Positive Hamiltonians can give purely exponential decay

Paolo Facchi
Università di Bari & INFN Italy

Joint work with Daniel Burgarth, Aberystwyth University

Policeta - San Rufo
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Exponential decay law

\[ N(t) = N_0 e^{-\gamma t} \]
Exponential decay

\[ N(t) = N_0 e^{-\gamma t} \]

\( ^{56}\text{Mn} \) radioactive decay

Testing the exponential decay law of gold $^{198}$Au


*Institute of Nuclear Sciences "Vinča", 11001 Belgrade, PB 522, Serbia and Montenegro

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Abstract

A long time scale experimental testing of the exponential decay law of gold, $^{198}$Au, is presented. The experiment has been processed by using a contemporary digital spectrum analyzer. Within the limits of the experimental error, no deviation from the exponential decay law of $^{198}$Au has been observed.

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Keywords: Non-exponential decay; Zeno effect
Quantum theory

• Gamow
• Weisskopf - Wigner
• Friedrichs - Lee
• Fermi
• Fock - Krylov
Quantum systems

Survival amplitude \[ a(t) = \langle \psi | e^{-itH} \psi \rangle \]

Survival probability \[ p(t) = |a(t)|^2 \]

\[ a(t) = \langle \psi | e^{-itH} \psi \rangle = \int e^{-iEt} d\mu_\psi(E) \quad \text{Fourier transform} \]

\[ \mu_\psi(\Omega) = \text{Prob} \left( E \in \Omega \right) \]

\[ \mu_\psi = \mu_{pp} + \mu_{ac} + \mu_{sc} \quad \text{spectral measure} \]

Lebesgue decomposition theorem
Pure point spectrum

Survival amplitude \[ a(t) = \langle \psi | e^{-itH} \psi \rangle \]
\[ a(t) = \langle \psi | e^{-itH} \psi \rangle = \int e^{-iEt} d\mu_\psi(E) \quad \text{Fourier transform} \]

\[ \mu_\psi = \mu_{pp} = \sum_j |c_j|^2 \delta(E - E_j) \quad c_j = \langle E_j | \psi \rangle \quad \text{pure point} \]

almost periodic motion \[ \overline{p(t)} \to \sum_j |c_j|^4 = \text{const} > 0, \quad \text{as} \quad t \to \infty \]
Continuous spectrum

Survival amplitude \[ a(t) = \langle \psi | e^{-iHt} \psi \rangle \]

\[ a(t) = \langle \psi | e^{-iHt} \psi \rangle = \int e^{-iEt} d\mu_{\psi}(E) \quad \text{Fourier transform} \]

\[ \mu_{\psi} = \mu_{ac} + \mu_{sc} \quad \text{continuous spectrum} \]

\[ \overline{p(t)} \to 0, \quad \text{as} \quad t \to \infty \quad \text{RAGE theorem} \]
Quantum unstable systems

Survival amplitude \( a(t) = \langle \psi | e^{-itH} \psi \rangle \)

Survival probability \( p(t) = |a(t)|^2 \)

\[
\langle \psi | e^{-itH} \psi \rangle = \int e^{-iEt} \, d\mu_\psi(E) \quad \mu_\psi = \mu_{ac}
\]

absolutely continuous

\[
p_\psi(E) = \frac{d\mu_\psi(E)}{dE} \quad p_\psi(E) = |\langle E | \psi \rangle|^2 \geq 0
\]

\[
\lim_{t \to \infty} a(t) = \lim_{t \to \infty} \int e^{-iEt} p_\psi(E) \, dE = 0 \quad \text{unstable system}
\]

Riemann-Lebesgue lemma
Short times

Survival amplitude \( a(t) = \langle \psi | e^{-itH} \psi \rangle \)

\[
p(t) = \left| a(t) \right|^2 = \left| \langle \psi | \exp(-itH)\psi \rangle \right|^2
\]

