A gentle introduction to Schwinger's picture and groupoids in Quantum Mechanics

Ongoing work in collaboration with A. Ibort and G. Marmo

Presented by F. M. Ciaglia

Alternative pictures of Quantum Mechanics

Schrodinger - Dirac

- Schrodinger equations for wave functions.
- Superposition principle leads to complex Hilbert spaces.
- States, observables, probability interpretations, dynamical evolutions, and composition of systems are built out of the Hilbert space of the system.

Heisenberg - von Neumann

- Heisenberg equation for matrix mechanics.
- C*-algebras to describe the algebra of observables.
- States, observables, probability interpretations, dynamical evolutions, and composition of systems are built out of the C*-algebra of the system.

The Gelfand-Naimark-Segal (GNS) construction "connects" these alternative pictures.

The C*-algebra is represented on a Hilbert space built out of a choice of a fiducial reference state in the algebra of observables of the system.

Different states lead to different Hilbert spaces/representations.

As it is customary in Quantum Mechanics, we start quoting Dirac:

What is the precise mathematical nature of the dynamical variables?
It is better, for the present, to keep an open mind about these dynamical variables and just call them q-numbers.

The classical theory of measurement is implicitly based upon the concept of an (...) idealized experiment that disturbs no property of the system. (...) It is characteristic of atomic phenomena, however, that (...) a measurement on one property can produce unavoidable changes in the value previously assigned to another property, and it is without meaning to ascribe numerical values to all the attributes of a microscopic system. The mathematical language that is appropriate to the atomic domain is found in the symbolic transcription of the laws of microscopic measurement.

Julian Schwinger, The Algebra of Microscopic Measurement, Proc. N. A. S. 1959.

Schwinger's picture of QM and groupoids

- **Fix a family $\mathscr{A}$ of experimental devices (e.g., Stern-Gerlach devices along the z-axis) with space of outcomes Ω.**

- **The outcome "a" of the measurement of the property associated with $\mathscr{A}$ is compatible with different values (a', a'', ...) of the same property before the act of measurement.**

- **Introduce the transitions among the outcomes of experiments denoted by M(a', a).**

Roughly speaking, M(a', a) stands for: the experimental device "accepts" the system with property a and let it "emerges" with property a'.

Schwinger's picture of QM and groupoids

Transitions are composable:

$$M(a''', a'') \odot M(a', a) = \begin{cases} M(a''', a) & \text{if } \; a'' = a' \\ \text{not defined} & \text{if } \; a'' \neq a \end{cases}$$

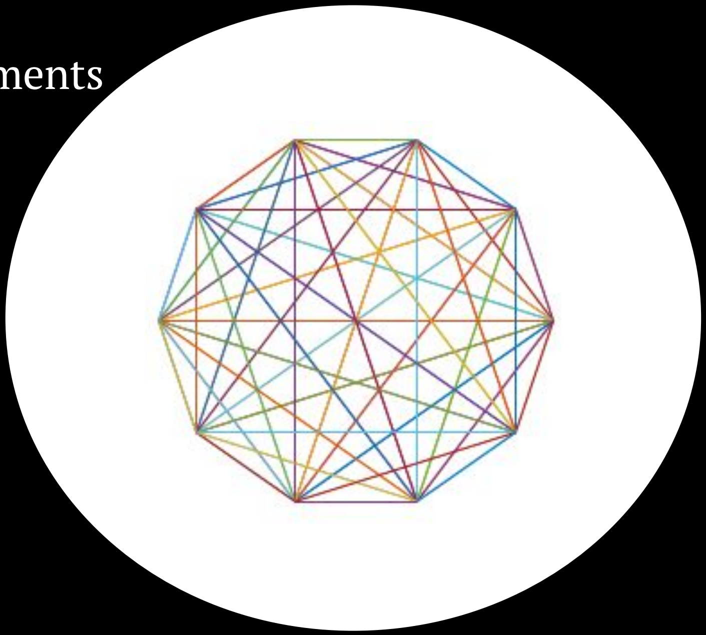
- **The composition rule ⊙ is in general non-commutative.**

- **The composition rule ⊙ depends on the class $\mathscr{A}$ of experimental devices we have selected.**

Question: What is the mathematical structure underlying the set of transitions among outcomes of experiments?

