Classical and Quantum Dynamics for a Parametric Oscillator with Analytical Solutions and Comparison with the Damped Harmonic Oscillator

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1. Introduction
2. Nonlinear (NL) formulation of time-dependent (TD) quantum mechanics (QM)
3. Analytical solutions for parametric oscillator with $\omega \propto \frac{1}{t}$
4. Conclusions and perspectives

Classical Theories: **real** quantities **nonlinear** eqs. possible

Quantum Theory: **complex** quantities **linear** theory

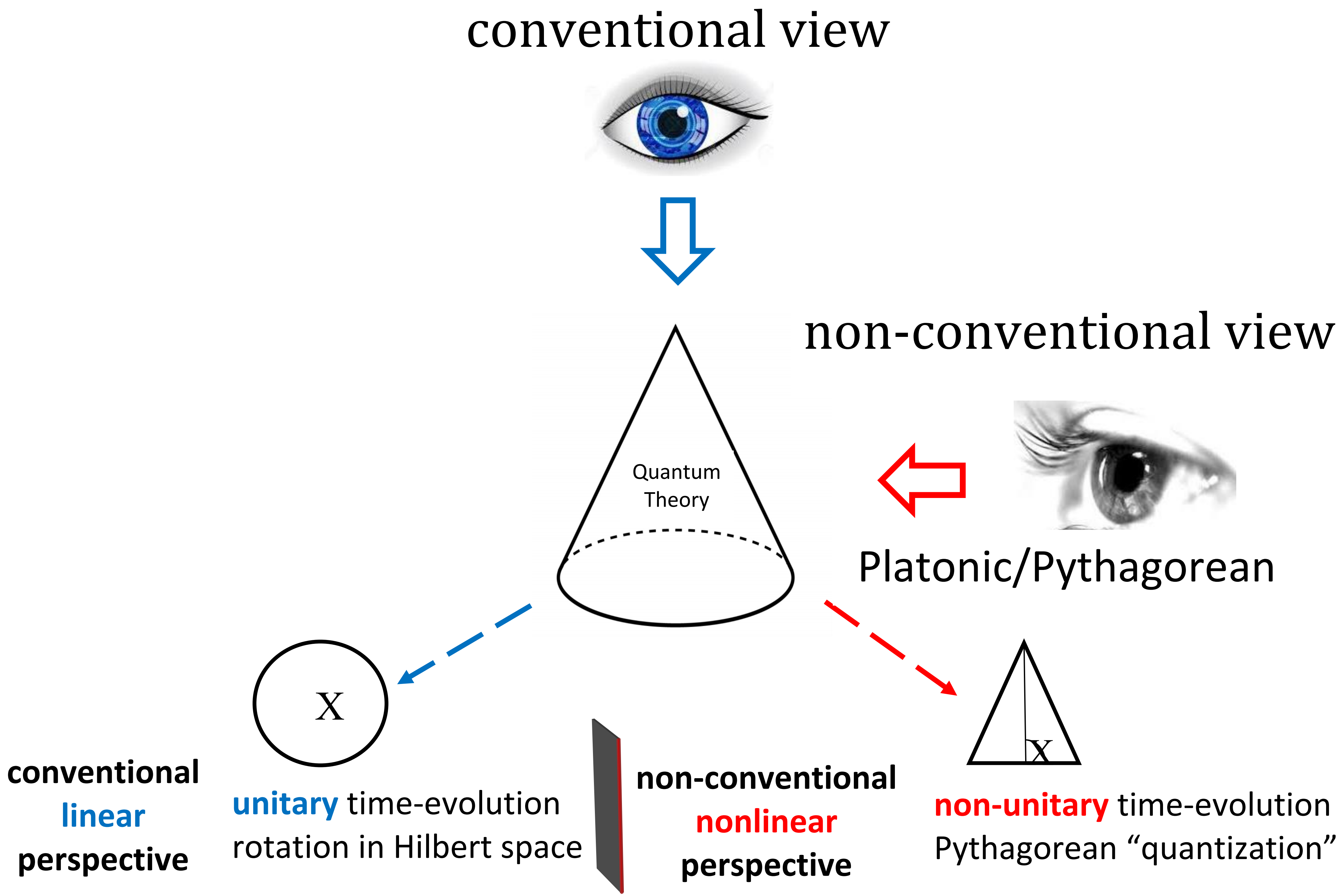
– ?**complexification**@ of **classical** theory (mechanics)

– plus information about **coupling** of real and imaginary parts

– obtained from complex **nonlinear** (Riccati) equation

$\rightarrow$ Same info about **quantum dynamics** as from TDSE by **solving** classical **Newtonian equation**

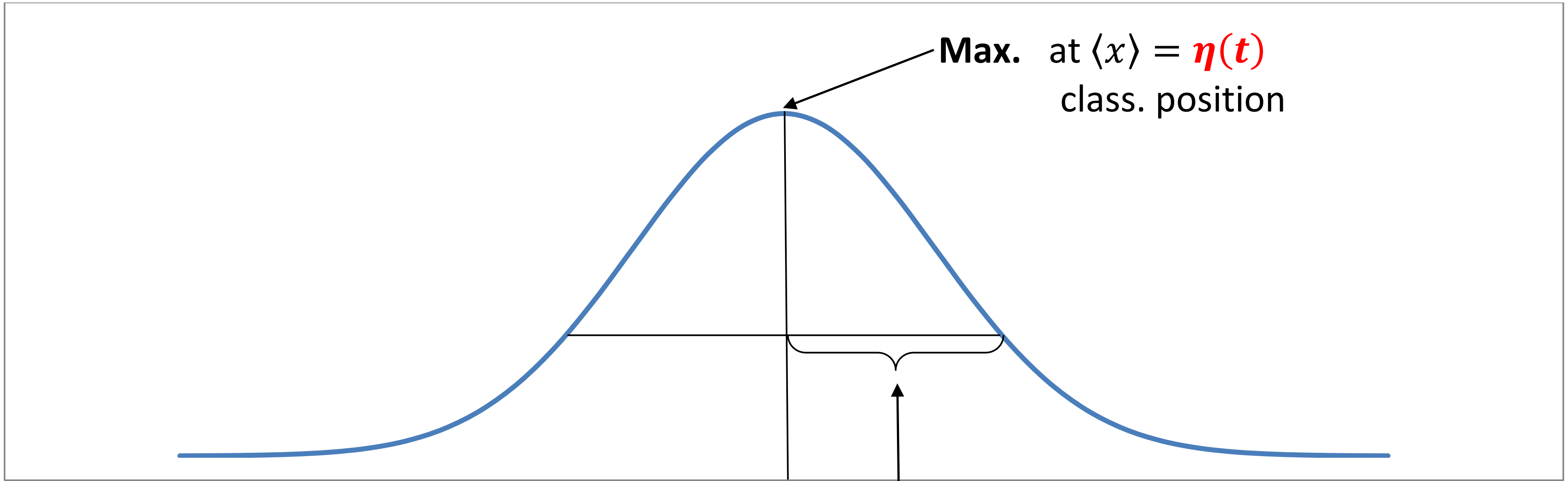
EXAMPLE: **Analytical solution** of class. **Newtonian eq.** of motion for **parametric oscillator** with TD **frequency** prop. to $\frac{1}{t}$


conventional view
non-conventional view
Quantum
Theory
Platonic/Pythagorean
X
X
conventional
linear
perspective
unitary time-evolution
rotation in Hilbert space
non-conventional
nonlinear
perspective
non-unitary time-evolution
Pythagorean "quantization"

TDSE – Gaussian Wave Packets (WPs)

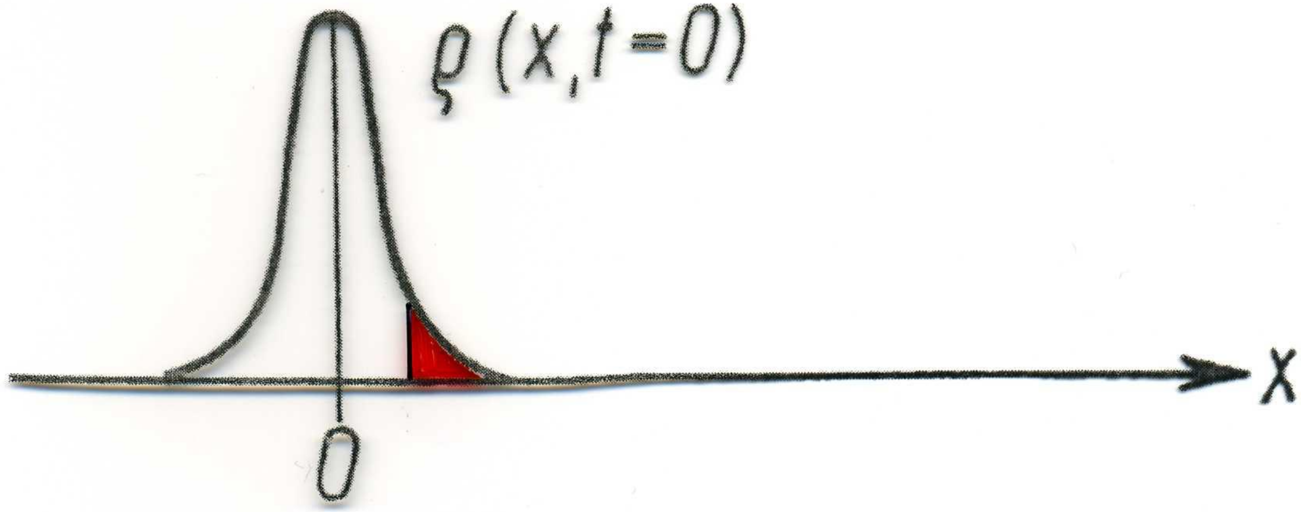
TDSE: $$i\hbar \frac{\partial}{\partial t}\Psi(\underline{r},t) = \left\{-\frac{\hbar^2}{2m}\Delta + V(\underline{r},t)\right\}\Psi(\underline{r},t)$$

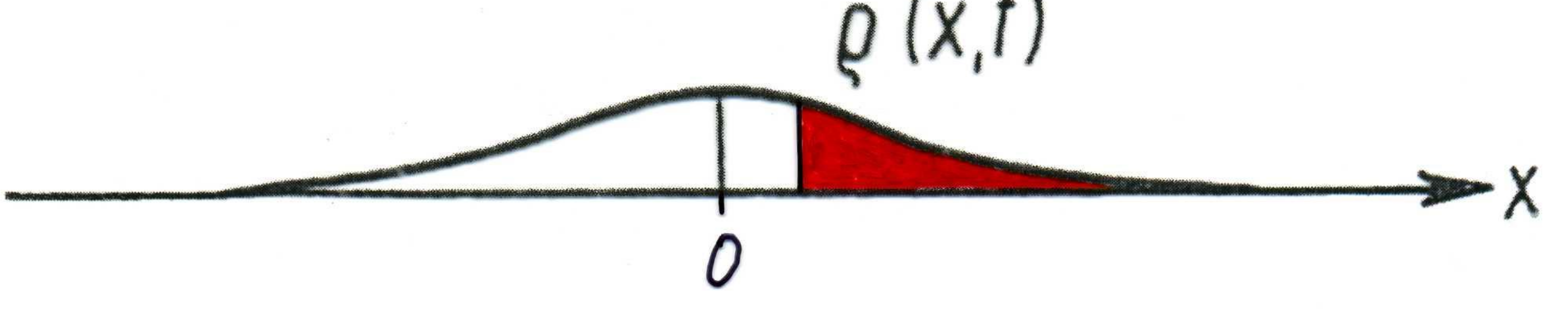
For quadratic Hamiltonians, e.g. HO, Gaussian WP solutions



$\Psi(x,t) \propto \exp\{i\boldsymbol{y}(x-\boldsymbol{\eta})^2 + \dots\dots\}$

Width prop. $\boldsymbol{\alpha_L} \propto \sqrt{\langle \tilde{x}^2 \rangle}$
position uncertainty


ϱ (x, t=0)
x
0


ϱ (x, t)
x
0

Time-Evolution of MAXIMUM and WIDTH of Gaussian WP Solution of TDSE

WP-ansatz: $$\boxed{\Psi(x,t) = N(t)\exp\left\{i\left[\boldsymbol{y(t)}\tilde{x}^2 + \frac{1}{\hbar}\langle p\rangle\tilde{x} + K(t)\right]\right\}}$$

$y(t)$ complex; $\tilde{x} = x - \eta(t)$, max. at $x = \boldsymbol{\eta(t)} = \langle \boldsymbol{x}\rangle$

in TDSE $\rightarrow$ terms prop. $\tilde{x}$: $$\boxed{\ddot{\eta} + \omega^2\eta = 0}$$

(with $\omega = 0, \omega_0$ *or* $\boldsymbol{\omega(t)}$)

$\rightarrow$ terms prop. $\tilde{x}^2$: $$\boxed{\dot{\mathbb{C}} + \mathbb{C}^2 + \omega^2 = 0}$$ **!**

complex NL Riccati eq.

with $\mathbb{C} = \frac{2\hbar}{m}\boldsymbol{y(t)}$, $\frac{2\hbar}{m}\boldsymbol{y_{\mathrm{I}}} = \frac{\hbar}{2m\langle\tilde{x}^2\rangle}$, $\sqrt{\langle\tilde{x}^2\rangle} \propto \boldsymbol{\alpha(t)}$

Direct Solution of the Complex Riccati Equation

Riccati: $\dot{\mathbb{C}} + \mathbb{C}^2 + \omega^2 = 0$ inhomogeneous, NL

Particular Solution: $\tilde{\mathbb{C}}$ $\rightarrow$ General Solution: $\mathbb{C} = \tilde{\mathbb{C}} + \mathbb{V}(t)$

Bernoulli: $\dot{\mathbb{V}} + 2\tilde{\mathbb{C}}\,\mathbb{V} + \mathbb{V}^2 = 0$ homogeneous

Linearization via $\mathbb{V}(t) = \frac{1}{\kappa(t)}$ $\Rightarrow$ $\dot{\kappa} - 2\tilde{\mathbb{C}}\,\kappa = 1$

General Solution $\mathbb{C} = \tilde{\mathbb{C}} + \frac{d}{dt}\ln\left(\boldsymbol{\kappa_0} + I(t)\right)$ depending on the **complex parameter**

For $\tilde{\mathbb{C}} = \text{const.}$: $\mathbb{C} = \tilde{\mathbb{C}} + \frac{e^{-2\tilde{\mathbb{C}}\cdot t}}{\frac{1}{2\tilde{\mathbb{C}}}\left(1-e^{-2\tilde{\mathbb{C}}\cdot t}\right)+\boldsymbol{\kappa_0}}$ with $\tilde{\mathbb{C}}$ generally **complex**

Complex Riccati Eqs. and Nonlinear (real) Ermakov Eqs.

