Positive Hamiltonians can give purely exponential decay

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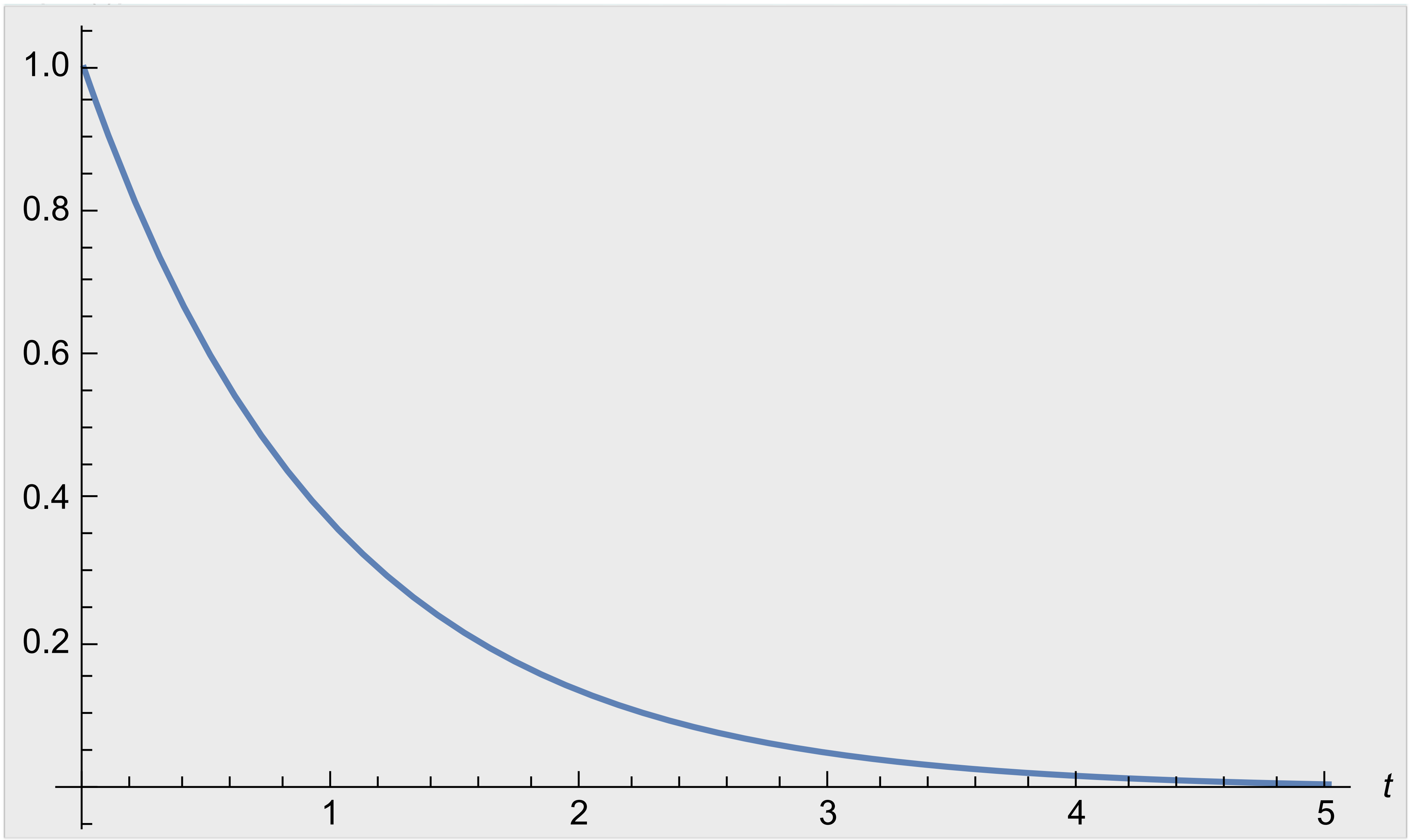
26 June 2018

Exponential decay law

$$N(t) = N_0 e^{-\gamma t}$$

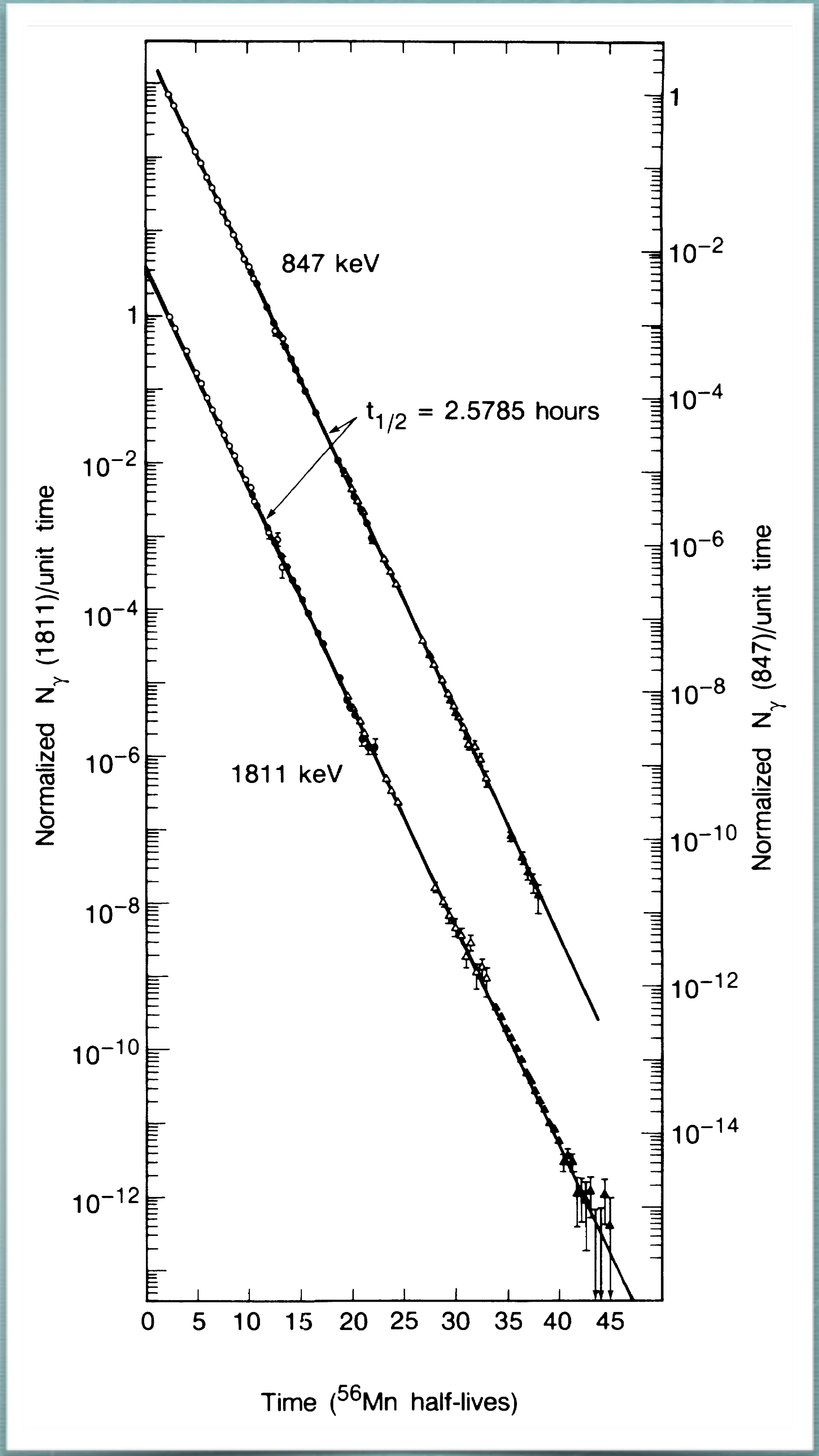


Exponential decay

$$N(t) = N_0\, e^{-\gamma t}$$

^{56}Mn radioactive decay

Norman, et al., Phys. Rev. Lett. **60**, 2246 (1988)