Groupoids

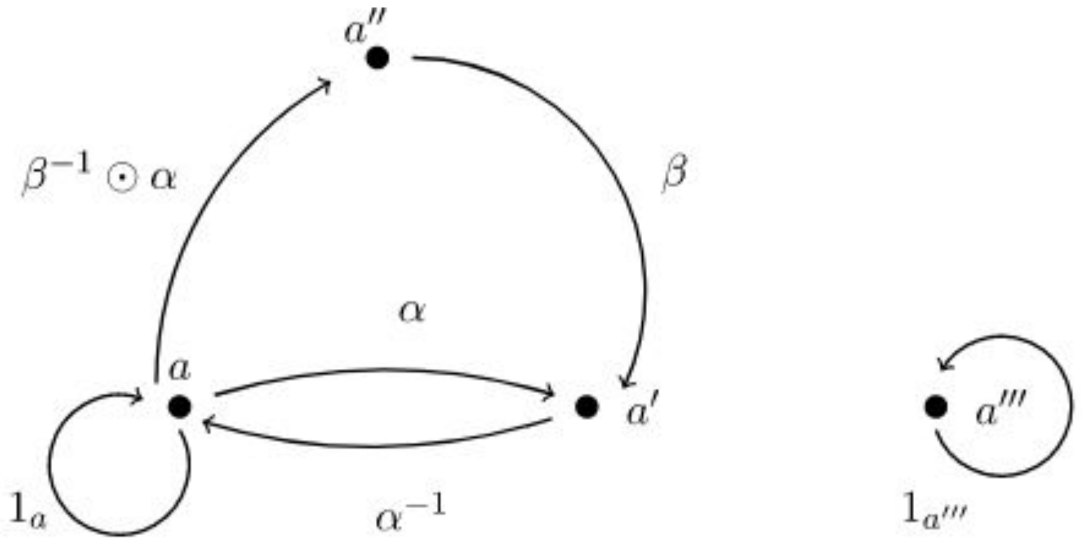
Groupoid of pairs of 10 elements



To every quantum system we associate a groupoid G over the space Ω of outcomes of experiments

Schwinger's picture of QM and groupoids

A simple example of groupoid G over the space of points Ω={a, a', a'', a'''}:



Another example is given by the Ritz-Rydberg combination principle of frequencies in spectral lines:

$$\nu_{(ik)} = \nu_{(ij)} + \nu_{(jk)}$$

Here it is Ω = ℕ and the frequencies of transition play the role of arrows.

Schwinger's picture of QM and groupoids

The set G of transitions among outcomes of experiments α=M(a', a) is a groupoid over the space Ω of outcomes of the experiments:

- **Source 's' and target 't' maps from G to Ω:**

$$s(\alpha) = s(M(a', a)) = a \quad \text{and} \quad t(\alpha) = t(M(a', a)) = a'$$

- **Associative composition rule among transitions:**

$$M(a''', a'') \odot M(a', a) = \begin{cases} M(a''', a) & \text{if } a'' = a' \\ \text{not defined} & \text{if } a'' \neq a \end{cases}$$

- **Every α in G has an inverse, and the operation of taking the inverse is an involution.**

Schwinger's picture of QM and groupoids

Complex-valued functions on the groupoid form an associative algebra $\mathscr{A}$(G) with pointwise sum and "convolution product":

$$(f_1 * f_2)(\gamma_i) = \sum_{\gamma_j \circ \gamma_k = \gamma_i} f_1(\gamma_j) f_2(\gamma_k)$$

In the finite discrete case there is a basis of $\mathscr{A}$(G) the elements of which are:

$$\delta_{\gamma_j}(\gamma_k) = \begin{cases} 1 \text{ if } \gamma_j = \gamma_k \\ 0 \text{ if } \gamma_j \neq \gamma_k \end{cases} \longrightarrow \quad f = \sum_k f(\gamma_k)\, \delta_{\gamma_k}$$

There is an involution * making $\mathscr{A}$(G) a *-algebra:

$$f^*(\gamma) := \overline{f(\gamma^{-1})}$$

The observables are the real elements in $\mathscr{A}$(G) w.r.t. the involution *.