New variable $\alpha(t)$ via $\boxed{\mathbb{C}_\mathrm{I} = \frac{1}{\alpha^2}}$ into $\mathcal{Im} \rightarrow$ $\boxed{\mathbb{C}_\mathrm{R} = \frac{\dot{\alpha}}{\alpha}}$

into $\mathcal{Re} \rightarrow$ $\boxed{\ddot{\alpha} + \boldsymbol{\omega^2(t)}\alpha = \frac{1}{\alpha^3}}$ **Ermakov** equation (Steen, Milne, Pinney, Lewis, Riesenfeld)

$\alpha(t)$ prop. WP width;

$\eta(t) = \langle x \rangle$: WP maximum, obeys

$$\boxed{\ddot{\eta} + \boldsymbol{\omega^2(t)}\eta = 0}$$

Elimination of $\omega^2(t)$ from the two eqs. leads to a **dynamical invariant**

$$\boxed{I_\mathrm{L} = \frac{1}{2}\left[(\dot{\eta}\alpha - \dot{\alpha}\eta)^2 + \left(\frac{\eta}{\alpha}\right)^2\right] = \text{const.}}$$

Using the definitions of $\mathbb{C}_\mathrm{I}$ and $\mathbb{C}_\mathrm{R}$, this can be rewritten as

or $$\boxed{I_\mathrm{L} = \frac{1}{2}\alpha^2[(\dot{\eta} - \mathbb{C}_\mathrm{R}\eta)^2 + (\mathbb{C}_\mathrm{I}\eta)^2] = \frac{1}{2}\alpha^2[(\dot{\eta} - \mathbb{C}\eta)(\dot{\eta} - \mathbb{C}^*\eta)]}$$

Note: $mI_\mathrm{L} = \textbf{\textit{action}}$

Generalized Creation/Annihilation Operators and CS

$$H_{\mathrm{op}} = \frac{1}{2m} p_{\mathrm{op}}^2 + \frac{m}{2}\omega_0^2 x^2 = \hbar\omega_0\left(a^+ a + \frac{1}{2}\right)$$ **energy**, or $$\widehat{H}_{\mathrm{op}} = \frac{H_{\mathrm{op}}}{\hbar\omega_0} = \left(a^+ a + \frac{1}{2}\right)$$

with $a^+ a$: number operator; $[a, a^+]_- = 1$; Note: $\frac{H}{\omega_0} = \boldsymbol{action}$

$$a = i\sqrt{\frac{m}{2\hbar\omega_0}}\left(\frac{p_{\mathrm{op}}}{m} - \boldsymbol{i\omega_0} x\right) \qquad a^+ = -i\sqrt{\frac{m}{2\hbar\omega_0}}\left(\frac{p_{\mathrm{op}}}{m} + \boldsymbol{i\omega_0} x\right)$$

For **HO**: $\boldsymbol{i\omega_0} \mathrel{\widehat{=}} \boldsymbol{i}\mathbb{C}_{\mathrm{I}} = i\frac{1}{\alpha^2}$, i.e., imaginary part of Riccati variable

Operator corresponding to Ermakov invariant via replacements $\eta \to x, \dot{\eta} \to \frac{p_{\mathrm{op}}}{m}$ and non-commutativity of x and $p_{\mathrm{op}} = \frac{\hbar}{i}\frac{\partial}{dx}$:

$$I_{\mathrm{L,op}} = \frac{\hbar}{m}\left(a^+(t)a(t) + \frac{1}{2}\right)$$

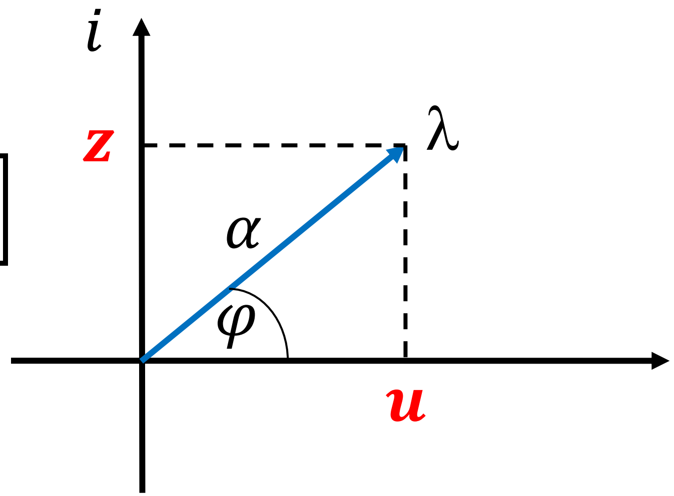
with $$a(t) = i\sqrt{\frac{m}{2\hbar}}\,\boldsymbol{\alpha(t)}\left(\frac{p_{\mathrm{op}}}{m} - \mathbb{C}x\right) \qquad a^+(t) = -i\sqrt{\frac{m}{2\hbar}}\,\boldsymbol{\alpha(t)}\left(\frac{p_{\mathrm{op}}}{m} - \mathbb{C}^* x\right)$$

Linearization of the Complex Riccati Equation

NL Riccati eq. $\boxed{\dot{\mathbb{C}} + \mathbb{C}^2 + \omega^2 = 0}$ can be **linearized** by $\boxed{\mathbb{C} = \frac{\dot{\lambda}}{\lambda}}$

to yield $\boxed{\ddot{\lambda} + \omega^2 \lambda = 0}$ **complex** Newtonian eq.

for complex variable $\boxed{\lambda(t) = u + iz = \alpha \cdot e^{i\varphi}}$



Insert **polar** form in def.: $\rightarrow$ $\boxed{\mathbb{C} = \frac{\dot{\alpha}}{\alpha} + i\dot{\varphi}}$

Inserting $\mathbb{C}$ in terms of α, $\dot{\alpha}$ and $\dot{\varphi}$ into $\mathcal{Im}$ of Riccati eq. yields

$\boxed{\dot{\varphi} = \frac{1}{\alpha^2}} = \mathbb{C}_{\mathrm{I}}$, i.e. consistent with original definition

Conservation of "angular momentum" in the **complex plane**!

Conservation law in **Cartesian** coordinates:

$$\boxed{\dot{z}u - \dot{u}z = 1}$$

Equivalent to the **Wronskian** determinant of the **two linear independent solutions** u and z of the complex Newtonian equation.

Knowing one solution (e.g. $z(t)$) it is possible to calculate the other via

$$\boxed{u = -z \int^{t} \frac{1}{z^2} dt'}$$

Knowing both provides $\boxed{\alpha^2 = u^2 + z^2}$

Quantum Uncertainties Expressed in Terms of $\mathbb{C}$ or $\boldsymbol{\alpha}$

$$\langle\tilde{x}^2\rangle(t) = \frac{\hbar}{2m}\frac{1}{\mathbb{C}_\mathbf{I}} = \frac{\hbar}{2m}\boldsymbol{\alpha}^\mathbf{2}$$

$$\langle\tilde{p}^2\rangle(t) = \frac{m\hbar}{2}\frac{1}{\mathbb{C}_\mathbf{I}}\left(\mathbb{C}_\mathbf{R}^\mathbf{2} + \mathbb{C}_\mathbf{I}^\mathbf{2}\right) = \frac{m\hbar}{2}\left[\dot{\boldsymbol{\alpha}}^\mathbf{2} + \frac{1}{\boldsymbol{\alpha}^\mathbf{2}}\right]$$

$$\langle[\tilde{x},\tilde{p}]_+\rangle(t) = \hbar\frac{\mathbb{C}_\mathbf{R}}{\mathbb{C}_\mathbf{I}} = \hbar\boldsymbol{\alpha}\dot{\boldsymbol{\alpha}}$$

$$U = \langle\tilde{x}^2\rangle\langle\tilde{p}^2\rangle = \frac{\hbar^2}{4}\left\{1 + \left(\frac{\mathbb{C}_\mathbf{R}}{\mathbb{C}_\mathbf{I}}\right)^2\right\} = \frac{\hbar^2}{4}\{1 + (\boldsymbol{\alpha}\dot{\boldsymbol{\alpha}})^2\}$$

$$\tilde{E} = \frac{\langle\tilde{p}^2\rangle}{2m} + \frac{m}{2}\omega^2\langle\tilde{x}^2\rangle = \frac{\hbar}{4}\frac{1}{\mathbb{C}_\mathrm{I}}\left\{\mathbb{C}_\mathbf{R}^\mathbf{2} + \mathbb{C}_\mathbf{I}^\mathbf{2} + \omega^2\right\} = \frac{\hbar}{4}\left\{\dot{\boldsymbol{\alpha}}^\mathbf{2} + \frac{1}{\boldsymbol{\alpha}^\mathbf{2}} + \omega^2\boldsymbol{\alpha}^\mathbf{2}\right\}$$

Classical Trajectory from WP Width

Solution of Ermakov equation : $\alpha(t)$

Conservation law $\dot{\varphi} = \frac{1}{\alpha^2}$

$$\varphi = \int \frac{1}{\alpha^2} dt' \longrightarrow \lambda = \alpha e^{i\varphi} = u + iz$$

Classical Trajectory : $z = \alpha \sin \varphi$

As $\alpha^2 = \frac{2m}{\hbar} \langle \tilde{x}^2 \rangle$ → **From WP-width** → α

TD Green Function / Feynman Kernel

$$\Psi(x,t) = \int_{-\infty}^{+\infty} dx' \boldsymbol{K(x,x',t,t')}\, \Psi(x',t')$$

Feynman: K via path integral method

$$K(x,x',t,t') \propto \exp\left\{\frac{im}{2\hbar}\left[\frac{\dot{\boldsymbol{z}}}{\boldsymbol{z}}x^2 - 2\frac{x}{\boldsymbol{z}}\left(\frac{x'}{\alpha_0}\right) + \frac{\boldsymbol{u}}{\boldsymbol{z}}\left(\frac{x'}{\alpha_0}\right)^2\right]\right\}$$

Inserting K yields

$$\Psi(x,t) \propto \exp\left\{\frac{im}{2\hbar}\left[\frac{\dot{\boldsymbol{z}}}{\boldsymbol{z}}x^2 - \frac{1}{\boldsymbol{z}\lambda}\left(x - \frac{p_0\alpha_0}{m}\boldsymbol{z}\right)^2\right]\right\}$$

Comparison with Gaussian WP-ansatz shows

$$\boldsymbol{z(t)} = \frac{m}{\alpha_0 p_0}\eta(t)$$

Wigner Function

$$W(\boldsymbol{x},\boldsymbol{p},t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dz\,\boldsymbol{\Psi}^*\left(x+\frac{z}{2},t\right)\boldsymbol{\Psi}\left(x-\frac{z}{2},t\right)e^{-\frac{i}{\hbar}pz}$$

$\Psi_{\mathrm{WP}}(x,t)$ and use relations between $y_{\mathrm{I}}, y_{\mathrm{R}}$ and uncertainties $\langle\tilde{x}^2\rangle$ etc.:

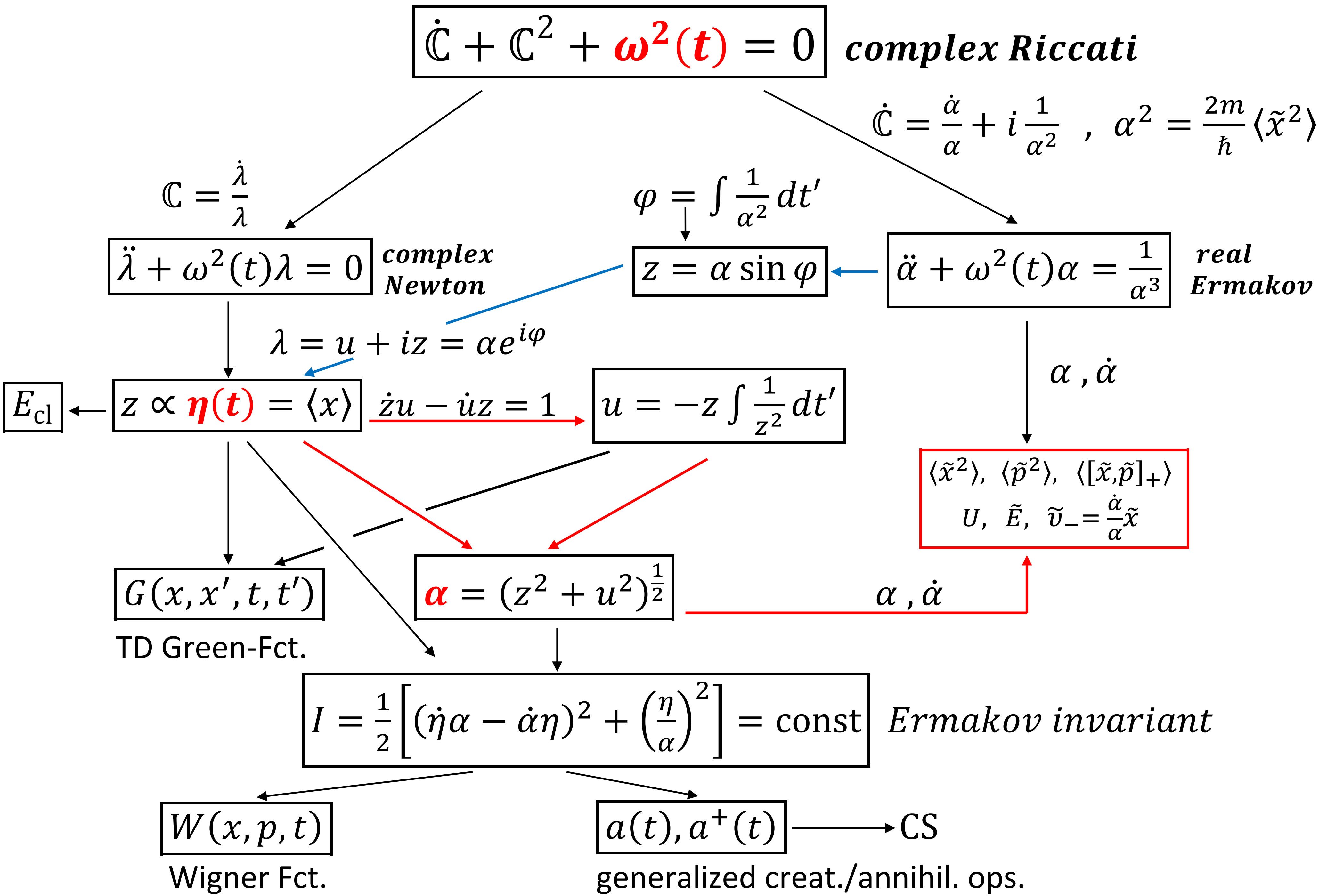
$$W(x,p,t)=\frac{1}{\pi\hbar}\exp\left\{-\frac{2}{\hbar^2}\left[\langle\tilde{p}^2\rangle\tilde{x}^2-\langle[\tilde{x},\tilde{p}]_+\rangle\tilde{x}\tilde{p}+\langle\tilde{x}^2\rangle\tilde{p}^2\right]\right\}$$

