Topics in mathematical relativity

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Rome, February 2019
Recent progress on mathematical GR

Plan of the talk

- Asymptotics in the radiation regime
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- Positivity of ADM energy \textit{in higher dimensions}
Asymptotics in the radiation regime
Stability of Schwarzschild (Kerr?) and Kerr-de Sitter
Positivity of hyperbolic energy \textit{in higher dimensions}
Definition (Penrose, 1962)

- Gravitational radiation can be defined by adding smoothly a **conformal boundary at infinity**, called **Scri**, to the space-time.

- A space-time is called **asymptotically simple** if every **maximally extended null geodesic** has an initial point and a final point at the conformal boundary at infinity.

**Key idea:** the large-distance gravitational field is studied by local analysis near the conformal boundary at infinity.

Theorem (PTC, Delay, 2003)

*There exists a large class of non-trivial vacuum asymptotically simple space-times.*

The proof relies heavily upon deep results of **Friedrich**, and of **Corvino and Schoen**. This is a small data result.
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Gravitational radiation à la Penrose
What about generic initial data? Penrose uses smooth functions throughout

Definition

A function is called polyhomogeneous at null infinity if it admits an asymptotic expansion of the form

$$f(u, r, \theta, \phi) = \sum_{i \geq 0, j \leq N} f_{ij}(u, \theta, \phi) r^{-i} \ln^{j}(r) = f_{00}(u, \theta, \phi) + \ldots,$$

for some sequence $N(i)$, $i(0) = 0$.

A metric is called polyhomogeneous at null infinity if the metric functions $g_{\mu\nu} - \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric, are polyhomogeneous and tend to zero as $r \to \infty$. 
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**Gravitational radiation à la Penrose**

What about *generic* initial data? Penrose uses smooth functions throughout. Andersson, PTC & Friedrich 1992

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(New proof, alternative to those of Christodoulou & Klainerman or of Lindblad & Rodnianski, of stability of Minkowski spacetime; cf. also Keir arXiv:1808.09982.)
Black hole Stability, $\Lambda = 0$: (on the 25th anniversary of the Christodoulou-Klainerman theorem, almost there...)

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*Schwarzschild black holes are stable under a*xi-*symmetric polarised non-linear perturbations*
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Theorem (Hintz 2017)

*Kerr-de Sitter black holes are the only stationary black holes near Kerr-de Sitter*
Positive energy?
Theorem (Lohkamp 2016; Schoen, Yau, 2017)

The ADM mass of $n$-dimensional asymptotically flat Riemannian manifolds, $n \geq 3$, is non-negative, and vanishes only for Euclidean space.
Positive energy for asymptotically flat manifolds

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1. Known previously in dimensions \( n \leq 7 \) (Schoen & Yau 1981, 1989)
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3. \( m \geq 0 \) for AF metrics \( \implies \) suitably regular static black holes are Schwarzschild in all dimensions

4. Hollands and Wald (2016): variational identities involving total mass for AF metrics can be used to prove existence of instabilities in “black strings”
Asymptotically Anti-de Sitter metrics

Geometric formulae for total energy (Ashtekar Romano 1992; Herzlich 2015; PTC, Barzegar, Höerzinger 2017), space-dimension $n$

\[ g \rightarrow_{r \rightarrow \infty} \bar{g} = - V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2, \quad V = r^2 + 1. \]

- For any Killing vector $X$ of $\bar{g}$ we have

\[ H_b (X, \mathcal{L}) = \frac{1}{16(n - 2) \pi} \lim_{R \rightarrow \infty} \int_{t=0, r=R} X^\nu Z^\xi W^{\alpha\beta}_{\nu\xi} dS_{\alpha\beta}, \]

where $W^{\alpha\beta}_{\nu\xi}$ is the Weyl tensor of $g$ and $Z = r \partial_r$ is the dilation vector field.
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• Riemannian version, asymptotically hyperbolic Riemannian metrics $g$, $R^i_j$ is the Ricci tensor of $g$:

\[ H_b(X, \mathcal{L}) = -\frac{1}{16(n-2)\pi} \lim_{R \to \infty} \int_{r=R} X^0 V Z^j (R^i_j - \frac{R}{n} \delta^j_i) dS_i. \]
Theorem (PTC, Delay, arXiv:1901.05263)

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Energy-momentum vector and localised Maskit gluing

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Energy-momentum vector and localised Maskit gluing
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$$g = x^{-2} \left( dx^2 + \left( h_{AB}(y^C) + x^n \mu_{AB}(y^C) \right) dy^A dy^B + \text{lower order} \right),$$

where $y^A$ are coordinates at the conformal boundary at infinity,
Positive energy for \textbf{asymptotically hyperbolic manifolds} space-dimension $n$

\textbf{Theorem (Andersson, Cai, Galloway 2008, PTC, Galloway, Nguyen, Paetz 2018)}

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where $y^A$ are coordinates at the conformal boundary at infinity,

\[ m = \int_{S^{n-1}} \mu \, d^{n-1}y, \]

where the \textbf{mass aspect function} is defined as

\[ \mu = h^{AB} \mu_{AB}. \]

The mass aspect function of $n$-dimensional asymptotically hyperbolic Riemannian manifolds, $3 \leq n \leq 7$, cannot be negative.

1. Uses a deformation argument independent of dimension, and a positivity theorem valid for $3 \leq n \leq 7$
2. Different story if conformal infinity is not spherical
3. One can use the Lohkamp – Schoen-Yau theorem to remove the dimension assumption
4. This (equivalent) version uses another deformation argument for $n \geq 4$
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