Schwinger's picture of QM and groupoids

From the groupoid to a Hilbert space $\mathscr{H}$:

$$|\psi\rangle = \sum_{j=1}^{\#(\Omega)} \psi_j \, |a_j\rangle \ \text{ with } \psi_j \in \mathbb{C} \text{ and } \ a_j \in \Omega$$

$$\langle \phi | \psi \rangle := \sum_{j=1}^{\#(\Omega)} \overline{\phi_j} \psi_j$$

There is a *-representation of $\mathscr{F}$(G) as linear operators in $\mathscr{B}$($\mathscr{H}$):

$$\pi(f)|\psi\rangle := \sum_{j=1}^{\#(\Omega)} \sum_{\gamma : s(\gamma) = a_j} \psi_j \, f(\gamma) \, |t(\gamma)\rangle$$

$$||f|| \, := \, ||\pi(f)||_{\mathcal{B}(\mathcal{H})}$$

Schwinger's picture of QM and groupoids

States are the normalized linear functionals on $\mathscr{F}$(G). In particular, there are particular states associated with the unit transitions 1_a in G:

$$\rho_a(f) := f(1_a)$$

$$\rho_a(f^* \star f) = \sum_{\alpha \odot \gamma = 1_a} f^*(\alpha) f(\gamma) = \sum_{\gamma} f^*(\gamma^{-1}) f(\gamma) = \sum_{\gamma} \overline{f(\gamma)} f(\gamma)$$

In the Hilbert space $\mathscr{H}$ carrying the fundamental representation of $\mathscr{F}$(G):

$$\rho_a(f) = \mathrm{Tr}\left(|a\rangle\langle a|\, \pi(f)\right)$$

The Hilbert space of the GNS represenation for ρ_a is isomorphic to the Hilbert space $\mathscr{H}$ carrying the fundamental representation of $\mathscr{F}$(G).

EXAMPLES

Example: the bit

A simple example with $\Omega = \{+, -\}$ and $G = \{1_+, 1_-\}$:

The Hilbert space $\mathscr{H}$ is 2-dim. and the representation of $\mathscr{A}(G)$ reads:

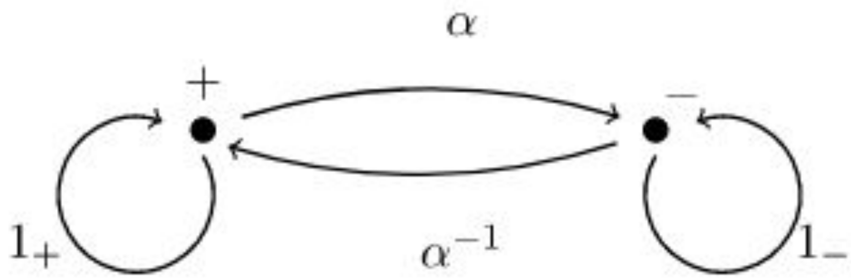
$$\pi(f)|+\rangle = \sum_{\gamma:\, s(\gamma)=+} f(\gamma)|t(\gamma)\rangle = f(1_+)|+\rangle$$

$$\pi(f)|-\rangle = \sum_{\gamma:\, s(\gamma)=-} f(\gamma)|t(\gamma)\rangle = f(1_-)|-\rangle$$

The algebra $\mathscr{A}(G)$ is a maximally commutative subalgebra of $\mathscr{B}(\mathscr{H})$.