Express uncertainties in terms of α and $\dot{\alpha}$ and rearrange:

$$W(x,p,t)=\frac{1}{\pi\hbar}\exp\left\{-\frac{m}{\hbar}\left[\left(\dot{\alpha}\tilde{x}-\alpha\frac{\tilde{p}}{m}\right)^2+\left(\frac{\tilde{x}}{\alpha}\right)^2\right]\right\}$$

For $x=p=0$:

$$W(0,0,t)=\frac{1}{\pi\hbar}\exp\left\{-\frac{2m}{\hbar}\boldsymbol{I}_{\mathrm{L}}\right\}$$



$\dot{\mathbb{C}} + \mathbb{C}^2 + \omega^2(t) = 0$
complex Riccati
$\dot{\mathbb{C}} = \frac{\dot{\alpha}}{\alpha} + i\frac{1}{\alpha^2}$, $\alpha^2 = \frac{2m}{\hbar}\langle\tilde{x}^2\rangle$
$\mathbb{C} = \frac{\dot{\lambda}}{\lambda}$
$\varphi = \int \frac{1}{\alpha^2} dt'$
$\ddot{\lambda} + \omega^2(t)\lambda = 0$
complex Newton
$z = \alpha \sin\varphi$
$\ddot{\alpha} + \omega^2(t)\alpha = \frac{1}{\alpha^3}$
real Ermakov
$\lambda = u + iz = \alpha e^{i\varphi}$
$\alpha , \dot{\alpha}$
E_{cl}
$z \propto \eta(t) = \langle x \rangle$
$\dot{z}u - \dot{u}z = 1$
$u = -z \int \frac{1}{z^2} dt'$
$\langle\tilde{x}^2\rangle$, $\langle\tilde{p}^2\rangle$, $\langle[\tilde{x},\tilde{p}]_+\rangle$
U, $\tilde{E}$, $\tilde{v}_- = \frac{\dot{\alpha}}{\alpha}\tilde{x}$
$G(x, x', t, t')$
TD Green-Fct.
$\alpha = (z^2 + u^2)^{\frac{1}{2}}$
$\alpha , \dot{\alpha}$
$I = \frac{1}{2}\left[(\dot{\eta}\alpha - \dot{\alpha}\eta)^2 + \left(\frac{\eta}{\alpha}\right)^2\right] = \mathrm{const}$
Ermakov invariant
$W(x, p, t)$
Wigner Fct.
$a(t), a^+(t)$
CS
generalized creat./annihil. ops.

Comparison: SE – Diffusion Equation

SE: WP solution $\Psi(x,t) \propto \exp\left\{i\frac{\hbar}{2m}\mathbb{C}(\boldsymbol{t})\tilde{x}^2\right\}$, $\boxed{\dot{\mathbb{C}} + \mathbb{C}^2 + \omega^2 = 0}$

$\rightarrow$ **CE**: $\varrho_S(x,t) = \Psi^*\Psi$: $\boxed{\frac{\partial}{\partial t}\varrho_S + \frac{\partial}{\partial x}(\varrho_S \boldsymbol{\upsilon_-}) = 0}$ $\boxed{\upsilon_- = \dot{\eta} + \frac{\dot{\alpha}}{\alpha}\tilde{x}}$

$$\varrho_S \propto \exp\left\{-\frac{m}{\hbar}\frac{\tilde{x}^2}{\alpha^2}\right\} = \exp\left\{-\frac{\tilde{x}^2}{2\langle\tilde{x}^2\rangle_S}\right\}$$

Diffusion Eq.: $\boxed{\frac{\partial}{\partial t}\varrho_D - D\frac{\partial^2}{\partial x^2}\varrho_D = 0}$ $\boxed{\upsilon_D = -D\frac{\frac{\partial}{\partial x}\varrho_D}{\varrho_D} = \frac{D}{\hbar/2m}\frac{1}{\alpha^2}\tilde{x}}$

$\varrho_D \propto \exp\{-\mathbf{R}x^2\} = \exp\left\{-\frac{x^2}{4Dt}\right\} = \exp\left\{-\frac{x^2}{2\langle\tilde{x}^2\rangle_D}\right\}$, $\boxed{\dot{R} + 4DR^2 = 0}$

$R = \frac{1}{4Dt}$ $\rightarrow$ $\boxed{\langle\tilde{x}^2\rangle_D(t) = 2Dt}$ compare : $\boxed{\langle\tilde{x}^2\rangle_S = \frac{\hbar}{2m}\alpha^2}$

for (see free SE) $\boxed{\mathbf{D} = \frac{\hbar}{2m}}$ $\rightarrow$ $\boxed{\alpha^2 = \mathbf{2t}}$

Assume: $\boxed{\mathbb{C}_\mathrm{I} = \frac{1}{\alpha^2} = \frac{1}{2t}}$ $\xrightarrow{\mathcal{Im}\,\{\text{Ric.}\}}$ $\boxed{\mathbb{C}_\mathrm{R} = \frac{\dot{\alpha}}{\alpha} = \frac{1}{2t}}$

$\mathcal{Re}\,\{\text{Ric.}\}$ $\dot{\mathbb{C}}_\mathrm{R} + \mathbb{C}_\mathrm{R}^2 - \mathbb{C}_\mathrm{I}^2 + \boldsymbol{\omega_{\mathrm{Ri}}^2(t)} = \dot{\mathbb{C}}_\mathrm{R} + \omega_{\mathrm{Ri}}^2(t) = 0 \rightarrow \boxed{\omega_{\mathrm{Ri}}(t) = \frac{1}{\sqrt{2}t}}$

Linearization via $\mathbb{C} = \frac{\dot{\lambda}}{\lambda}$

$\rightarrow$ $\boxed{\ddot{\lambda} + \boldsymbol{\omega_{\mathrm{Ne}}^2(t)}\lambda = 0}$ complex Newton , $\lambda = u + iz$

Ansatz: $z = t^{\frac{1}{2}}$, $\dot{z} = \frac{1}{2} t^{-\frac{1}{2}}$, $\frac{\dot{z}}{z} = \frac{1}{2t}$

$\ddot{z} = -\frac{1}{4} t^{-\frac{3}{2}} = -\frac{1}{4t^2} t^{\frac{1}{2}} = -\frac{1}{4t^2} z$ $\rightarrow$ $\boxed{\omega_{\mathrm{Ne}}(t) = \frac{1}{2t} \neq \omega_{\mathrm{Ri}}(t)}$

Further: 2^{nd} linear independent solution of λ-Eq. missing**!**

2nd Approach to Acomplexify@ Riccati

Starting point: **Real Newton** $\boxed{\ddot{z} + \omega^2(t) z = 0}$

with $\boxed{\omega(t) = \omega_{\mathrm{Ne}}(t) = \frac{1}{2t}}$ $\rightarrow$ $\boxed{z(t) = t^{\frac{1}{2}}}$ corresponding to a

real Riccati Eq. $\boxed{\dot{\mathrm{R}} + \mathrm{R}^2 + \omega^2(t) = 0}$ with $\boxed{\mathrm{R} = \frac{\dot{z}}{z} = \frac{1}{2t}}$

2nd linear independent solution $u(t)$ of Newtonian eq. via

$$u = -z \int^t \frac{1}{z^2} dt' \quad \rightarrow \quad \boxed{u(t) = -t^{\frac{1}{2}} \ln t}$$

$$\rightarrow \quad \boxed{\alpha^2 = u^2 + z^2 = t[1 + \ln^2 t]}$$

$$\rightarrow \quad \boxed{\mathbb{C}_{\mathrm{I}} = \frac{1}{\alpha^2} = \frac{1}{t[1+\ln^2 t]}} \qquad \boxed{\mathbb{C}_{\mathrm{R}} = \frac{\dot{\alpha}}{\alpha} = \frac{1}{2} \frac{(1+\ln t)^2}{t(1+\ln^2 t)} = \frac{1}{2t} + \frac{\ln t}{t(1+\ln^2 t)}}$$

Problem: Divergence for $t \rightarrow 0$ (also for $\omega(t)$) **!**

Modified TD Frequency: $\boxed{\omega(t) = \frac{1}{2(t+b)}}$ $\quad b = \text{const} \neq 0$, dim. : time

$\rightarrow$ Modified complex Newton: $\boxed{\ddot{\lambda} + \left(\frac{1}{2(t+b)}\right)^2 \lambda = 0}$ $\quad \lambda = u + iz$

$\boxed{z = (t+b)^{\frac{1}{2}}}$ $\quad \rightarrow \quad$ $\boxed{u = -(t+b)^{\frac{1}{2}} \ln(t+b)}$

$$\boxed{\alpha^2(t) = (t+b)[1 + \ln^2(t+b)]}$$

with $\quad \alpha^2(0) = b[1 + \ln^2 b]$

and $\quad \boxed{\mathbb{C}_{\mathrm{I}}(t) = \frac{1}{(t+b)[1+\ln^2(t+b)]}}$, $\boxed{\mathbb{C}_{\mathrm{R}}(t) = \frac{1}{2(t+b)} + \frac{\ln(t+b)}{(t+b)[1+\ln^2(t+b)]}}$

with $\quad \mathbb{C}_{\mathrm{I}}(0) = \frac{1}{b[1+\ln^2 b]}$, $\mathbb{C}_{\mathrm{R}}(0) = \frac{1}{2b} + \frac{\ln b}{b[1+\ln^2 b]}$

Proof via Direct Solution of Complex Riccati

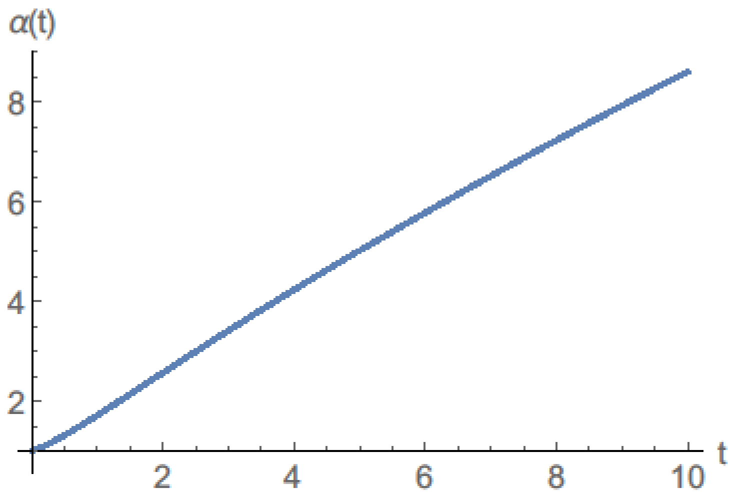
Assume: $\boxed{\tilde{\mathbb{C}} = \frac{1}{2(t+b)}}$ is **particular solution** (real)

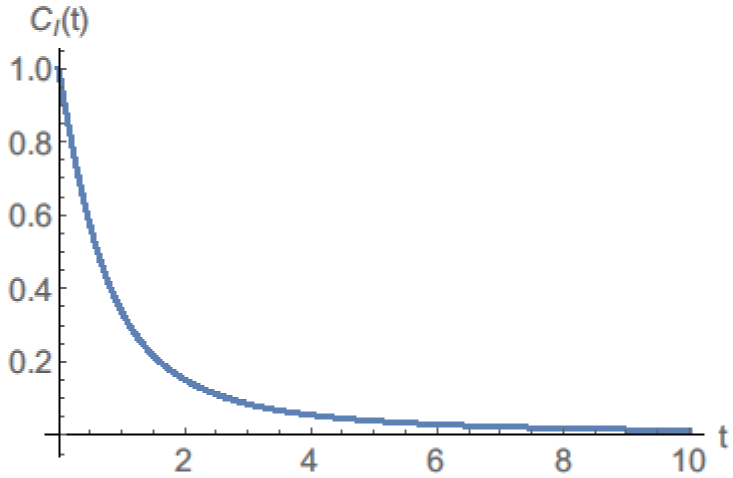
$\Rightarrow$ General solution: $\boxed{\mathbb{C} = \tilde{\mathbb{C}} + \mathbb{V}(t)}$ with $\boxed{\dot{\mathbb{V}} + \frac{1}{(t+b)}\mathbb{V} + \mathbb{V}^2 = 0}$

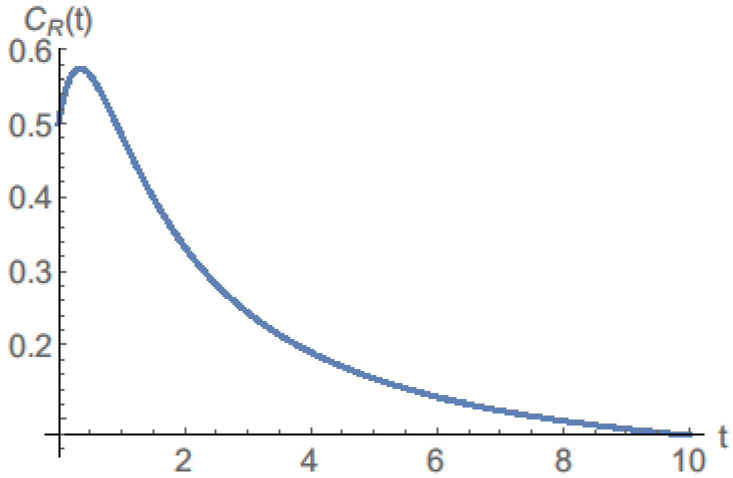
with $\boxed{\mathbb{V}_\mathrm{I} = \mathbb{C}_\mathrm{I} = \frac{1}{(t+b)[1+\ln^2(t+b)]}}$ follows from $\mathcal{Im}\,\{\mathrm{Ber.}\}$ $\left[\mathbb{V}_\mathrm{R} = -\frac{1}{2}\frac{\dot{\mathbb{V}}_\mathrm{I}}{\mathbb{V}_\mathrm{I}} - \frac{1}{2(t+b)}\right]$

$\Rightarrow$ $\boxed{\mathbb{V}_\mathrm{R} = \frac{\ln(t+b)}{(t+b)[1+\ln^2(t+b)]}}$

fulfilling $\boxed{\dot{\mathbb{V}}_\mathrm{R} + \frac{1}{(t+b)}\mathbb{V}_\mathrm{R} + \mathbb{V}_\mathrm{R}^2 - \mathbb{V}_\mathrm{I}^2 = 0}$ $\mathcal{Re}\,\{\mathrm{Ber.}\}$