Exponential decay

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Testing the exponential decay law of gold ^{198}Au

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Abstract

A long time scale experimental testing of the exponential decay law of gold, ^{198}Au, is presented. The experiment has been processed by using a contemporary digital spectrum analyzer. Within the limits of the experimental errors no deviation from the exponential decay law of ^{198}Au has been observed.

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Keywords: Non-exponential decay; Zeno effect

Quantum theory

- Gamow
- Weisskopf - Wigner
- Friedrichs - Lee
- Fermi
- Fock - Krylov

Quantum systems

Survival amplitude $\quad a(t) = \langle \psi \,|\, \mathrm{e}^{-\mathrm{i}tH} \psi \rangle$

Survival probability $\quad p(t) = |\, a(t) \,|^2$

$$a(t) = \langle \psi \,|\, \mathrm{e}^{-\mathrm{i}tH} \psi \rangle = \int \mathrm{e}^{-\mathrm{i}Et} \, \mathrm{d}\, \mu_\psi(E)$$ Fourier transform

spectrum

$$\mu_\psi(\Omega) = \mathrm{Prob}\,(\, E \in \Omega \,)$$

$$\mu_\psi = \mu_{\mathrm{pp}} + \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}}$$ spectral measure

Lebesgue decomposition theorem

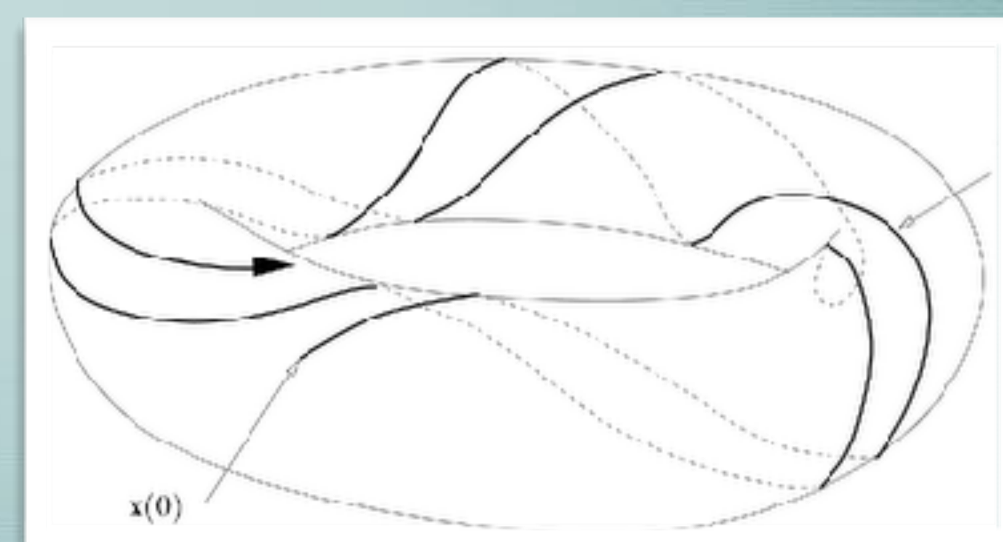
Pure point spectrum

Survival amplitude $\quad a(t) = \langle \psi | \mathrm{e}^{-\mathrm{i}tH} \psi \rangle$

$$a(t) = \langle \psi | \mathrm{e}^{-\mathrm{i}tH} \psi \rangle = \int \mathrm{e}^{-\mathrm{i}Et} \, \mathrm{d}\,\mu_\psi(E)$$ Fourier transform

$$\mu_\psi = \mu_{\mathrm{pp}} = \sum_j |c_j|^2 \delta(E - E_j) \qquad c_j = \langle E_j | \psi \rangle$$ pure point

$$a(t) = \langle \psi | \mathrm{e}^{-\mathrm{i}tH} \psi \rangle = \sum_j |c_j|^2 \mathrm{e}^{-\mathrm{i}E_j t}$$



almost periodic motion

$$\overline{p(t)} \to \sum_j |c_j|^4 = \mathrm{const} > 0, \qquad \text{as} \quad t \to \infty$$

Continuous spectrum

Survival amplitude $\quad a(t) = \langle \psi | \mathrm{e}^{-\mathrm{i}tH} \psi \rangle$

$$a(t) = \langle \psi | \mathrm{e}^{-\mathrm{i}tH} \psi \rangle = \int \mathrm{e}^{-\mathrm{i}Et} \, \mathrm{d}\, \mu_\psi(E) \quad \text{Fourier transform}$$

$$\mu_\psi = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} \qquad \text{continuous spectrum}$$

$$\overline{p(t)} \to 0, \qquad \text{as} \quad t \to \infty \qquad \text{RAGE theorem}$$

Quantum unstable systems

Survival amplitude $a(t) = \langle \psi | \mathrm{e}^{-\mathrm{i}tH} \psi \rangle$

Survival probability $p(t) = |a(t)|^2$

$$\langle \psi | \mathrm{e}^{-\mathrm{i}tH} \psi \rangle = \int \mathrm{e}^{-\mathrm{i}Et} \, \mathrm{d}\mu_\psi(E) \qquad \mu_\psi = \mu_{\mathrm{ac}}$$

absolutely continuous

$$p_\psi(E) = \frac{\mathrm{d}\mu_\psi(E)}{\mathrm{d}E} \qquad p_\psi(E) = |\langle E | \psi \rangle|^2 \geq 0$$

$$\lim_{t\to\infty} a(t) = \lim_{t\to\infty} \int \mathrm{e}^{-\mathrm{i}Et} p_\psi(E) \mathrm{d}E = 0$$

unstable system

Riemann-Lebesgue lemma

Short times

Survival amplitude $\quad a(t) = \langle \psi | \mathrm{e}^{-\mathrm{i}tH} \psi \rangle$