Example: the q-bit

A simple example with $\Omega = \{+, -\}$ and $G = \{1_+, 1_-, \alpha, \alpha^{-1}\}$:



The Hilbert space $\mathscr{H}$ is 2-dim. and the representation of $\mathscr{F}(G)$ reads:

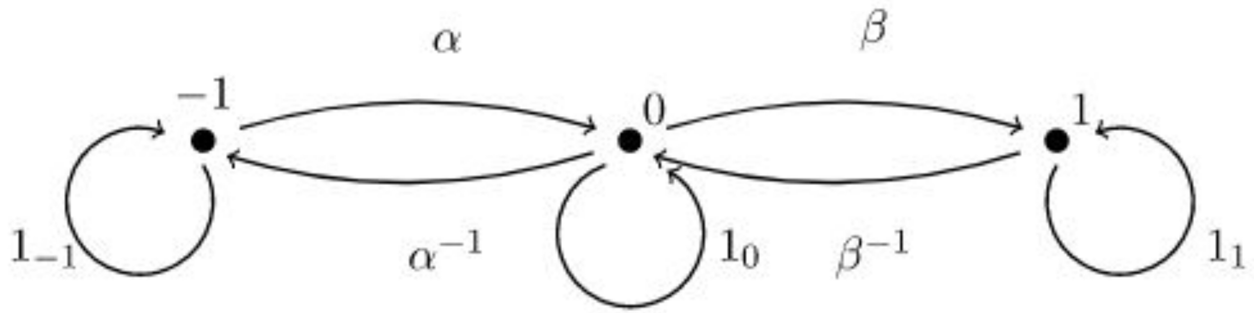
$$\pi(f)\,|+\rangle = \sum_{\gamma:\, s(\gamma)=\{+\}} f(\gamma)\,|t(\gamma)\rangle = f(1_+)\,|+\rangle + f(\alpha)\,|-\rangle$$

$$\pi(f)\,|-\rangle = \sum_{\gamma:\, s(\gamma)=\{-\}} f(\gamma)\,|t(\gamma)\rangle = f(1_-)\,|-\rangle + f(\alpha^{-1})\,|+\rangle$$

The algebra $\mathscr{F}(G)$ is in one-to-one correspondence with $\mathscr{B}(\mathscr{H})$.

Example: the q-trit

Next, we take Ω = {-1,0,1} and G = {1_{-1}, 1_0, 1_1, α, α^{-1}, β, β^{-1}, $\alpha\odot\beta$, $\beta^{-1}\odot\alpha^{-1}$}:



The Hilbert space $\mathscr{H}$ is 3-dim. and the representation of $\mathscr{F}$(G) is again $\mathscr{B}(\mathscr{H})$:

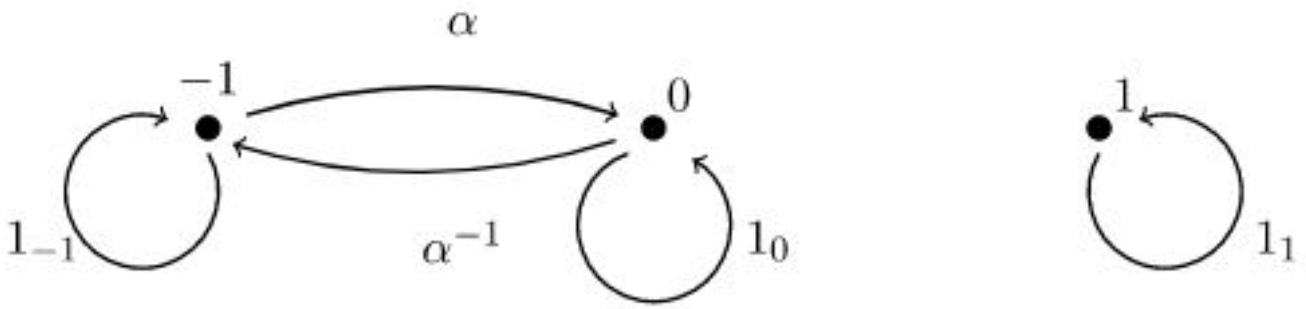
$$\pi(f)\,|-1\rangle = \sum_{\gamma:\, s(\gamma)=\{-1\}} f(\gamma)\,|t(\gamma)\rangle = f(1_{-1})\,|-1\rangle + f(\alpha)\,|0\rangle + f(\alpha\odot\beta)\,|1\rangle$$

$$\pi(f)\,|0\rangle = \sum_{\gamma:\, s(\gamma)=\{0\}} f(\gamma)\,|t(\gamma)\rangle = f(\alpha^{-1})\,|-1\rangle + f(1_0)\,|0\rangle + f(\beta)\,|1\rangle$$

$$\pi(f)\,|1\rangle = \sum_{\gamma:\, s(\gamma)=\{1\}} f(\gamma)\,|t(\gamma)\rangle = f(\beta^{-1}\odot\alpha^{-1})\,|-1\rangle + f(\beta^{-1})\,|0\rangle + f(1_1)\,|1\rangle$$