α(t)
8
6
4
2
t
2
4
6
8
10


$C_I(t)$
1.0
0.8
0.6
0.4
0.2
t
2
4
6
8
10

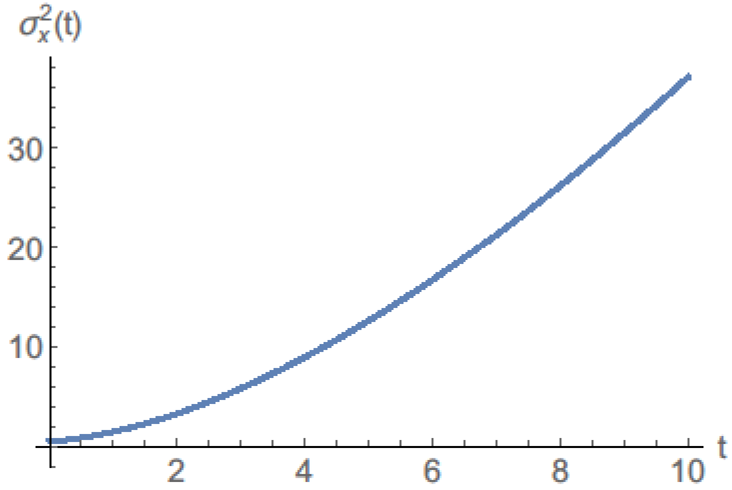

$C_R(t)$
0.6
0.5
0.4
0.3
0.2
t
2
4
6
8
10

Corresponding QM Contributions

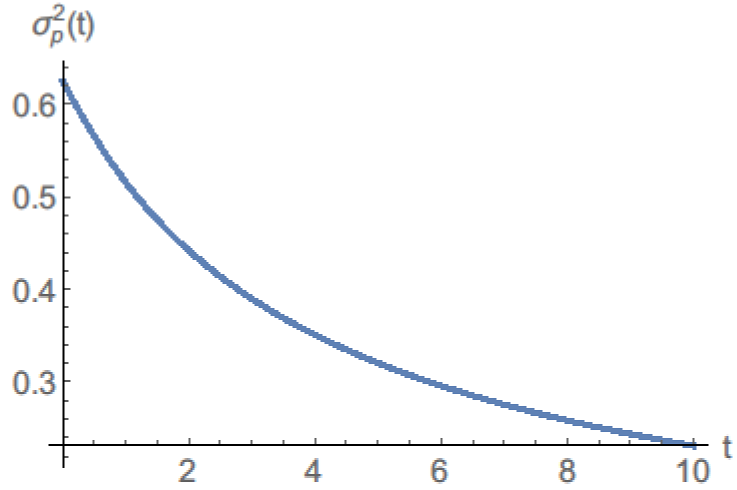
1. $$\langle \tilde{x}^2 \rangle(t) = \frac{\hbar}{2m} \boldsymbol{\alpha^2} = \frac{\hbar}{2m}(t+b)[1+\ln^2(t+b)]$$
   $$\langle \tilde{x}^2 \rangle_0 = \frac{\hbar}{2m} b[1+\ln^2 b]$$
   $$\langle \tilde{x}^2 \rangle_\infty \to \infty$$

2. $$\langle [\tilde{x},\tilde{p}]_+ \rangle(t) = \hbar \boldsymbol{\dot{\alpha}\alpha} = \frac{\hbar}{2}[1+\ln(t+b)]^2$$
   $$\langle [\tilde{x},\tilde{p}]_+ \rangle_0 = \frac{\hbar}{2}[1+\ln b]^2$$
   $$\langle [\tilde{x},\tilde{p}]_+ \rangle_\infty \to \infty$$

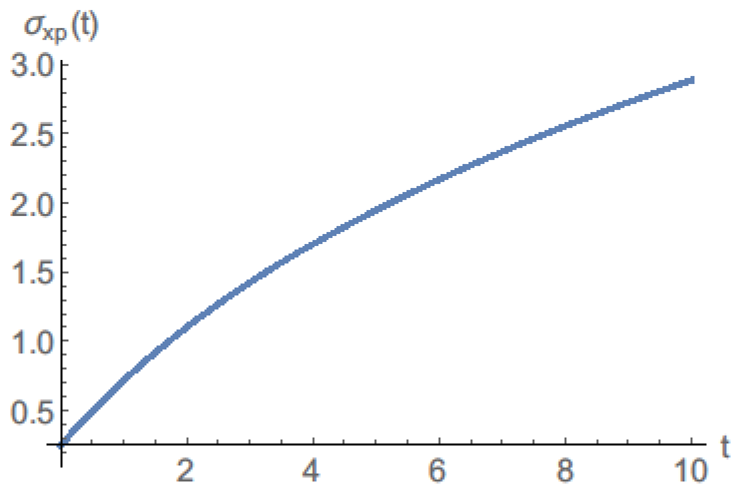
3. $$\langle \tilde{p}^2 \rangle(t) = \frac{\hbar m}{2}\left(\boldsymbol{\dot{\alpha}^2} + \boldsymbol{\frac{1}{\alpha^2}}\right) = \frac{\hbar m}{2}\frac{1}{(t+b)}\left\{\frac{1}{4}[1+\ln(t+b)]^2 + \frac{1}{2}\ln(t+b) + 1\right\}$$
   $$\langle \tilde{p}^2 \rangle_0 = \frac{\hbar m}{2}\frac{1}{b}\left\{\frac{1}{4}[1+\ln b]^2 + \frac{1}{2}\ln b + 1\right\}$$
   $$\langle \tilde{p}^2 \rangle_\infty \to 0$$



$\sigma_x^2(t)$
30
20
10
2
4
6
8
10
t



$\sigma_p^2(t)$
0.6
0.5
0.4
0.3
2
4
6
8
10
t



$\sigma_{xp}(t)$
3.0
2.5
2.0
1.5
1.0
0.5
2
4
6
8
10
t

4. $$\tilde{E} = \frac{\hbar}{4}\left\{\dot{\alpha}^2 + \frac{1}{\alpha^2} + \omega^2(t)\,\alpha^2\right\} = \frac{\hbar}{4}\left\{\frac{1}{(t+b)}\left[\frac{1}{2}(1+\ln(t+b))^2+1\right]\right\}$$

$$\tilde{E}_0 = \frac{\hbar}{4}\left\{\frac{1}{b}\left[\frac{1}{2}(1+\ln b)^2+1\right]\right\}$$

$$\tilde{E}_\infty \to 0$$

5. $$U = \langle\tilde{x}^2\rangle\langle\tilde{p}^2\rangle = \frac{\hbar^2}{4}\left\{1+(\alpha\dot{\alpha})^2\right\} = \frac{\hbar^2}{4}\left\{1+\frac{1}{4}[1+\ln(t+b)]^4\right\}$$

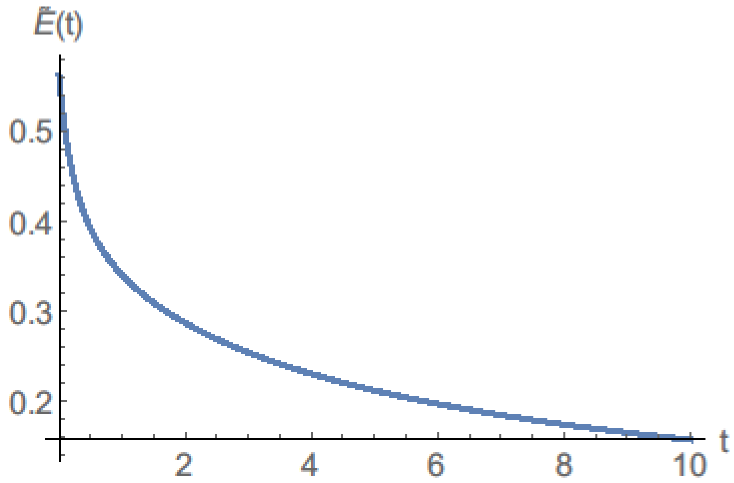
$$U_0 = \frac{\hbar^2}{4}\left\{1+\frac{1}{4}[1+\ln b]^4\right\}$$

$$U_\infty \to \infty \quad , \quad U_{\min,\,0} = U\left(t=0, b=\frac{1}{e}\right) = \frac{\hbar^2}{4}$$

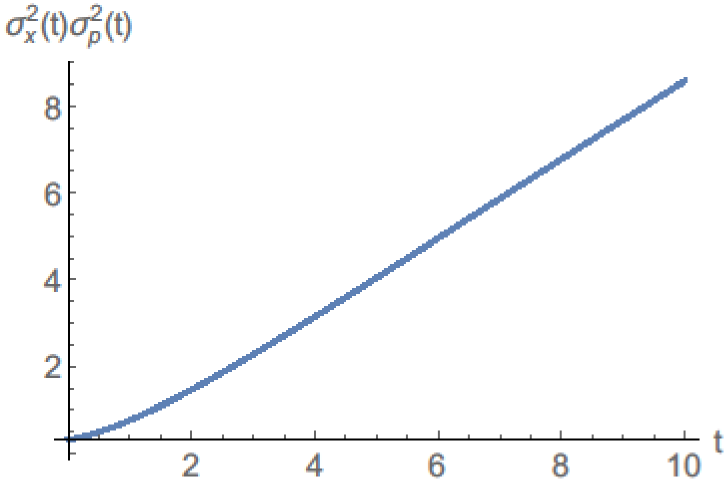
6. $$\tilde{\upsilon}_- = \upsilon_- - \langle\upsilon\rangle = \frac{\dot{\alpha}}{\alpha}\tilde{x} = \left\{\frac{1}{2(t+b)} + \frac{\ln(t+b)}{(t+b)[1+\ln^2(t+b)]}\right\}\tilde{x}$$

$$\tilde{\upsilon}_-(0) = \left\{\frac{1}{2b} + \frac{\ln b}{b[1+\ln^2 b]}\right\}(x-\eta_0)$$

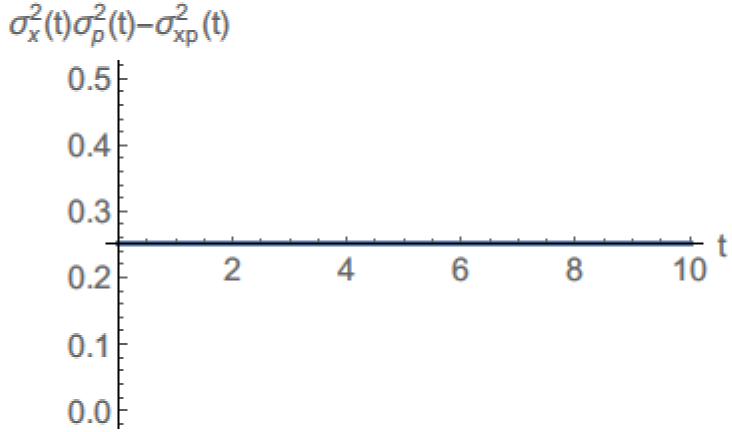
$$\tilde{\upsilon}_-(\infty) \to 0$$



$\tilde{E}(t)$
0.5
0.4
0.3
0.2
2
4
6
8
10
t



$\sigma_x^2(t)\sigma_p^2(t)$
8
6
4
2
2
4
6
8
10
t



$\sigma_x^2(t)\sigma_p^2(t)-\sigma_{xp}^2(t)$
0.5
0.4
0.3
0.2
0.1
0.0
2
4
6
8
10
t

Corresponding Classical Contributions

$z \propto \eta(t) = \langle x \rangle(t)$ with $\boxed{\ddot{\eta} + \left(\frac{1}{2(t+b)}\right)^2 \eta = 0}$

$\eta(t) = v_0 \tau^{\frac{1}{2}} (t+b)^{\frac{1}{2}}\,,$ $\quad \eta_0 = v_0 \tau^{\frac{1}{2}}\, b^{\frac{1}{2}}\,,$ $\quad \tau$: dim[time]

$\dot{\eta}(t) = \frac{1}{2} v_0 \tau^{\frac{1}{2}} (t+b)^{-\frac{1}{2}} = v_0 \frac{\tau^{\frac{1}{2}}}{2(t+b)^{\frac{1}{2}}}\,,$ $\quad \dot{\eta}_0 = v_0 \frac{\tau^{\frac{1}{2}}}{2b^{\frac{1}{2}}} \rightarrow \boxed{b = \frac{\tau}{4}}$

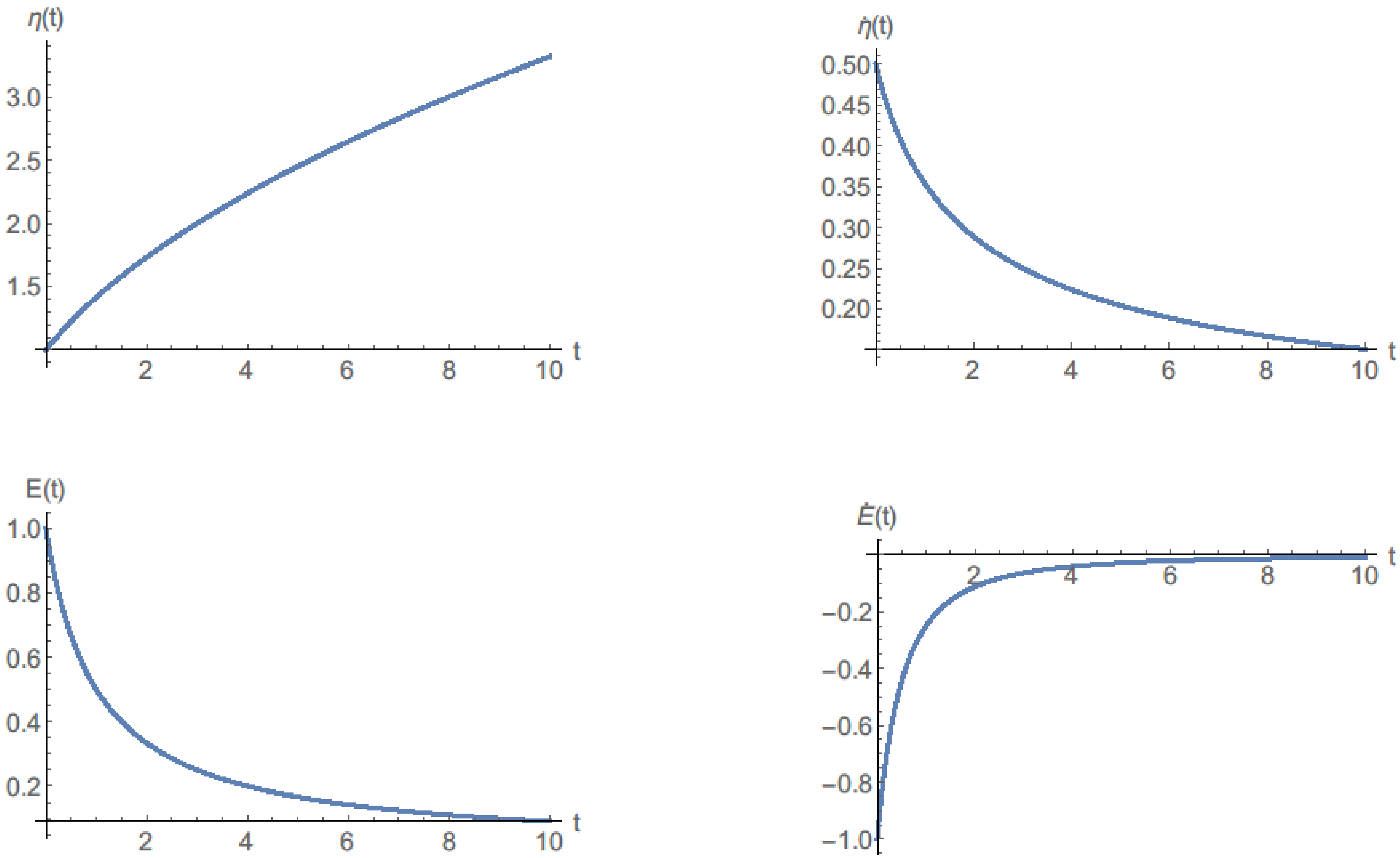
$$\boxed{E = T + V = \frac{m}{2}\dot{\eta}^2 + \frac{m}{2}\left(\frac{1}{2(t+b)}\right)^2 \eta^2 = \frac{m}{2} v_0^2 \frac{\tau}{2(t+b)} \propto \frac{1}{t}}$$