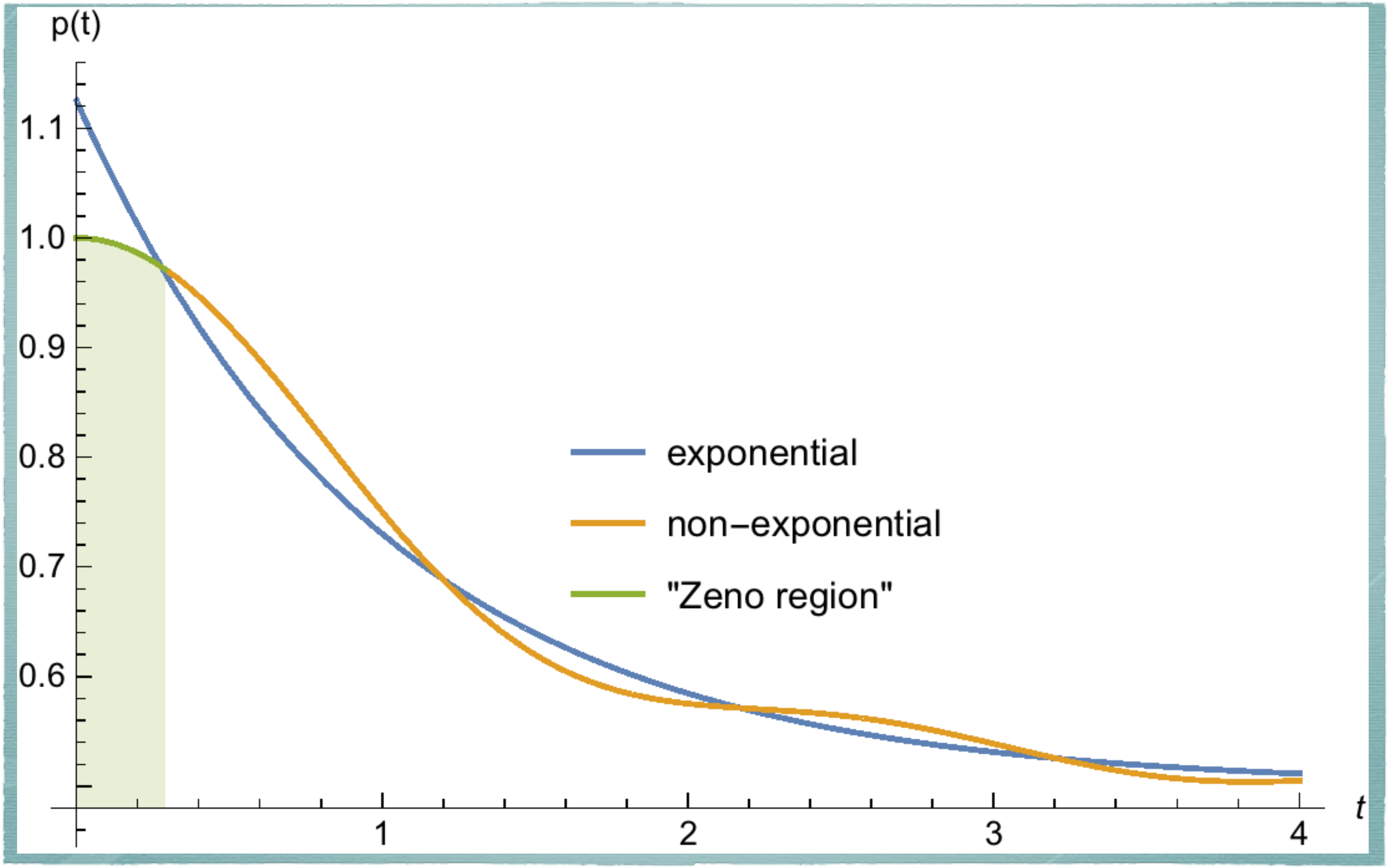
$$p(t) = \left| a(t) \right|^2 = \left| \langle \psi | \exp(-\mathrm{i}tH)\psi \rangle \right|^2$$

$\psi \in D(H) \qquad \|H\psi\| < \infty \qquad$ finite variance

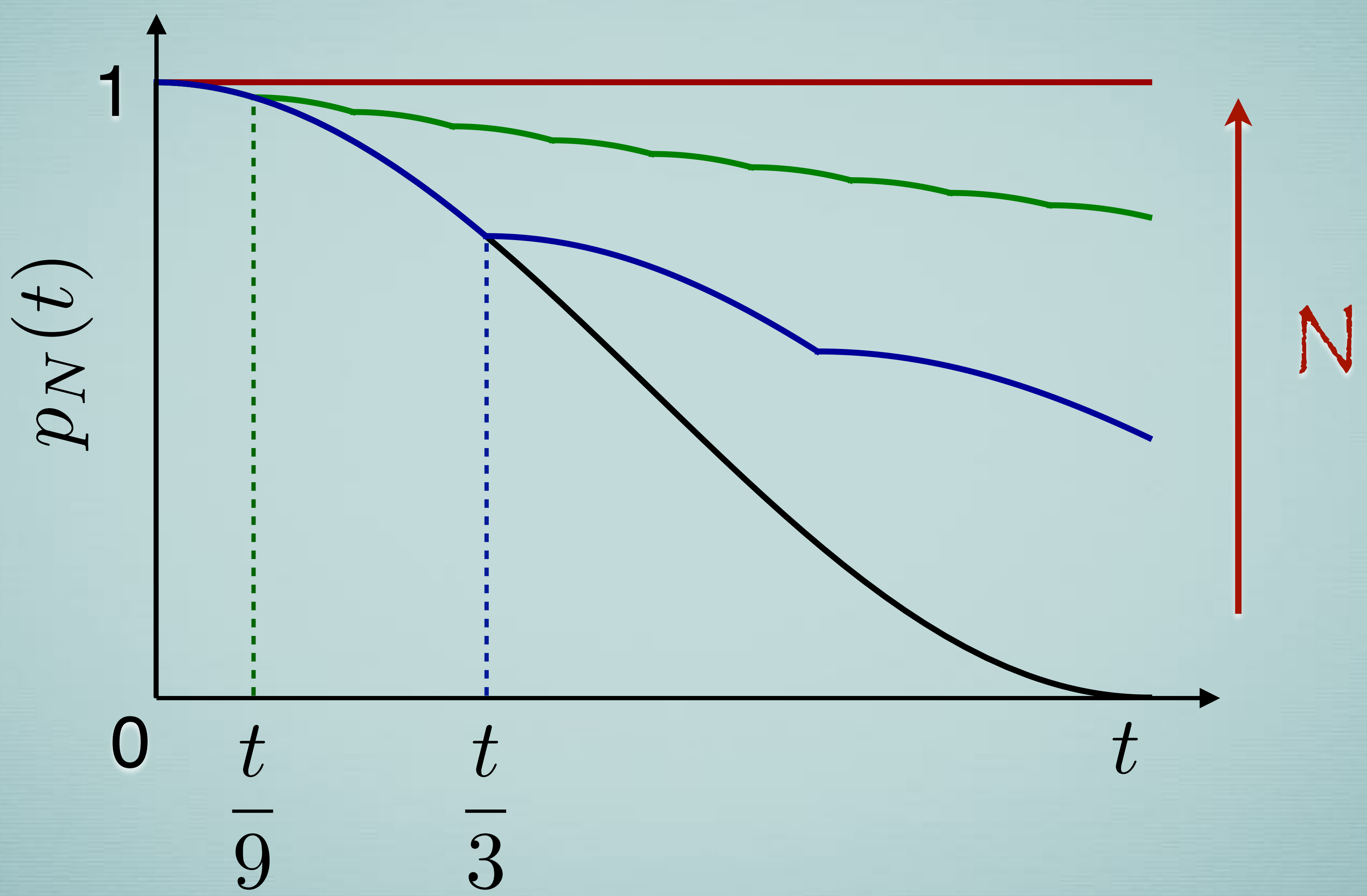
$$p(t) = 1 - t^2 \left(\langle H\psi | H\psi \rangle - \langle \psi | H\psi \rangle^2 \right) + o\left(t^2\right)$$

as $\quad t \to 0$

Short times


p(t)
1.1
1.0
0.9
0.8
0.7
0.6
1
2
3
4
t
exponential
non-exponential
"Zeno region"