Example: the "superselected q-trit"

Taking again $\Omega = \{-1,0,1\}$ but $G = \{1_{-1}, 1_0, 1_1, \alpha, \alpha^{-1}\}$:



The Hilbert space $\mathscr{H}$ is 3-dim. but the representation of $\mathscr{F}$(G) presents "superselection sectors":

$$\pi(f)\,|-1\rangle = \sum_{\gamma:\, s(\gamma)=\{-1\}} f(\gamma)\,|t(\gamma)\rangle = f(1_{-1})\,|-1\rangle + f(\alpha)\,|0\rangle$$

$$\pi(f)\,|0\rangle = \sum_{\gamma:\, s(\gamma)=\{0\}} f(\gamma)\,|t(\gamma)\rangle = f(\alpha^{-1})\,|-1\rangle + f(1_0)\,|0\rangle$$

$$\pi(f)\,|1\rangle = \sum_{\gamma:\, s(\gamma)=\{1\}} f(\gamma)\,|t(\gamma)\rangle = f(1_1)\,|1\rangle$$

QUESTION

What happens when we use a different experimental setup?

For instance, when we consider a Stern-Gerlach apparatus with a different orientation.

Schwinger's picture of QM and groupoids

The groupoid G of transitions over the space Ω of outcomes of the experiments is always associated with a specific family $\mathscr{A}$ experimental devices (e.g., Stern-Gerlach apparatus).

Selecting another family $\mathscr{B}$ of experimental devices (e.g., Stern-Gerlach apparatus with a different orientation) we obtain another groupoid G' of transitions over the space Ω' of outcomes of experiments.

- **Physical consistency requires the existence of a C*-algebra isomorphism between $\mathscr{F}$(G') and $\mathscr{F}$(G).**
- **A transition in G is not necessarily a transition in G'.**
- **The isomorphism between $\mathscr{F}$(G') and $\mathscr{F}$(G) is "dual" to the isomorphism between the groupoid algebras $\mathbb{C}$[G] and $\mathbb{C}$[G'].**

Schwinger's picture of QM and groupoids

The groupoid algebra ℂ[G] is the free vector space built out of G:

$$\mathbf{A} \in \mathbb{C}[G]\,, \quad \mathbf{A} := \sum_j A_j \gamma_j \;\; \text{with} \;\; A_j \in \mathbb{C} \;\; \text{and} \;\; \gamma_j \in G$$

The algebra structure comes from the composition rule in G:

$$\mathbf{A} \cdot \mathbf{B} := \sum_{j,k} A_j B_k \delta(\gamma_j, \gamma_k) \gamma_j \odot \gamma_k$$

The isomorphism between ℂ[G] and ℂ[G'] is denoted by τ:

$$\tau(\gamma) = \sum_{\alpha \in G'} c(\gamma, \alpha)\alpha\,, \quad \gamma \in G\,, \;\; \alpha \in G'\,, \;\; c(\gamma, \alpha) \in \mathbb{C}$$

The groupoid law of G and G' are, in general, not compatible.

Schwinger's picture of QM and groupoids

The isomorphism between $\mathscr{F}$(G') and $\mathscr{F}$(G) is denoted by τ*:

$$(\tau^*(f))(\gamma) = f(\tau(\gamma)), \quad f \in \mathfrak{F}(G'), \quad \gamma \in G$$

Since $\mathscr{F}$(G') and $\mathscr{F}$(G) are isomorphic, the space of states of $\mathscr{F}$(G') is isomorphic to the space of states of $\mathscr{F}$(G).

Furthermore, the fundamental representations of $\mathscr{F}$(G') and $\mathscr{F}$(G) are unitarily equivalent and there is an unitary intertwining operator U mapping $\mathscr{H}$ into $\mathscr{H}$' :

$$\pi_G(\tau^*(f)) = U^\dagger \pi_{G'}(f) U$$

Summary

- To every physical system we associate a groupoid of transitions among outcomes of experiments.