$E_0 = \frac{m}{2} v_0^2 \frac{\tau}{2b} = m v_0^2$

$\boldsymbol{E_\infty = 0}$

$$\dot{E} = -\frac{m}{2} v_0^2 \frac{\tau}{2} \left(\frac{1}{t+b}\right)^2 = -2\omega(t) E = -\boldsymbol{4\omega(t)}T < 0$$

Compare: $\boxed{\ddot{\eta} + \gamma\dot{\eta} + \omega_0^2 \eta = 0}$ $\quad \dot{E} = -2\gamma T = -\boldsymbol{4\frac{\gamma}{2}} T \rightarrow \boxed{\boldsymbol{\omega(t)} \triangleq \frac{\gamma}{2}}$



η(t)
3.0
2.5
2.0
1.5
t
2
4
6
8
10
η̇(t)
0.50
0.45
0.40
0.35
0.30
0.25
0.20
t
2
4
6
8
10
E(t)
1.0
0.8
0.6
0.4
0.2
t
2
4
6
8
10
Ė(t)
t
2
4
6
8
10
−0.2
−0.4
−0.6
−0.8
−1.0

Dissipation and Irreversibility

$$\boxed{\ddot{\eta} + \left(\frac{1}{2(t+b)}\right)^2 \eta = \ddot{\eta} + \omega^2(t)\eta = 0}$$

$\boxed{\mathrm{b} = 0}$: $\boxed{\omega(t) = \omega(-t)}$ $\rightarrow$ Eq. of motion **invariant** under **time-reversal!**

BUT : $\boxed{\dot{E} < 0}$ $\rightarrow$ Adissipation@ of energy

$i.e.$: **Dissipation without irreversibility**

Compared to Diffusion process: **Irreversibility without dissipation**

$\boxed{\mathrm{b} \neq 0}$: $\boxed{\omega(t) \neq \omega(-t)}$ $\rightarrow$ Eq. of motion **no longer invariant** under **time-reversal!**

Addition of constant ***b*** **breaks symmetry!**

STILL : $\boxed{\dot{E} < 0}$ $\rightarrow$ Adissipation@

Comparison with Damped HO

Analytical solutions possible for parametric oscillator

with frequency $\boxed{\omega(t) = \frac{1}{a(t+b)}}$ for arbitrary constant b

However, qualitatively different behaviour depending on $\boldsymbol{a}$.

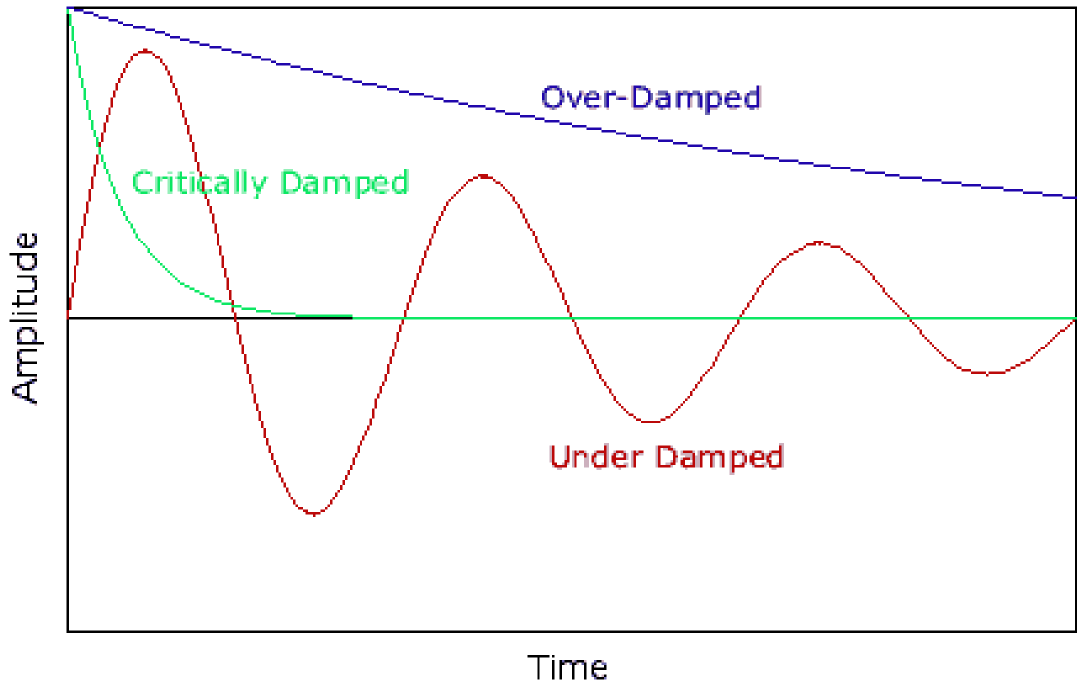
For $\boxed{a = 2}$ (see above) similar to **damped HO** with

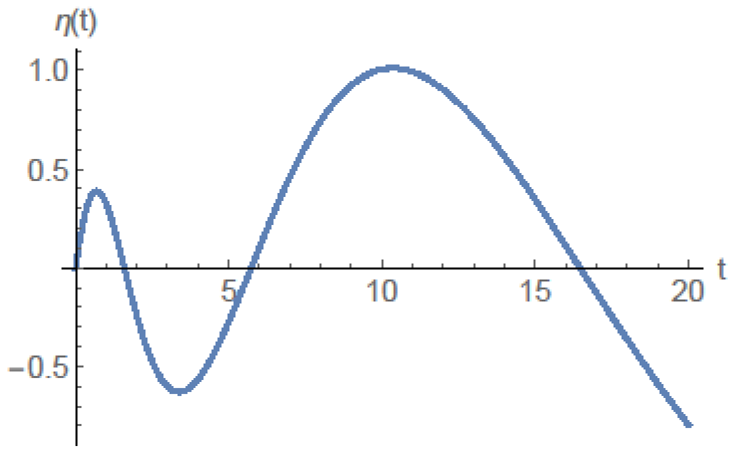
$\boxed{\omega_0 = \frac{\gamma}{2}}$, i.e. **aperiodic limit**

$\boxed{a > 2}$ similar to $\boxed{\omega_0 < \frac{\gamma}{2}}$, i.e. **overcritical damping**

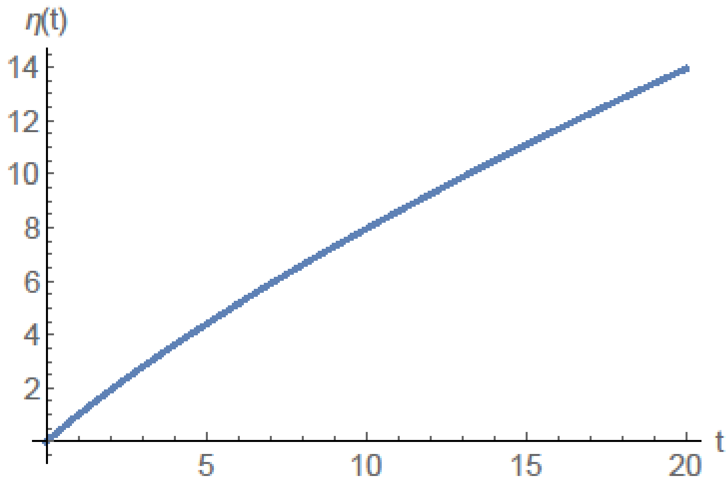
$\boxed{a < 2}$ similar to $\boxed{\omega_0 > \frac{\gamma}{2}}$, i.e. oscillatory behaviour