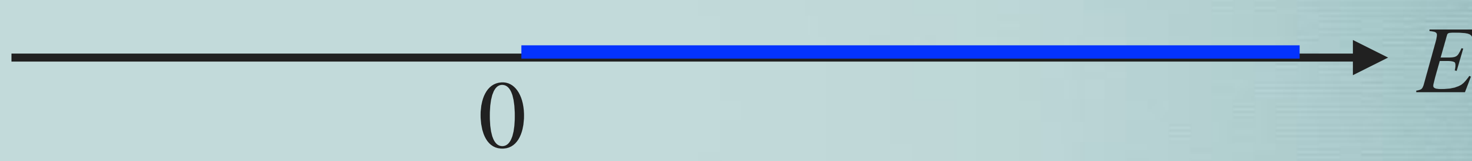
Quantum Zeno effect



Large times

Paley-Wiener theorem

$p_\psi(E) = |\langle E|\psi\rangle|^2 = 0$ for $E < 0$ Ground energy

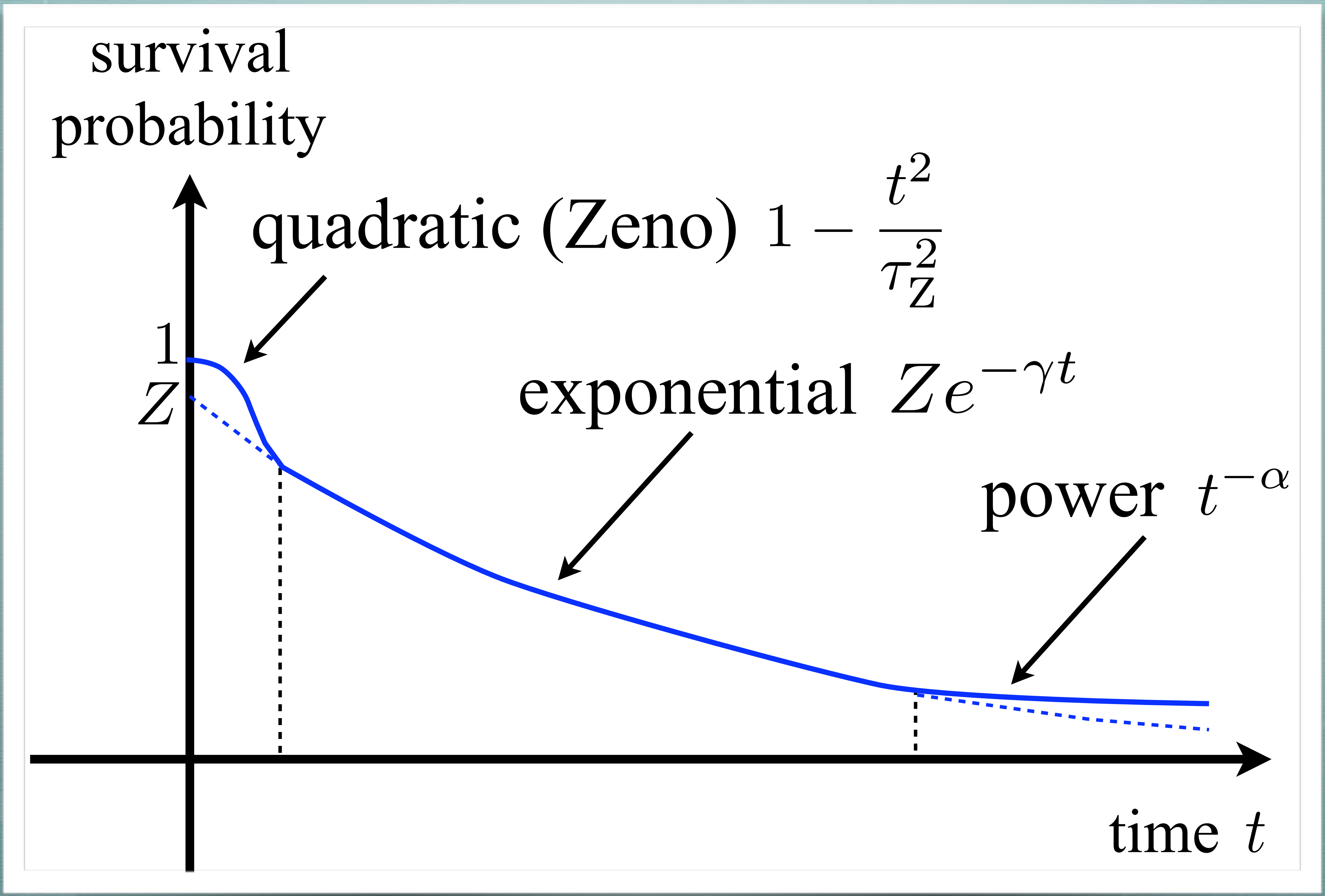


$$a(t) = \int_0^\infty \mathrm{e}^{-\mathrm{i}Et} p_\psi(E)\,\mathrm{d}E$$

$$\int_{\mathbb{R}} \frac{-\ln|a(t)|}{1+t^2}\,\mathrm{d}t < \infty$$

$$a(t) \sim \frac{C}{t^\alpha}$$

Quantum decay



Exponential decay?

$$a(t) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}Et} p_\psi(E)\,\mathrm{d}E \qquad p_\psi(E) = |\langle E|\psi\rangle|^2$$