\( \psi \in D(H) \quad \|H\psi\| < \infty \quad \text{finite variance} \)

\[
p(t) = 1 - t^2 \left( \langle H\psi | H\psi \rangle - \langle \psi | H\psi \rangle^2 \right) + o \left( t^2 \right)
\]

as \( t \to 0 \)
Short times

\[ p(t) \]

- **exponential**
- **non-exponential**
- "Zeno region"

![Graph showing exponential and non-exponential decay with a shaded "Zeno region".](image)
Quantum Zeno effect

\[ \rho_N(t) \]

\[ 0 \quad t \quad \frac{9}{9} \quad \frac{3}{3} \quad t \quad \frac{1}{1} \]
Large times

Paley-Wiener theorem

\[ p_\psi(E) = |\langle E | \psi \rangle|^2 = 0 \quad \text{for } E < 0 \quad \text{Ground energy} \]

\[ a(t) = \int_0^\infty e^{-iEt} p_\psi(E) \, dE \]

\[ \int_{\mathbb{R}} \frac{-\ln |a(t)|}{1 + t^2} \, dt < \infty \quad a(t) \sim \frac{C}{t^\alpha} \]
Quantum decay

Quantum evolutions at short, intermediate and long times

Saverio Pascazio

1, 2, 3

1 Dipartimento di Fisica and MECENAS, Università di Bari, I-70126 Bari, Italy
2 Istituto Nazionale di Ottica (INO-CNR), I-50125 Firenze, Italy
3 INFN, Sezione di Bari, I-70126 Bari, Italy

I. INTRODUCTION

The decay of unstable quantum mechanical systems is only approximately exponential [2, 3]. At short times, the decay is quadratic, while at long times the decay is governed by a power law. A typical quantum evolution is displayed in Fig. 1.

The above-mentioned features of the quantum evolution are consequences of first principles. The quadratic behaviour stems from a short-time expansion of the Schrödinger evolution. The mathematical hypotheses that justify such expansion are the normalizability of the wave function and the finite energy of the initial state: these conditions are so sacrosant from a physical point of view that they are considered to be indispensable. The ensuing "Zeno" region [4] has been experimentally confirmed on different physical systems [5, 6].

The familiar exponential decay sets in at intermediate times. It is always the consequence of approximations of some sort (usually weak-coupling and/or Markovianity). A proper treatment of this difficult problem requires a quantum field-theoretical analysis of the propagator, and its analytic continuation in the second Riemann sheet of the complex energy plane.

The long-time evolution is a consequence of the lower-boundedness of the Hamiltonian. This condition is also considered to be not renounceable from a physical perspective. Under this hypothesis, a straightforward application of the Paley-Wiener theorem on Fourier transforms yields long-time power tails. Unlike the short-time Zeno region, power-like decays have never been observed (to the best of our knowledge). They are difficult to detect, essentially because the long-time evolution is preceded by an exponential regime, that drastically depletes the initial state, making any subsequent observation prohibitively difficult.

The duration of the three afore-mentioned temporal regimes and the transition times that separate them depend on dimensionality and the parameters that characterize the physical system. Typically (but not always), the duration of the Zeno region scales with the square of the coupling constant, while the transition to a power law takes place at a time of the order of its logarithm (and can be, say, a hundred lifetimes). In typical decaying systems (such as an unstable particle or an excited atom) the Zeno region is so short as to result unobservable, and the power law sets in after such a long time that virtually all systems have already (exponentially) decayed.