undercritical damping

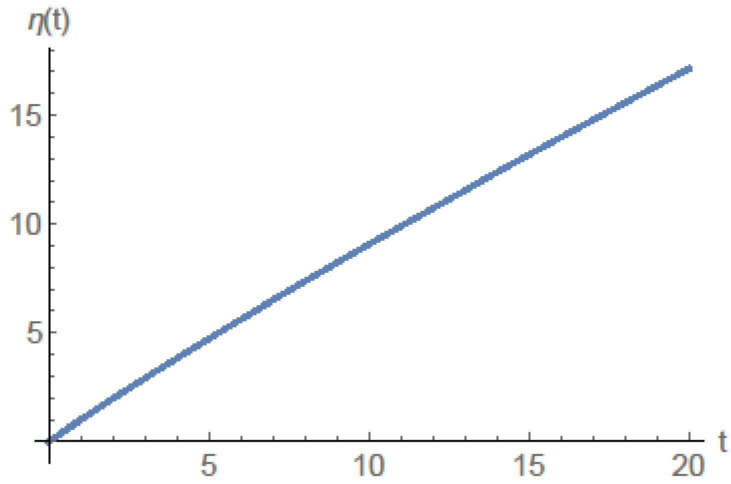

Over-Damped
Critically Damped
Under Damped
Amplitude
Time



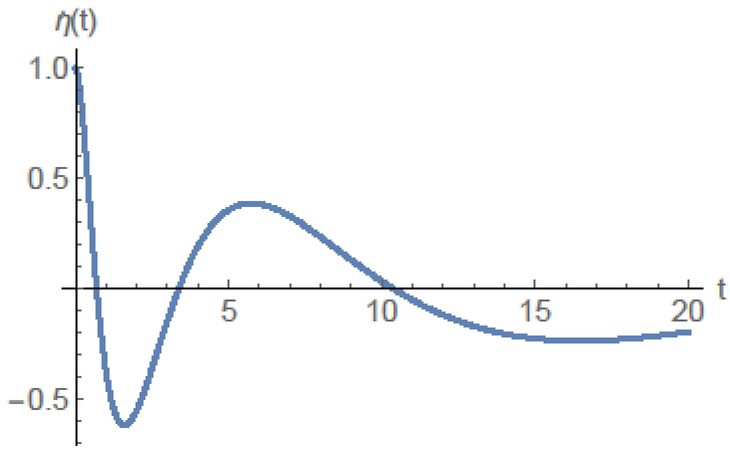
η(t)
1.0
0.5
−0.5
5
10
15
20
t



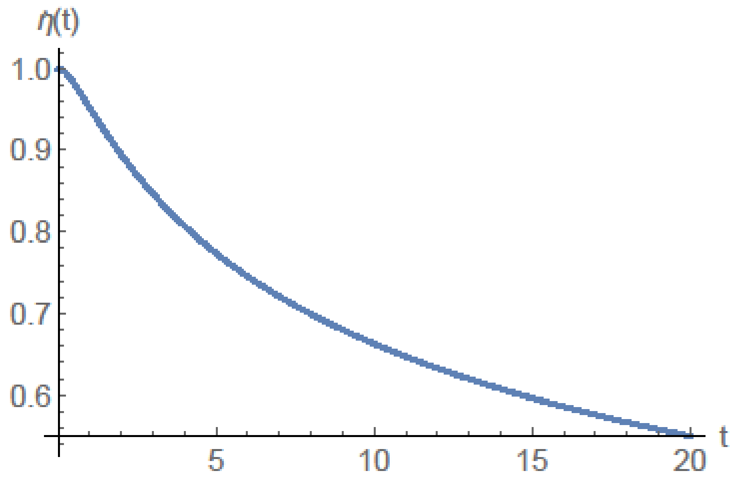
η(t)
14
12
10
8
6
4
2
5
10
15
20
t



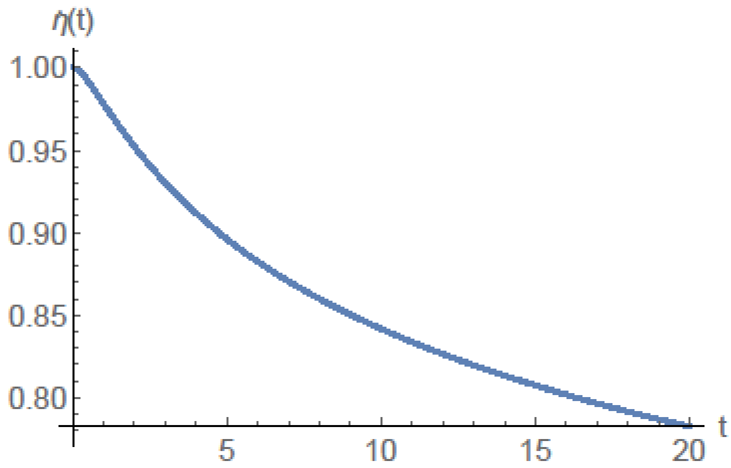
η(t)
15
10
5
5
10
15
20
t



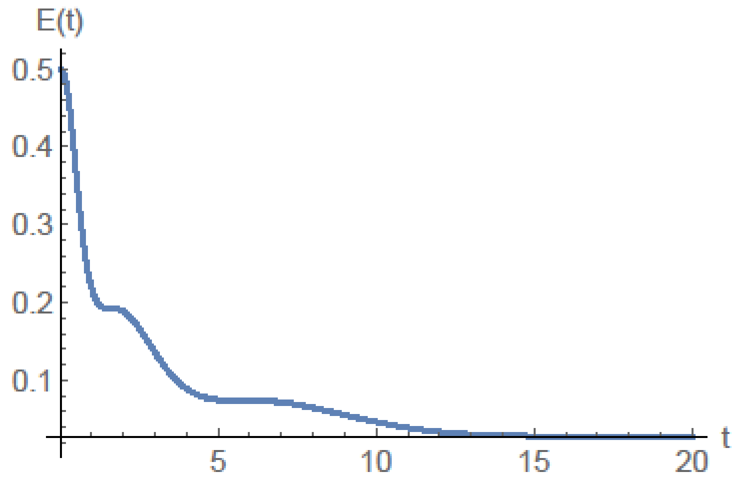
η(t)
1.0
0.5
−0.5
5
10
15
20
t

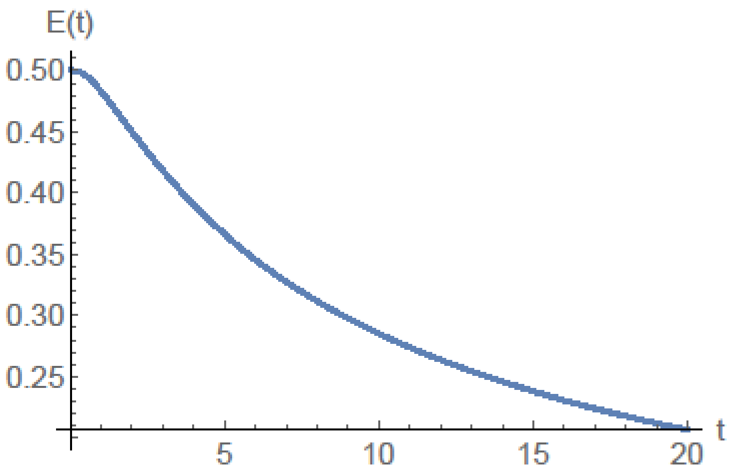


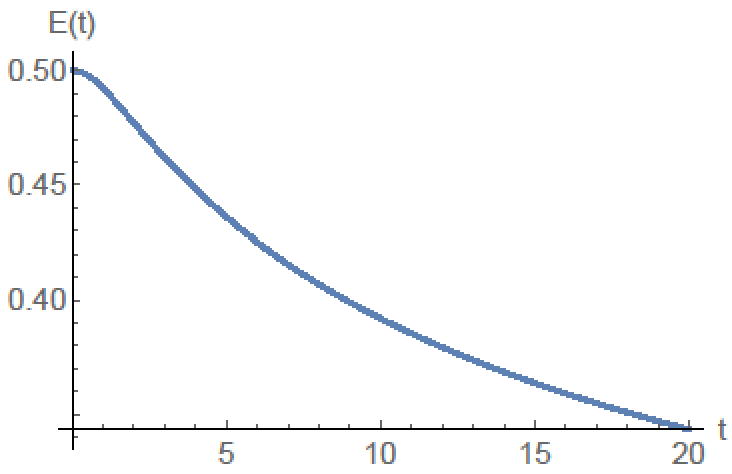
η(t)
1.0
0.9
0.8
0.7
0.6
5
10
15
20
t

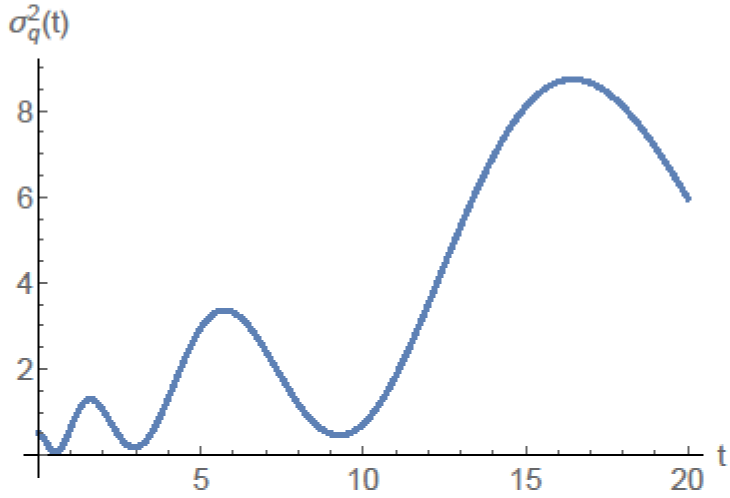


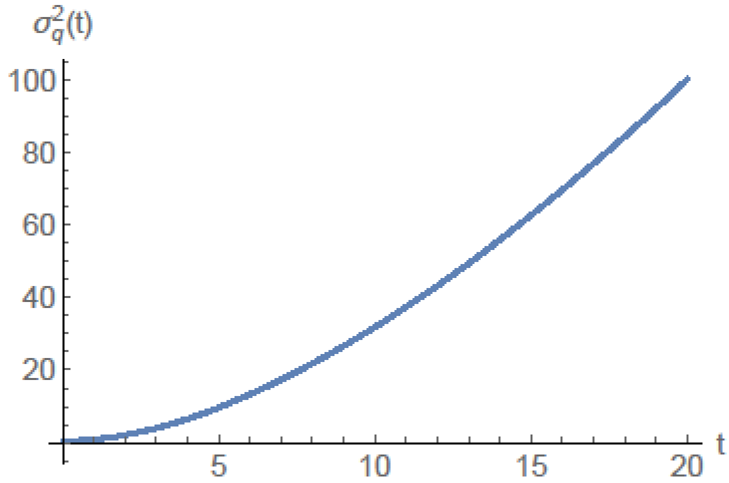
η(t)
1.00
0.95
0.90
0.85
0.80
5
10
15
20
t

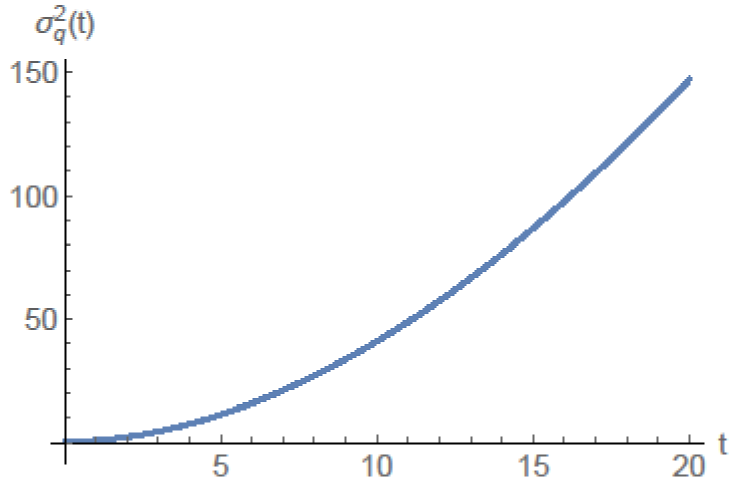

E(t)
0.5
0.4
0.3
0.2
0.1
5
10
15
20
t

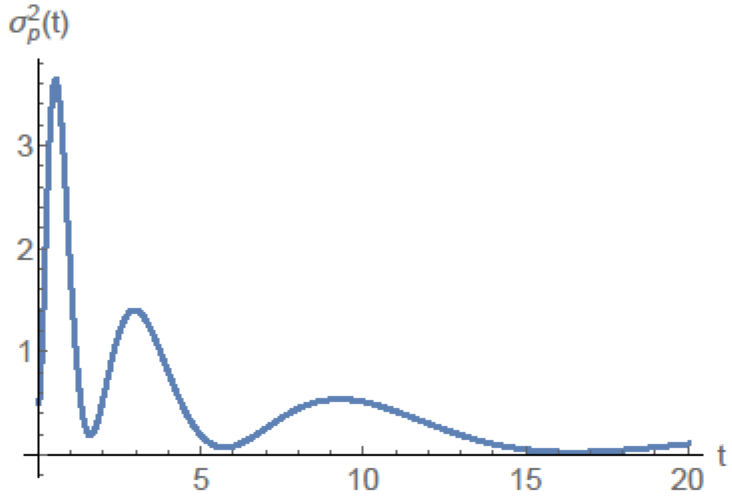

E(t)
0.50
0.45
0.40
0.35
0.30
0.25
5
10
15
20
t


E(t)
0.50
0.45
0.40
5
10
15
20
t

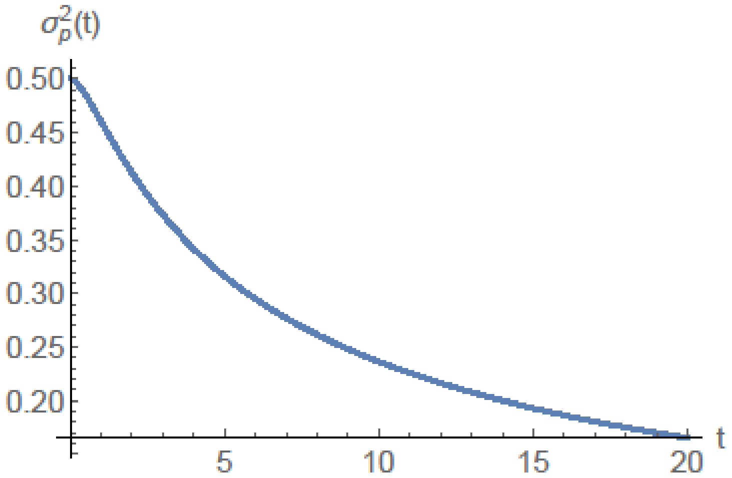

$\sigma_q^2(t)$
8
6
4
2
5
10
15
20
t


$\sigma_q^2(t)$
100
80
60
40
20
5
10
15
20
t

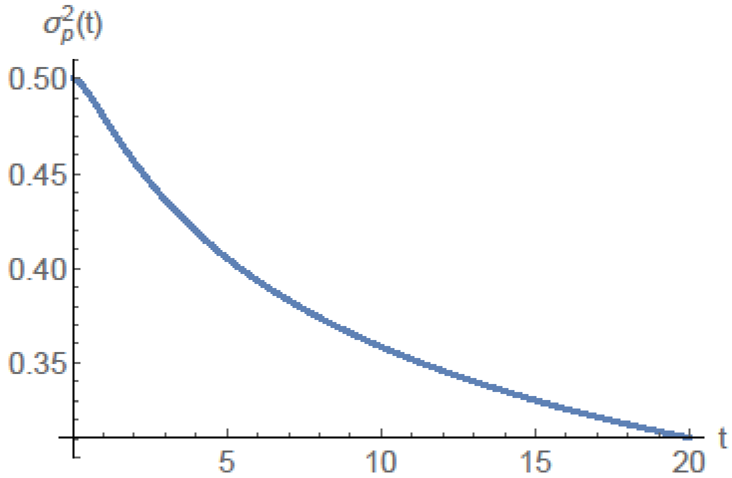

$\sigma_q^2(t)$
150
100
50
5
10
15
20
t



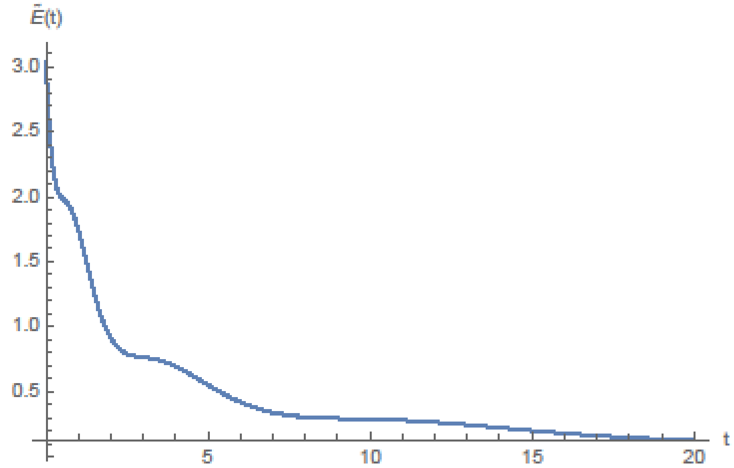
σp2(t)
3
2
1
t
5
10
15
20



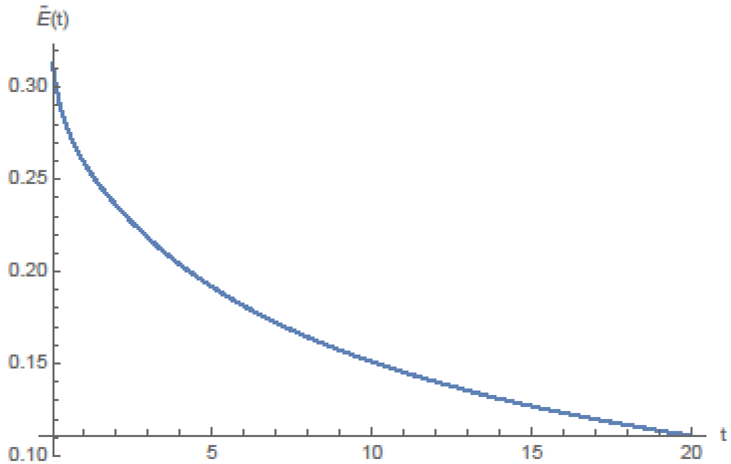
σp2(t)
0.50
0.45
0.40
0.35
0.30
0.25
0.20
t
5
10
15
20



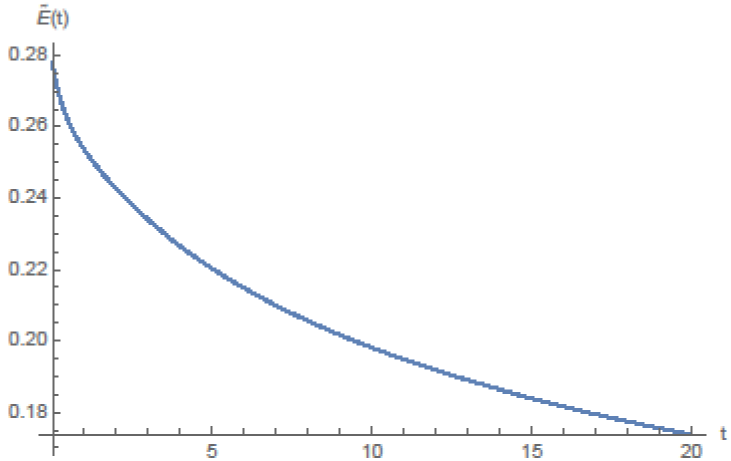
σp2(t)
0.50
0.45
0.40
0.35
t
5
10
15
20



$\bar{E}(t)$
3.0
2.5
2.0
1.5
1.0
0.5
5
10
15
20
t



$\bar{E}(t)$
0.30
0.25
0.20
0.15
0.10
5
10
15
20
t



$\bar{E}(t)$
0.28
0.26
0.24
0.22
0.20
0.18
5
10
15
20
t

Conclusions

- Knowledge of **two linear independent solutions** of **classical Newtonian** equation makes it possible to obtain time-dependence of WP width and, from there, all **dynamical properties** of a **quantum system**
- Knowledge of **time-dependence** of **WP width** makes it possible to obtain **time-dependence** of **classical dynamics**
- **TD width** of **diffusion equation** suggests **parametric oscillator** with TD frequency $\boldsymbol{\omega(t) = \frac{1}{2t}}$
- Divergence for $t \to 0$: modified frequency $\boldsymbol{\omega(t) = \frac{1}{a(t+b)}}$ with $a = 2$
  Analytical solutions for **classical** and **QM** equations of motion
- Energy dissipated as in case of damped HO:
  - $a < 2$ corresponds to **undercritical** damping $\omega_0 > \frac{\gamma}{2}$
  - $a = 2$ to **aperiodic limit** $\omega_0 = \frac{\gamma}{2}$
  - $a > 2$ to **overdamped** case $\omega_0 < \frac{\gamma}{2}$