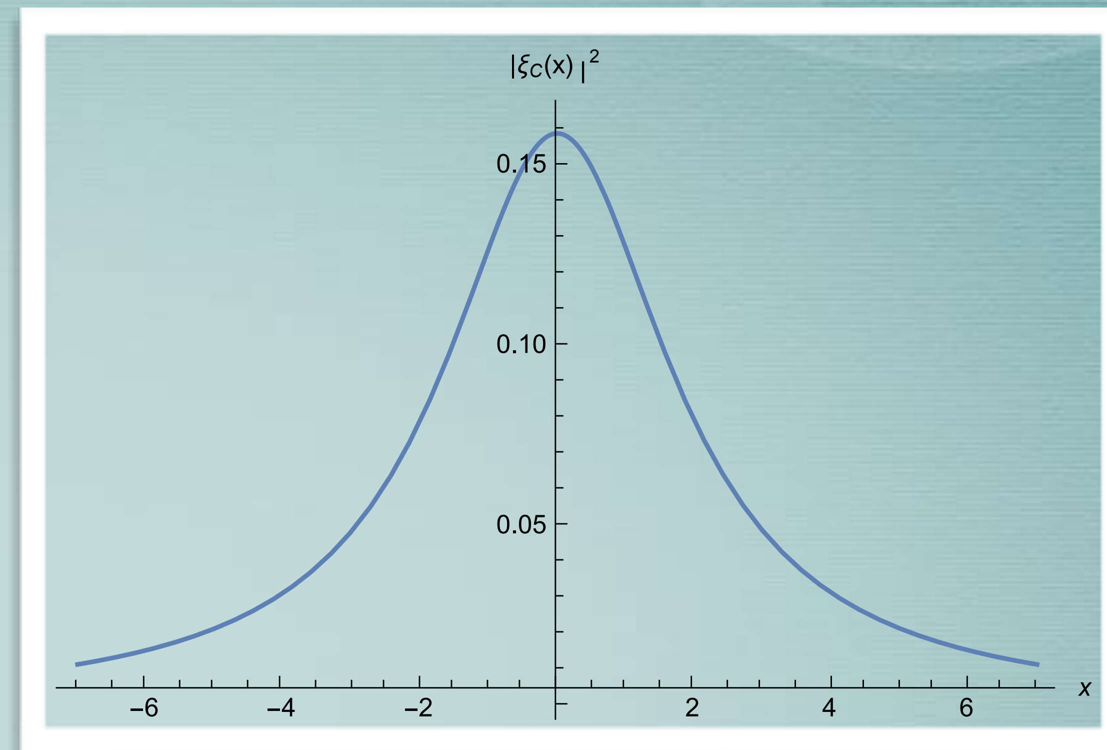
$$a(t) = \exp\left(-\frac{\gamma}{2}|t| - \mathrm{i}\,\omega_0 t\right)$$

Fourier transform

$$p_\psi(E) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}Et} a(t)\,\mathrm{d}t = \frac{\gamma}{2\pi}\frac{1}{(E-\omega_0)^2 + \frac{\gamma^2}{4}}$$

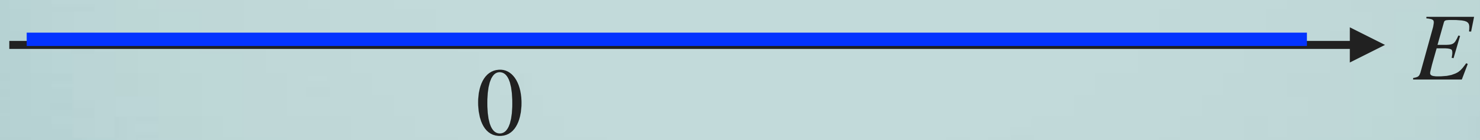
$$\langle E|\psi\rangle = \phi_{\mathrm{C}}(E) = \sqrt{p_\psi(E)}\,\mathrm{e}^{\mathrm{i}\alpha(E)} \qquad \phi_{\mathrm{C}}(E) = \sqrt{\frac{\gamma}{2\pi}}\frac{1}{E-\omega_0 - i\frac{\gamma}{2}}$$

Exponential decay



Cauchy

$$\phi_{\mathrm{C}}(E) = \sqrt{\frac{\gamma}{2\pi}}\,\frac{1}{E - \omega_0 - i\frac{\gamma}{2}}$$



$H = q$ position operator spec $H = \mathbb{R}$

$$\langle \phi_{\mathrm{C}} \,|\, \mathrm{e}^{-\mathrm{i}tq} \phi_{\mathrm{C}} \rangle = \exp\left(-\frac{\gamma}{2}\,|\,t\,| - \mathrm{i}\,\omega_0 t \right)$$

exponential

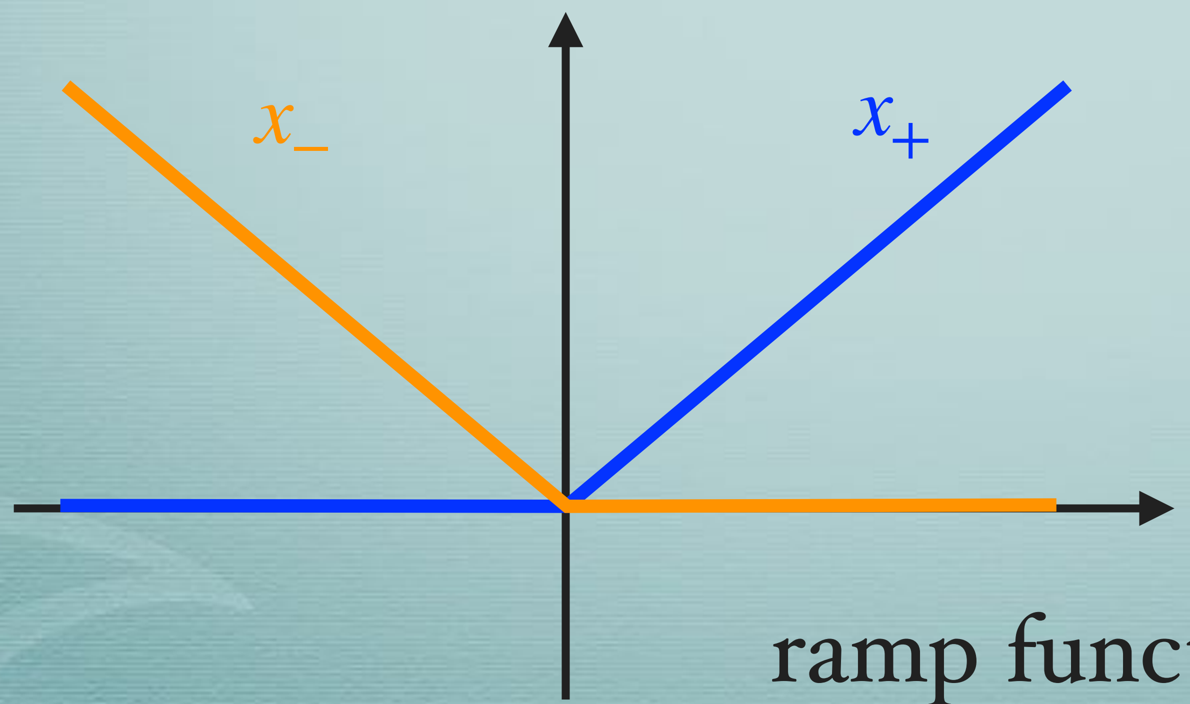
Subsystem evolution

$\mathscr{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R})$ spin-1/2 particle

$$\psi = |0\rangle \otimes \phi_0 + |1\rangle \otimes \phi_1 \simeq \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$$