II. QUANTUM MECHANICAL EVOLUTIONS

A. Short times

Let $H$ be the Hamiltonian of a quantum system and
Exponential decay?

\[ a(t) = \int_{\mathbb{R}} e^{-iEt} p_{\psi}(E) \, dE \quad p_{\psi}(E) = |\langle E | \psi \rangle|^2 \]

\[ a(t) = \exp \left( -\frac{\gamma}{2} |t| - i \omega_0 t \right) \]

**Fourier transform**

\[ p_{\psi}(E) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iEt} a(t) \, dt = \frac{\gamma}{2\pi} \frac{1}{(E - \omega_0)^2 + \frac{\gamma^2}{4}} \]

\[ \langle E | \psi \rangle = \phi_C(E) = \sqrt{p_{\psi}(E)} \, e^{i\alpha(E)} \quad \phi_C(E) = \sqrt{\frac{\gamma}{2\pi}} \frac{1}{E - \omega_0 - i\frac{\gamma}{2}} \]
\[ \phi_C(E) = \sqrt{\frac{\gamma}{2\pi}} \frac{1}{E - \omega_0 - i \frac{\gamma}{2}} \]

**Cauchy**

\[ \langle \phi_C | e^{-iqt} \phi_C \rangle = \exp \left( -\frac{\gamma}{2} |t| - i \omega_0 t \right) \]

Exponential decay

**position operator**

\[ H = q \]

\[ \text{spec } H = \mathbb{R} \]
Subsystem evolution

\( \mathcal{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R}) \) spin-\( \frac{1}{2} \) particle

\[ \psi = |0\rangle \otimes \phi_0 + |1\rangle \otimes \phi_1 \simeq \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \]

Hamiltonian

\[ H = |0\rangle\langle 0| \otimes q_+ + |1\rangle\langle 1| \otimes q_- \simeq \begin{pmatrix} q_+ \\ q_- \end{pmatrix} \]

\[ q_+ \phi(x) = x_+ \phi(x) \]

\[ q_- \phi(x) = x_- \phi(x) \]
Subsystem evolution

\[ H = \begin{pmatrix} q_+ \\ q_- \end{pmatrix} \quad \psi = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \]

\[ \langle \psi | H \psi \rangle = \langle \phi_0 | q_+ \phi_0 \rangle + \langle \phi_1 | q_- \phi_1 \rangle \]

\[ = \int_{0}^{+\infty} x (|\phi_0(x)|^2 + |\phi_1(-x)|^2) \, dx \geq 0, \]

positive Hamiltonian

\[ q = q_+ - q_- \]
Subsystem evolution

\[ H = |0⟩⟨0| \otimes q_+ + |1⟩⟨1| \otimes q_- \]

\[ U(t) = |0⟩⟨0| \otimes e^{-itq_+} + |1⟩⟨1| \otimes e^{-itq_-} \quad \text{unitary group} \]

\[ \rho \otimes |φ_C⟩⟨φ_C| \quad \text{factorized initial state} \]

\[ ρ(t) = \text{tr}_2 \left( U(t)(\rho \otimes |φ_C⟩⟨φ_C|)U(t)^† \right) \quad \text{reduced density matrix} \]

\[ ρ(t) = \begin{pmatrix}
ρ_{00} & f(t) \rho_{01} \\
f(t) \rho_{10} & ρ_{11}
\end{pmatrix} \quad \text{subsystem evolution} \]

\[ f(t) = ⟨φ_C | e^{-itq} φ_C⟩ = e^{-\frac{γ}{2}|t|−iω_0t} \quad \text{exponential overlap function} \]
Subsystem evolution

\[
\rho(t) = \begin{pmatrix}
\rho_{00} & e^{-\frac{\gamma}{2} t - i \omega_0 t} \rho_{01} \\
e^{-\frac{\gamma}{2} t + i \omega_0 t} \rho_{10} & \rho_{11}
\end{pmatrix}
\]
dephasing channel

GKLS master equation

Markovian!