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- O. Castaños, D. Schuch and O. Rosas-Ortiz, Generalized creation and annihilation operators via complex nonlinear Riccati equations, *J. Phys. A.: Math. Theor.* (2013); arXiv:1211.5109
- D. Schuch, Is quantum theory intrinsically nonlinear?, *Phys. Scr*. **87**, 038117 (2013) or Physica Scripta **Highlights of 2013**
- H. Cruz, D. Schuch, O. Castaños and O. Rosas-Ortiz, Time-evoluton of quantum systems via a complex nonlinear Riccati equation. I. Conservative systems with time-independent Hamiltonian, *Ann. Physics* **360**, 44-60 (2015)
- H. Cruz, D. Schuch, O. Castaños and O. Rosas-Ortiz, Time-evoluton of quantum systems via a complex nonlinear Riccati equation. II. Dissipative systems, *Ann. Physics* **373**, 609-630 (2016)

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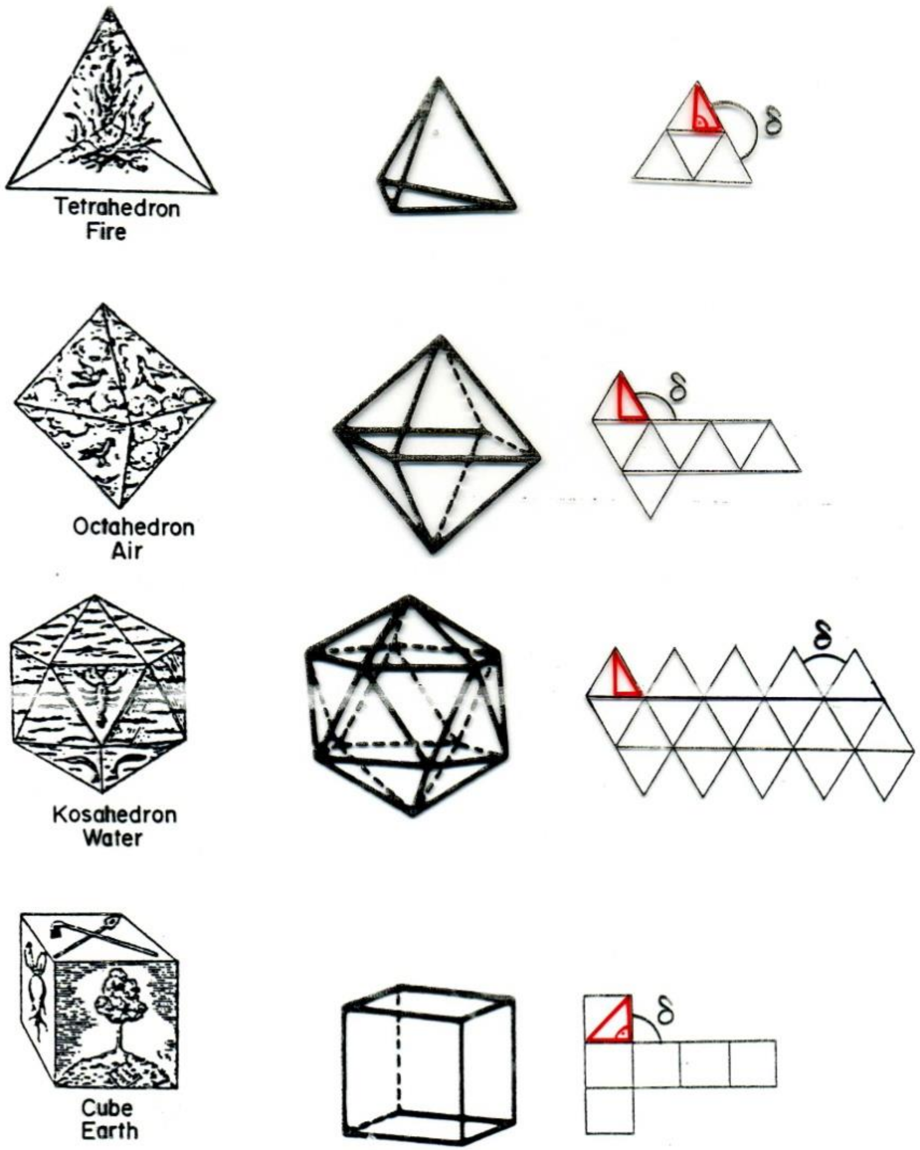
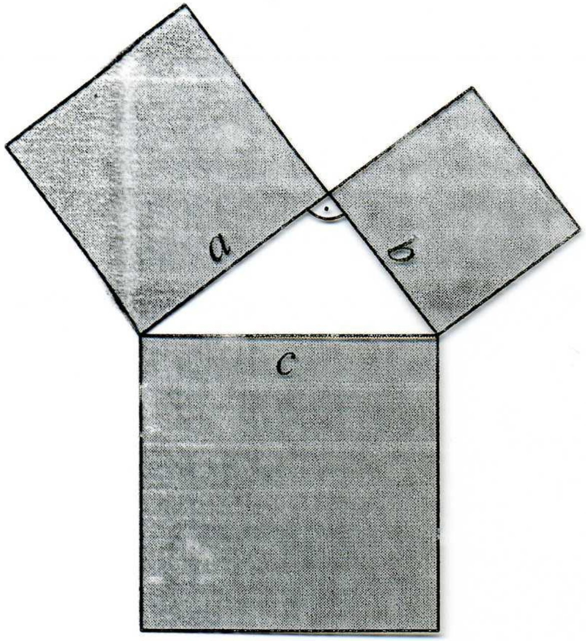


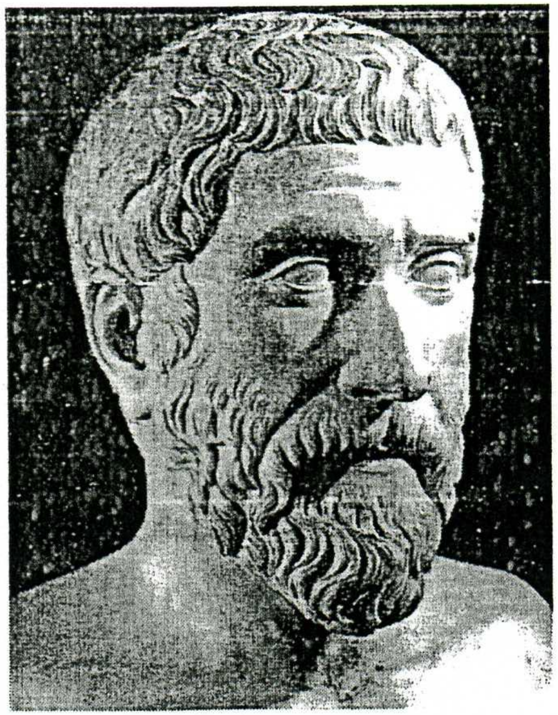
Jay Kappraff: „Connections“, 2nd Ed.,
World Scientific, Singapore, 2001, p. 259/260

Pythagorean Triples



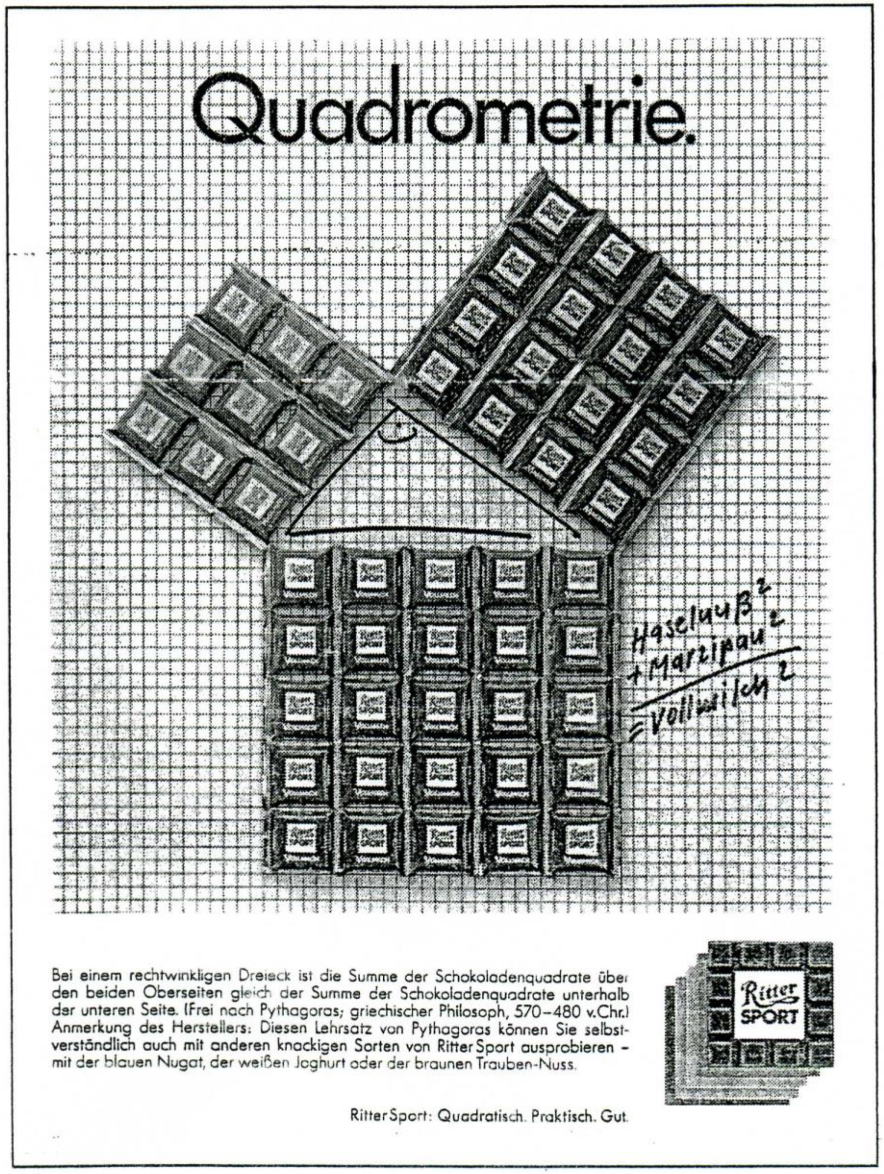
Pythagoras – Konfiguration

$$a^2 + b^2 = c^2$$



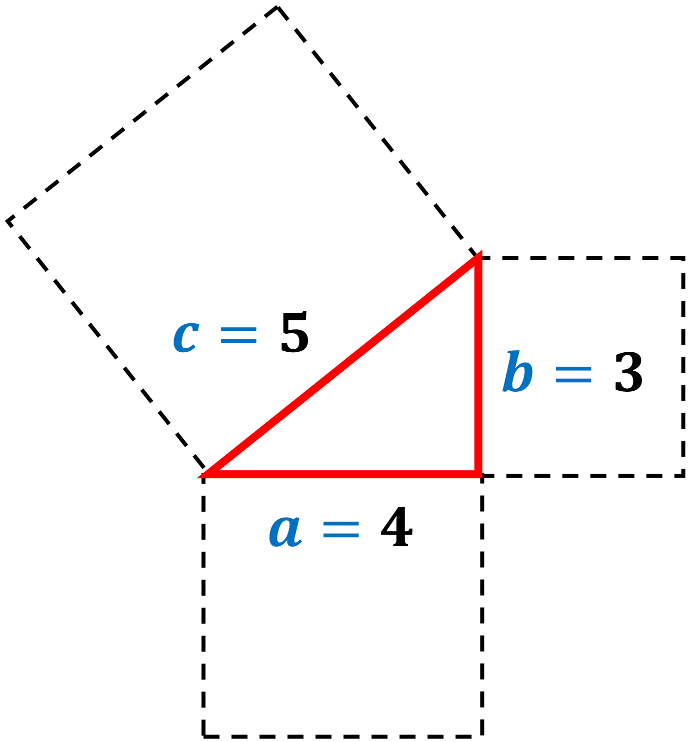
Pythagoras

Peter Baptist: „Pythagoras und kein Ende?“
Ernst Klett Schulbuchverlag, Leipzig, 1997, p. 21



Peter Baptist: „Pythagoras und kein Ende?“
Ernst Klett Schulbuchverlag, Leipzig, 1997, p. 26

$$a^2 + b^2 = c^2$$



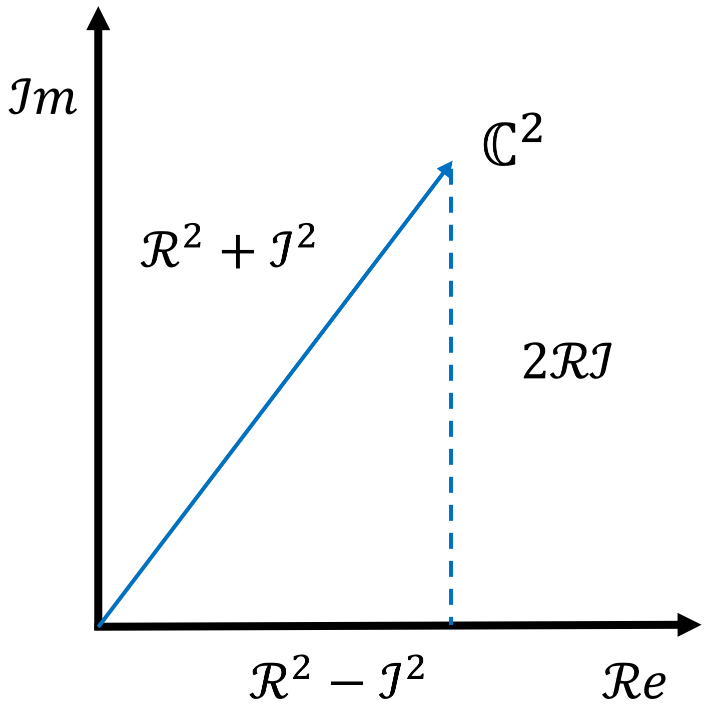
1. 3,4,5
2. 5,12,13
3. 7,24,25
4. 8,15,17
5. 9,40,41
6. 11,60,61
7. 12,35,37
8. 13,84,85
9. 16,63,65
10. 20,21,29
11. 20,99,101
12. 28,45,53
13. 33,56,65
14. 36,77,85
15. 39,80,89
16. 48,55,73
17. 60,91,109
20. 65,72,97