Hamiltonian

$$H = |0\rangle\langle 0| \otimes q_+ + |1\rangle\langle 1| \otimes q_- \simeq \begin{pmatrix} q_+ & \\ & q_- \end{pmatrix}$$



ramp functions

$$q_+\phi(x) = x_+\phi(x)$$

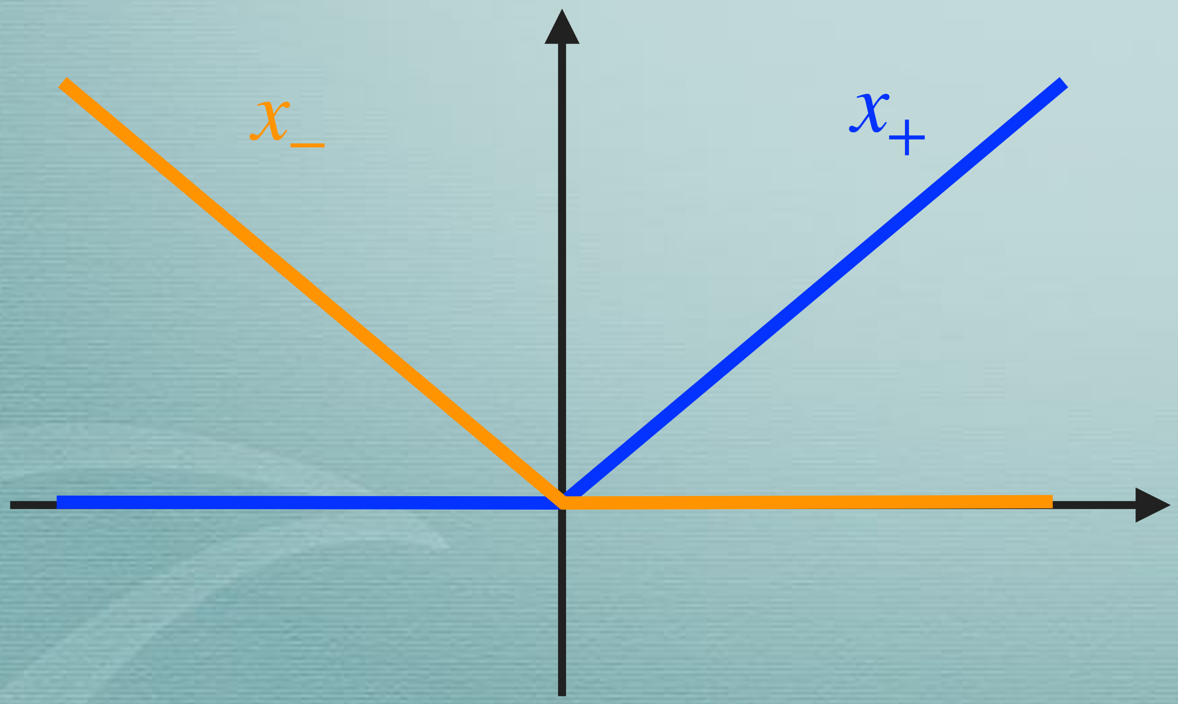
$$q_-\phi(x) = x_-\phi(x)$$

Subsystem evolution

$$H = \begin{pmatrix} q_+ & \\ & q_- \end{pmatrix} \qquad \psi = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$$

$$\begin{aligned} \langle \psi \,|\, H\psi \rangle &= \langle \phi_0 \,|\, q_+ \phi_0 \rangle + \langle \phi_1 \,|\, q_- \phi_1 \rangle \\ &= \int_0^{+\infty} x(|\phi_0(x)|^2 + |\phi_1(-x)|^2)\mathrm{d}x \geq 0, \end{aligned}$$

positive Hamiltonian



$q = q_+ - q_-$

Subsystem evolution

$$H = |0\rangle\langle 0| \otimes\ q_+ + |1\rangle\langle 1| \otimes\ q_-$$

$$U(t) = |0\rangle\langle 0| \otimes \mathrm{e}^{-\mathrm{i}tq_+} + |1\rangle\langle 1| \otimes \mathrm{e}^{-\mathrm{i}tq_-}$$ unitary group

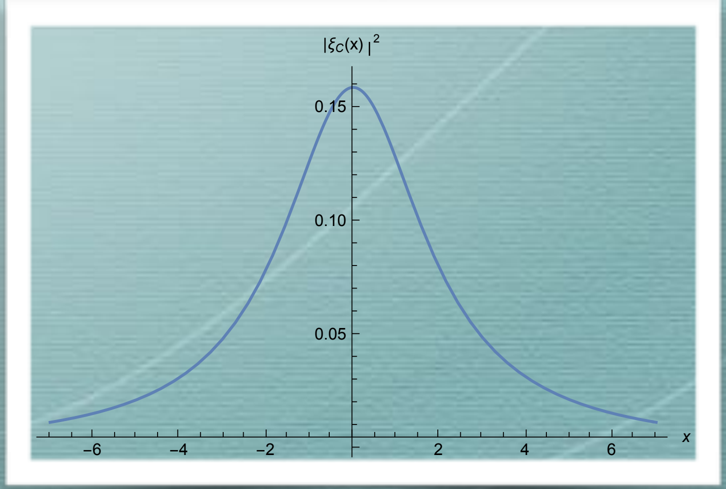
$$\rho \otimes |\phi_\mathrm{C}\rangle\langle\phi_\mathrm{C}|$$ factorized initial state

$$\rho(t) = \mathrm{tr}_2 \left(U(t)(\rho \otimes |\phi_\mathrm{C}\rangle\langle\phi_\mathrm{C}|)U(t)^\dagger \right)$$ reduced density matrix

$$\rho(t) = \begin{pmatrix} \rho_{00} & f(t)\,\rho_{01} \\ \overline{f(t)}\,\rho_{10} & \rho_{11} \end{pmatrix}$$ subsystem evolution

exponential overlap function

$$f(t) = \langle\phi_\mathrm{C}|\mathrm{e}^{-\mathrm{i}tq}\phi_\mathrm{C}\rangle = \mathrm{e}^{-\frac{\gamma}{2}|t| - \mathrm{i}\,\omega_0 t}$$



Subsystem evolution

$$\rho(t) = \begin{pmatrix} \rho_{00} & \mathrm{e}^{-\frac{\gamma}{2}t - \mathrm{i}\,\omega_0 t}\,\rho_{01} \\ \mathrm{e}^{-\frac{\gamma}{2}t + \mathrm{i}\,\omega_0 t}\,\rho_{10} & \rho_{11} \end{pmatrix}$$

dephasing channel

GKLS master equation

Markovian!

$$\rho(t) = \mathrm{e}^{t\mathscr{L}}\rho$$

$$\mathscr{L}\rho(t) = -\,\mathrm{i}\,\omega_0[\sigma_z, \rho(t)] - \frac{\gamma}{2}[\sigma_z, [\sigma_z, \rho(t)]]$$

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Exponential decay!

$$\rho(t) = \mathrm{e}^{t\mathscr{L}}\rho \qquad \mathscr{L}\rho(t) = -\,\mathrm{i}\,\omega_0[\sigma_z, \rho(t)] - \frac{\gamma}{2}[\sigma_z, [\sigma_z, \rho(t)]]$$

