - |g_A, e_B\rangle}{2} \]

- \( n \) even: decays to ground states + photon
- \( n \) odd: relaxes to the entangled bound state
- \( n \) even: decays to ground states + photon
- \( n \) odd: decays to ground states + photon
Entanglement by relaxation

\[ |e_A, g_B\rangle = \frac{|e_A, g_B\rangle + |g_A, e_B\rangle}{2} + \frac{|e_A, g_B\rangle - |g_A, e_B\rangle}{2} \]

\( n \) even

decays to ground states + photon

\( n \) odd

relaxes to the entangled bound state

\( n \) even

decays to ground states + photon

\( n \) odd

\[ \Sigma_{\pm}^{\Pi}(z - i0^+) = \Sigma_{\pm}(z + i0^+) \]

\[ U(t)\theta(t) = \frac{i}{2\pi} \int_{-\infty+i0^+}^{+\infty+i0^+} dz \ e^{-izt} G(z) \]

\[ E_p^{(+)} - i\gamma_p^{(+)} / 2 \]

\[ E_p^{(-)} - i\gamma_p^{(-)} / 2 \]
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For $N = 2$ the state reads

$$|\psi\rangle = c_{AB} |e_A, e_B\rangle + \sum_{s=\pm} \int dk \ B_s(k) |\Psi^s; k\rangle$$

$$+ \frac{1}{\sqrt{2}} \int dk dk' \ A(k, k') |k, k'\rangle \text{ with } B_s \in L^2(\mathbb{R}), A \in L^2(\mathbb{R}) \otimes 2.$$ 

This expression implies a dipolar interaction with rotated couplings:

$$V_d = \int dk \sum_{s=\pm} \frac{1 + se^{ikd}}{\sqrt{2\omega(k)}} b_s a^\dagger(k) + \text{H.c. with } g_s(k) = \frac{g_0(k) + sg_d(k)}{\sqrt{2}}$$

where $b_s^\dagger |g_A, g_B\rangle = \frac{\sigma_A^+ + s\sigma_B^+}{\sqrt{2}} |g_A, g_B\rangle = |\Psi^s\rangle$.

Properties: $(\sigma_{A,B})^2 = 0 \implies b_+ b_- = 0$ and $|e_A, e_B\rangle = \pm (b_\pm^\dagger)^2 |g_A, g_B\rangle$.
Non-perturbative weak coupling limit

Selected parameters: $\omega_0 = 1.1M$, $\lambda = 10^{-2}M$, $k_0d = \frac{\pi}{2}$

\[ \frac{k_0}{M} \simeq 0.458 \]

\[ \phi_- = \text{Arg}\{A_-(k_1, k_2)\} \]
\[ \phi_+ = \text{Arg}\{A_+(k_1, k_2)\} \]
\[ \text{Arg}\{A(k_1, k_2)\} \]
\[ \phi_+ - \phi_- \]

Self-energy non-perturbative effect splitting:

$2 \Re m\{\Sigma_\pm(\omega_0 + i0^+)\}$
Non-perturbative weak coupling limit

\[ \lambda = 10^{-3} M \]
\[ \lambda = 10^{-2} M \]
\[ \lambda = 2 \times 10^{-2} M \]

\[ \omega_0 = 1.1 M \]

\[ \omega_0 = 1.2 M \]
\[ \omega_0 = 1.3 M \]
\[ \omega_0 = 1.4 M \]

\[ k_0 d = \pi / 2 \]
\[ k_0 d = 3\pi / 2 \]
\[ k_0 d = 5\pi / 2 \]

 Peaks in the variation of \( k_0 d \) are points of the orange curve in the variation of \( \lambda \);

Increased distances implement stronger coupling regimes.
Non-perturbative weak coupling limit

$\lambda = 10^{-3} M$
$\lambda = 10^{-2} M$
$\lambda = 2 \times 10^{-2} M$

$\omega_0 = 1.1 M$
$\omega_0 = 1.2 M$
$\omega_0 = 1.3 M$
$\omega_0 = 1.4 M$

$R$ vs $k_0 d$

$R$ vs $\lambda / M (\times 10^{-3})$

$\omega_0 = 1.1 M$

peaks in the variation of $k_0 d$ are points of the orange curve in the variation of $\lambda$;

increased distances implement stronger coupling regimes.

$k_0 d = 5\pi /2$;

increasing values for $\omega_0$ push forward the plasmonic bound state.
Conclusions & Outlook

- An **entangled bound state** exists for discrete values of the inter-atomic distance, corresponding to the resonance condition in the one-excitation sector;

- in the two-excitation sector the **two photon state** is the asymptotic one in the weak coupling limit, showing direction correlation maximized in anti-resonance;
Thank you for your attention