\[
\rho(t) = e^{t\mathcal{L}} \rho \\
\mathcal{L} \rho(t) = -i \omega_0 [\sigma_z, \rho(t)] - \frac{\gamma}{2} [\sigma_z, [\sigma_z, \rho(t)]]
\]

\[
\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|
\]
\[ \rho(t) = e^{t \mathcal{L}} \rho \]
\[ \mathcal{L} \rho(t) = -i \omega_0 [\sigma_z, \rho(t)] - \frac{\gamma}{2} [\sigma_z, [\sigma_z, \rho(t)]] \]

\[ \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| \]

\[ \omega_0 = 0 \]

\[ \sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0| \quad \text{polarization} \]

\[ \langle \sigma_x(t) \rangle = \text{tr}(\sigma_x \rho(t)) = \langle \sigma_x(0) \rangle e^{-\frac{\gamma}{2}|t|} \quad \text{exponential} \]
Paley-Wiener?

\[ \rho_{\text{tot}}(t) = e^{-itH} \rho_{\text{tot}} e^{+itH} = e^{-it\mathcal{G}}(\rho_{\text{tot}}) \]

\[ \mathcal{G} \rho_{\text{tot}} = [H, \rho_{\text{tot}}] \quad \text{Liouvillian} \]

\[ \text{spec } \mathcal{G} = \text{spec } H - \text{spec } H \]

Hamiltonian spectrum

Liouvillian spectrum

\[ A = A^\dagger \quad \text{generic observable} \]

\[ \langle A(t) \rangle = \text{tr}(A e^{-itH} \rho_{\text{tot}} e^{+itH}) = \text{tr}(A e^{-it\mathcal{G}}(\rho_{\text{tot}})) \]
Paley-Wiener?

\[ \langle A(t) \rangle = \text{tr}(A e^{-itH} \rho_{\text{tot}} e^{+itH}) = \text{tr}(A e^{-it\mathcal{G}}(\rho_{\text{tot}})) \]

\[ \mathcal{G} \rho_{\text{tot}} = [H, \rho_{\text{tot}}] \]

**Survival probability**

\[ A = \rho_{\text{tot}} = |\psi\rangle \langle \psi| \]

\[ \langle A(t) \rangle = \text{tr}(|\psi\rangle \langle \psi| e^{-itH} |\psi\rangle \langle \psi| e^{+itH}) = |\langle \psi| e^{-itH} \psi \rangle|^2 = p(t) \]

Very special! It depends on absolute values of energy
Non-analitic potential?

ramp function

\[ H = |0\rangle\langle 0| \otimes V(q) + |1\rangle\langle 1| \otimes V(-q) \]

\[ W(x) = V(x) - V(-x) \]

increasing function
Non-analitic potential?

\[ H = |0\rangle\langle 0| \otimes V(q) + |1\rangle\langle 1| \otimes V(-q) \]

\[ W(x) = V(x) - V(-x) \quad \text{increasing function} \]

\[ \rho \otimes |\phi\rangle\langle \phi| \]

\[ \rho(t) = \begin{pmatrix} \rho_{00} & f(t) \rho_{01} \\ \overline{f(t)} \rho_{10} & \rho_{11} \end{pmatrix} \quad \text{subsystem evolution} \]

\[ f(t) = \langle \phi | e^{-itW(q)} \phi \rangle \]
\[ H = |0\rangle\langle 0| \otimes V(q) + |1\rangle\langle 1| \otimes V(-q) \]

\[ W(x) = V(x) - V(-x) \quad \rho \otimes |\phi\rangle\langle \phi| \]

\[ \rho(t) = \begin{pmatrix} \rho_{00} & f(t)\rho_{01} \\ \overline{f(t)}\rho_{10} & \rho_{11} \end{pmatrix} \quad f(t) = \langle \phi | e^{-itW(q)} \phi \rangle \]

\[ \phi(x) = |W'(W^{-1}(x))|^{-1/2} \phi_C(W^{-1}(x)) \quad \text{initial state} \]

**Example**

\[ V(x) = \exp(x) \quad W(x) = 2 \sinh x \]

\[ \phi(x) = \sqrt{\frac{\gamma}{4\pi \cosh y}} \frac{1}{y - \omega_0 - i\frac{\gamma}{2}} \quad y = \text{arcsinh} \left( \frac{x}{2} \right) \]
Thank you!