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\omega_0 = 0$$

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$$

polarization

$$\langle \sigma_x(t)\rangle = \mathrm{tr}(\sigma_x \rho(t)) = \langle \sigma_x(0)\rangle \mathrm{e}^{-\frac{\gamma}{2}|t|}$$

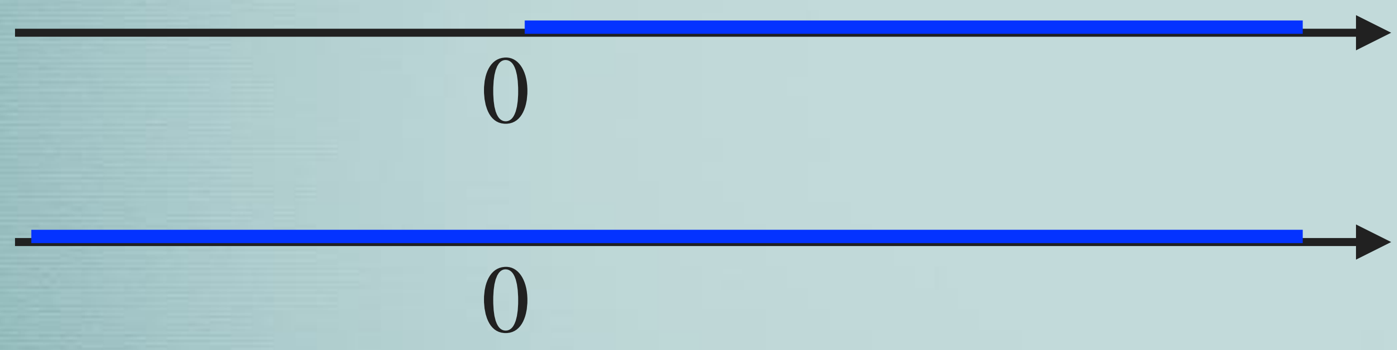
exponential

Paley-Wiener?

$$\rho_{\text{tot}}(t) = \mathrm{e}^{-\mathrm{i}tH}\rho_{\text{tot}}\mathrm{e}^{+\mathrm{i}tH} = \mathrm{e}^{-\mathrm{i}t\mathcal{G}}(\rho_{\text{tot}})$$

$$\mathcal{G}\rho_{\text{tot}} = [H, \rho_{\text{tot}}]$$ Liouvillian

$$\operatorname{spec}\mathcal{G} = \operatorname{spec}H - \operatorname{spec}H$$



Hamiltonian spectrum

Liouvillian spectrum

$$A = A^{\dagger}$$ generic observable

$$\langle A(t)\rangle = \mathrm{tr}(A\mathrm{e}^{-\mathrm{i}tH}\rho_{\text{tot}}\mathrm{e}^{+\mathrm{i}tH}) = \mathrm{tr}\big(A\,\mathrm{e}^{-\mathrm{i}t\mathcal{G}}(\rho_{\text{tot}})\big)$$

Paley-Wiener?

$$\langle A(t)\rangle = \mathrm{tr}(A\mathrm{e}^{-\mathrm{i}tH}\rho_{\mathrm{tot}}\mathrm{e}^{+\mathrm{i}tH}) = \mathrm{tr}\big(A\,\mathrm{e}^{-\mathrm{i}t\mathscr{G}}(\rho_{\mathrm{tot}})\big)$$

$$\mathscr{G}\rho_{\mathrm{tot}} = [H, \rho_{\mathrm{tot}}]$$

Survival probability

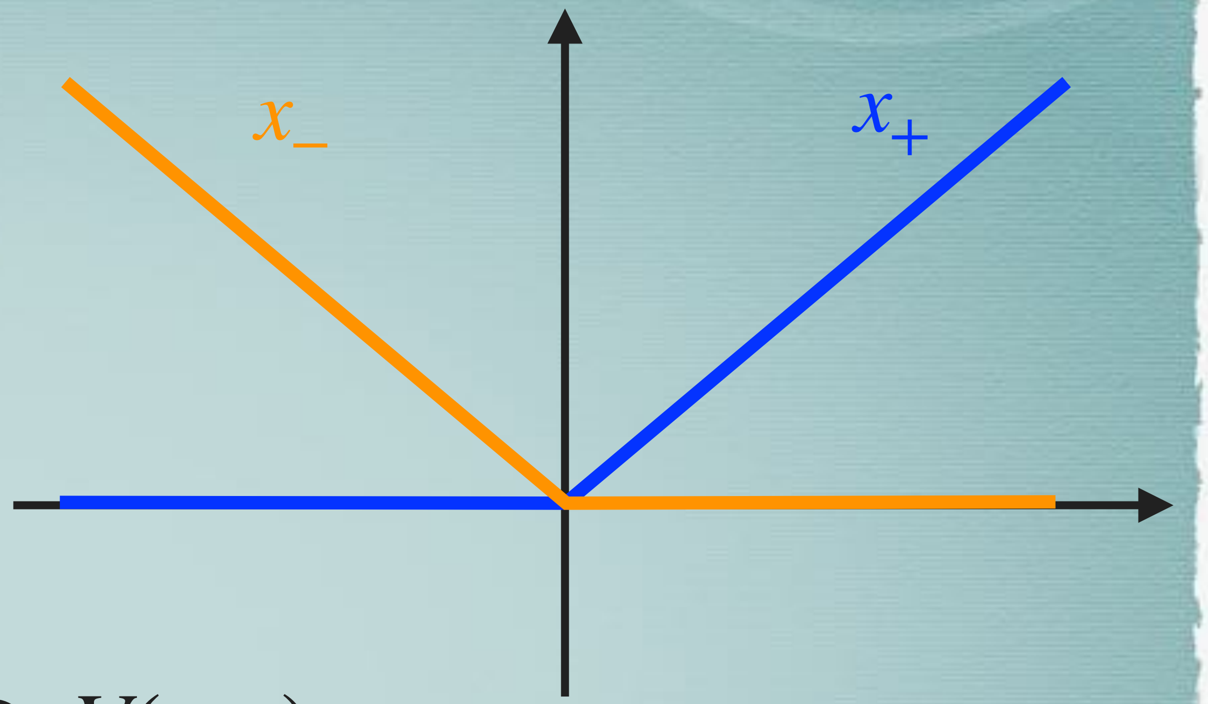
$$A = \rho_{\mathrm{tot}} = |\psi\rangle\langle\psi|$$

$$\langle A(t)\rangle = \mathrm{tr}(|\psi\rangle\langle\psi|\mathrm{e}^{-\mathrm{i}tH}|\psi\rangle\langle\psi|\mathrm{e}^{+\mathrm{i}tH}) = |\langle\psi|\mathrm{e}^{-\mathrm{i}tH}\psi\rangle|^2 = p(t)$$