- From the groupoid to the C*-algebra generated by the observables and to the Hilbert space of the system.

- Different families of experimental devices lead to different groupoids

- Equivalent descriptions are associated with isomorphic groupoid algebras.

THANK YOU

THANK YOU

There is no future. There is no past… Time is simultaneous.

An intricately structured jewel that humans insist on viewing one edge at a time, when the whole design is visible in every facet.

Dr. Manhattan

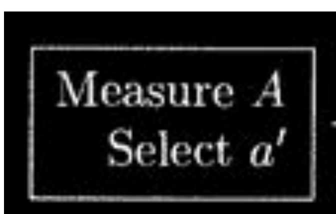
To do list

- Dynamical evolutions in the groupoid setting.
- Composition of systems and entanglement.
- Probability interpretation.

Schwinger’s picture of Quantum Mechanics

Start with the Stern-Gerlach experimental apparatus with only two possible outcomes:

For a general experimental apparatus A with n outcomes:



Symbol of atomic measurement → **M(a, a)**

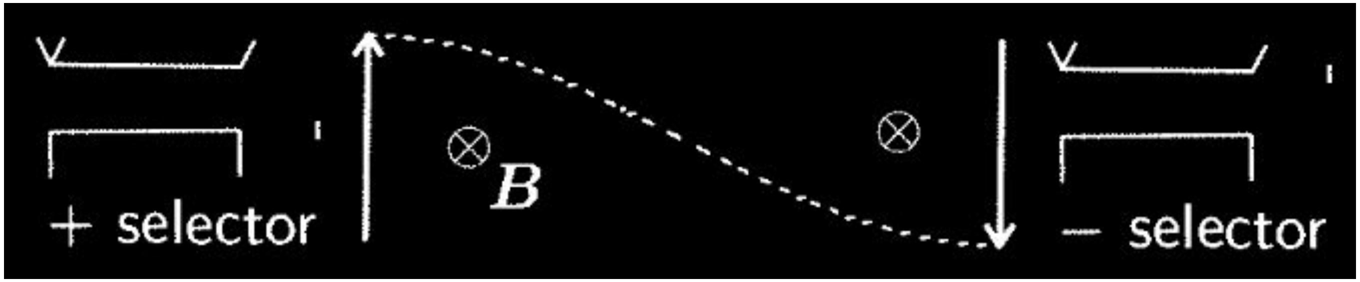
The system with property a is accepted, it then emerges with property a.

Schwinger's picture of Quantum Mechanics

The "ontological disturbance" of measurements suggests the existence of:

M(a', a)

The system with property a is accepted, it then emerges with property a'.



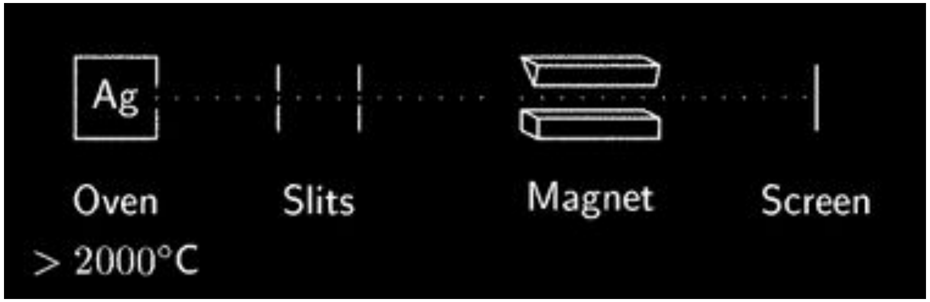
Two opposite Stern-Gerlach selectors with a homogeneous magnetic field in the middle.

The object M(a',a) is called a transition from a to a'.

$$M(a',a'')M(a''',a^{\mathrm{iv}}) = \delta(a'',a''')M(a',a^{\mathrm{iv}})$$

What is Schwinger's picture of Quantum Mechanics?

And why should we bother?



$M(a',a')$

$M(a',a'')$