Pythagorean Triples

$$\boxed{\mathbb{C} = \mathcal{R} + i\mathcal{J}} = \frac{\dot{\alpha}}{\alpha} + i\dot{\varphi} = \frac{2\hbar}{m}y$$

$$\boxed{-\frac{d}{dt}\mathbb{C} = \mathbb{C}^2}$$

$\mathbb{C}$: “quantized”, i.e. $\mathcal{R}$, $\mathcal{J}$ integer, $\mathcal{R} > \mathcal{J}$



a) $\mathcal{R} = 2 : \mathcal{R}e\{\mathbb{C}^2\} = \mathbf{3}$

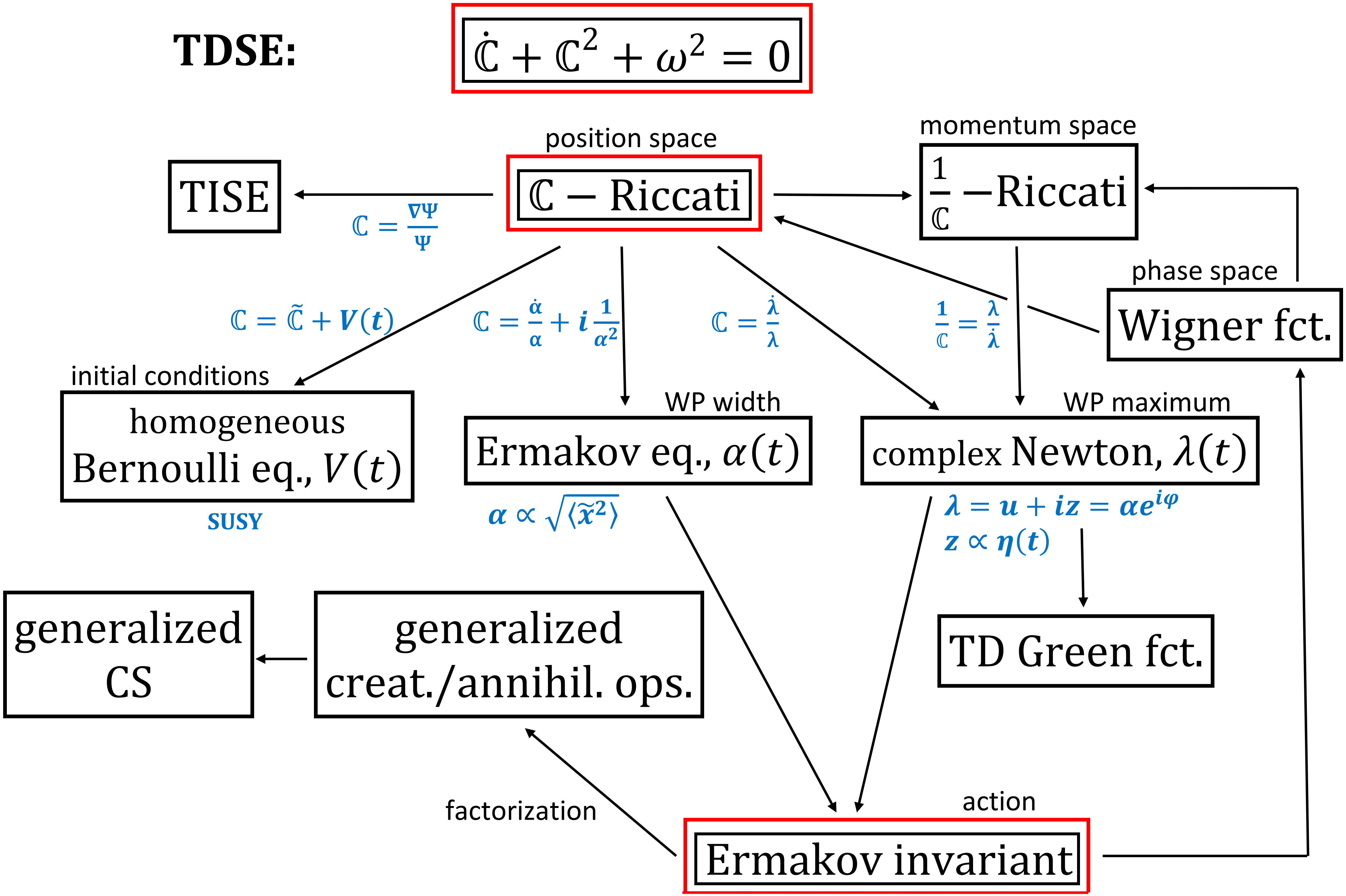
$\mathcal{J} = 1 : \mathcal{J}m\{\mathbb{C}^2\} = \mathbf{4}$

$|\mathbb{C}^2| = \mathbf{5}$

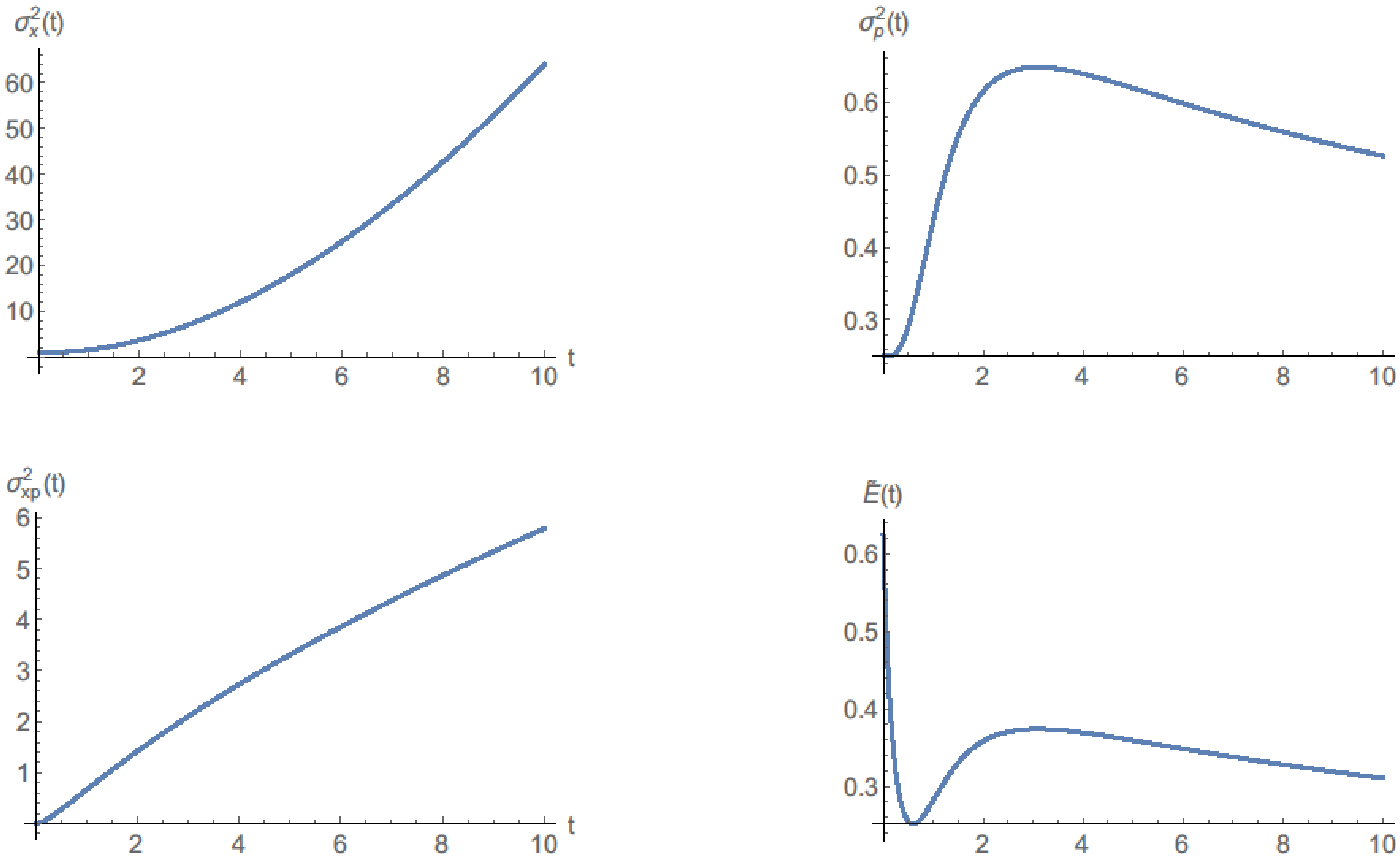
b) $\mathcal{R} = 3 : \mathcal{R}e\{\mathbb{C}^2\} = \mathbf{5}$

$\mathcal{J} = 2 : \mathcal{J}m\{\mathbb{C}^2\} = \mathbf{12}$

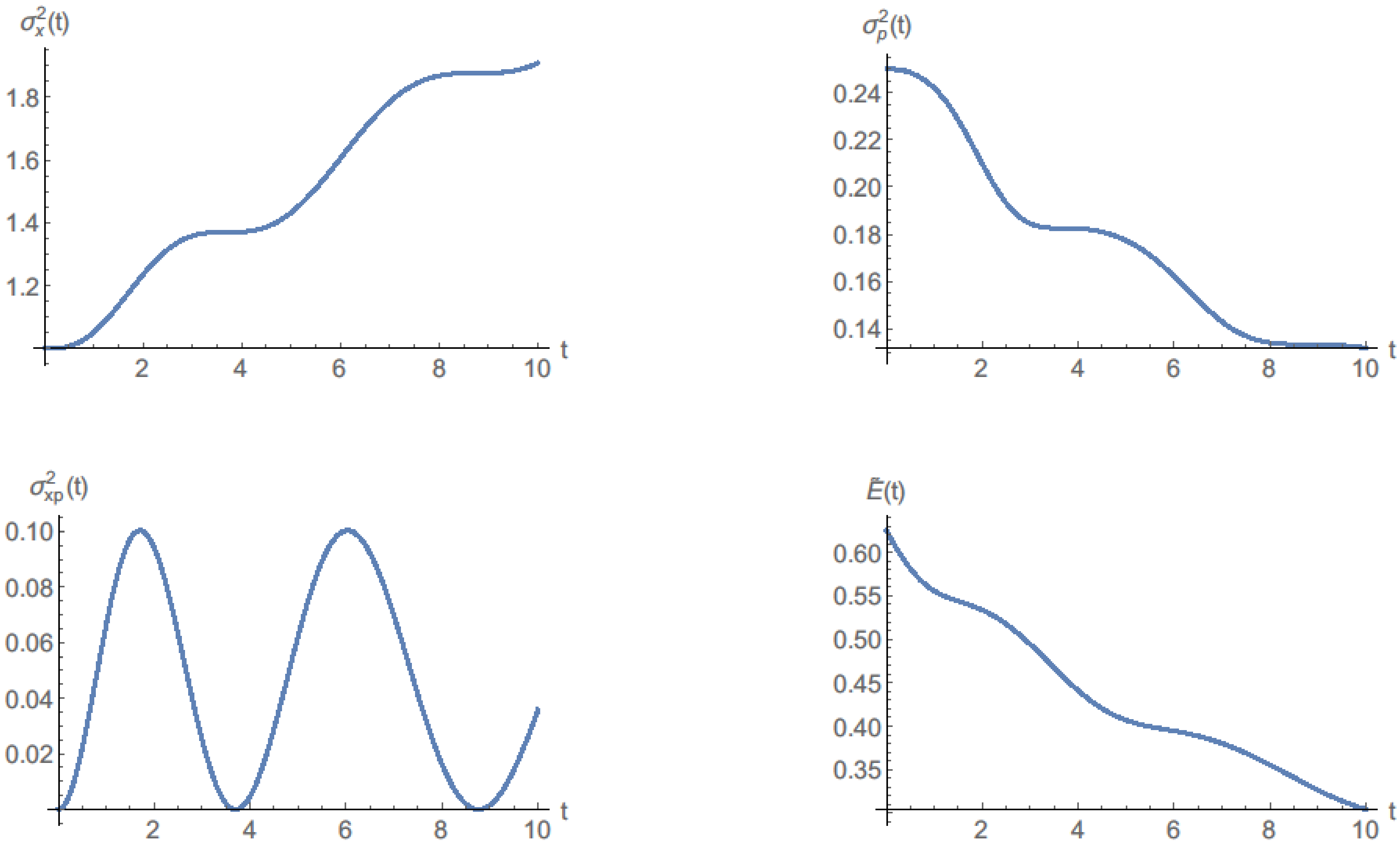
$|\mathbb{C}^2| = \mathbf{13}$



TDSE:
$\dot{\mathbb{C}} + \mathbb{C}^2 + \omega^2 = 0$
position space
$\mathbb{C}$ – Riccati
momentum space
$\frac{1}{\mathbb{C}}$ –Riccati
TISE
$\mathbb{C} = \frac{\nabla\Psi}{\Psi}$
phase space
Wigner fct.
$\mathbb{C} = \tilde{\mathbb{C}} + V(t)$
$\mathbb{C} = \frac{\dot{\alpha}}{\alpha} + i\frac{1}{\alpha^2}$
$\mathbb{C} = \frac{\dot{\lambda}}{\lambda}$
$\frac{1}{\mathbb{C}} = \frac{\lambda}{\dot{\lambda}}$
initial conditions
homogeneous Bernoulli eq., $V(t)$
SUSY
WP width
Ermakov eq., $\alpha(t)$
$\alpha \propto \sqrt{\langle \tilde{x}^2 \rangle}$
WP maximum
complex Newton, $\lambda(t)$
$\lambda = u + iz = \alpha e^{i\varphi}$
$z \propto \eta(t)$
TD Green fct.
generalized CS
generalized creat./annihil. ops.
factorization
action
Ermakov invariant



$\sigma_x^2(t)$
60
50
40
30
20
10
2
4
6
8
10
t
$\sigma_p^2(t)$
0.6
0.5
0.4
0.3
2
4
6
8
10
t
$\sigma_{xp}^2(t)$
6
5
4
3
2
1
2
4
6
8
10
t
$\tilde{E}(t)$
0.6
0.5
0.4
0.3
2
4
6
8
10
t



$\sigma_x^2(t)$
1.8
1.6
1.4
1.2
t
2
4
6
8
10
$\sigma_p^2(t)$
0.24
0.22
0.20
0.18
0.16
0.14
t
2
4
6
8
10
$\sigma_{xp}^2(t)$
0.10
0.08
0.06
0.04
0.02
t
2
4
6
8
10
$\tilde{E}(t)$
0.60
0.55
0.50
0.45
0.40
0.35
t
2
4
6
8
10