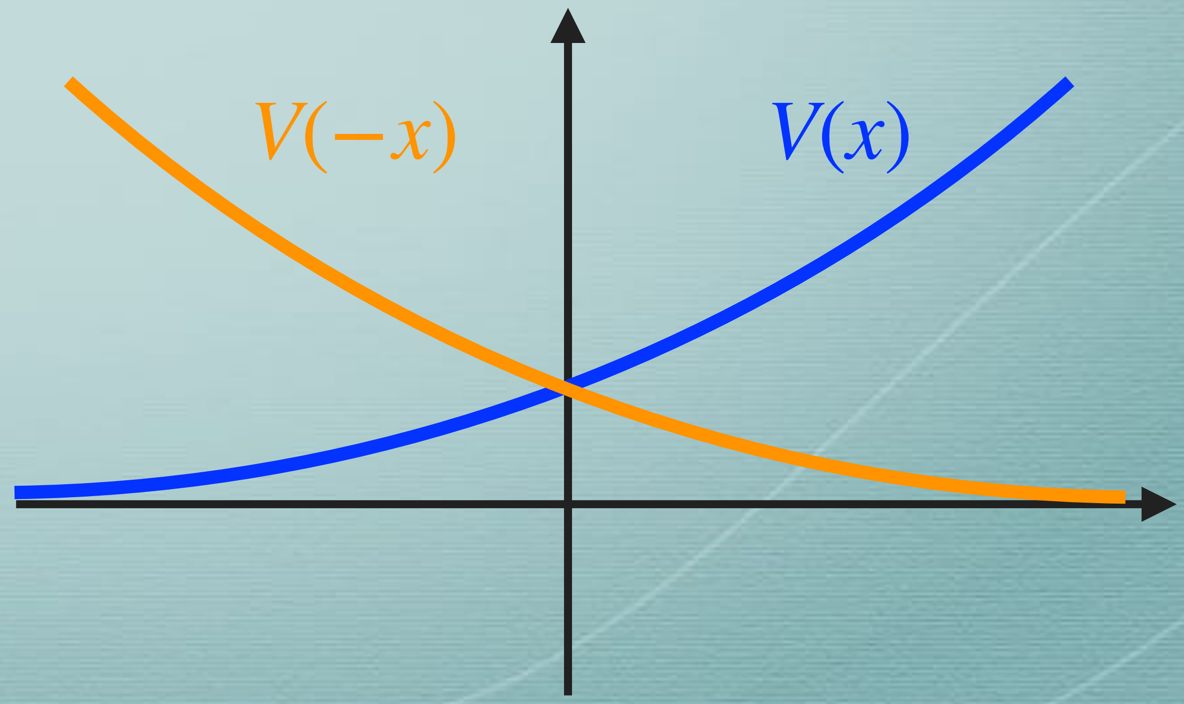
Very special! It depends on absolute values of energy

Non-analitic potential?

ramp function



$$H = |0\rangle\langle 0| \otimes V(q) + |1\rangle\langle 1| \otimes V(-q)$$



$$W(x) = V(x) - V(-x)$$

increasing function

Non-analitic potential?

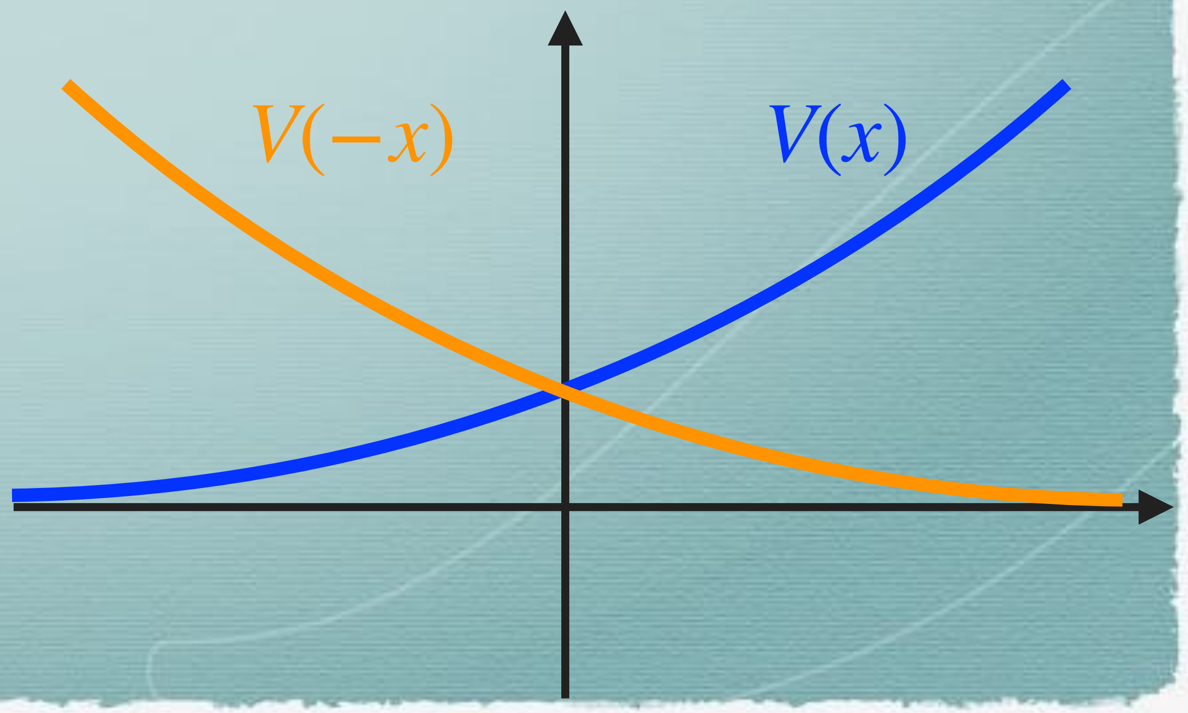
$$H = |0\rangle\langle 0| \otimes V(q) + |1\rangle\langle 1| \otimes V(-q)$$

$$W(x) = V(x) - V(-x)$$ increasing function

$$\rho \otimes |\phi\rangle\langle\phi|$$

$$\rho(t) = \begin{pmatrix} \rho_{00} & f(t)\,\rho_{01} \\ \overline{f(t)}\,\rho_{10} & \rho_{11} \end{pmatrix}$$ subsystem evolution

$$f(t) = \langle \phi | \mathrm{e}^{-\mathrm{i}tW(q)}\,\phi\rangle$$



$$H = |0\rangle\langle 0| \otimes V(q) + |1\rangle\langle 1| \otimes V(-q)$$

$$W(x) = V(x) - V(-x)$$ $$\rho \otimes |\phi\rangle\langle\phi|$$

$$\rho(t) = \begin{pmatrix} \rho_{00} & f(t)\,\rho_{01} \\ \overline{f(t)}\,\rho_{10} & \rho_{11} \end{pmatrix}$$ $$f(t) = \langle \phi \,|\, \mathrm{e}^{-\mathrm{i}tW(q)}\,\phi\rangle$$

$$\phi(x) = |W'(W^{-1}(x))|^{-1/2}\,\phi_{\mathrm{C}}(W^{-1}(x))$$ initial state

Example

$$V(x) = \exp(x) \qquad W(x) = 2\sinh x$$

$$\phi(x) = \sqrt{\frac{\gamma}{4\pi\cosh y}}\,\frac{1}{y - \omega_0 - i\frac{\gamma}{2}}$$ $$y = \operatorname{arcsinh}(x/2)$$

Thank you!