Topics in mathematical relativity

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University of Vienna

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- Asymptotics in the radiation regime

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- Stability of Schwarzschild

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- Positivity of hyperbolic energy *in higher dimensions*

Gravitational radiation à la Penrose

Definition (Penrose, 1962)

• Gravitational radiation can be defined by adding smoothly a *conformal boundary at infinity*, called Scri, to the space-time.

• A space-time is called asymptotically simple if every *maximally extended null geodesic* has an initial point and a final point at the conformal boundary at infinity.

Key idea: the large-distance gravitational field is studied by local analysis near the conformal boundary at infinity

Theorem (PTC, Delay, 2003)

There exists a large class of non-trivial vacuum asymptotically simple *space-times.*

The proof relies heavily upon deep results of *Friedrich*, and of *Corvino and Schoen*. This is a small data result.

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• A metric is called *polyhomogeneous at null infinity* if the metric functions

$$g_{\mu\nu} - \eta_{\mu\nu}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, are polyhomogeneous and tend to zero as $r \to \infty$

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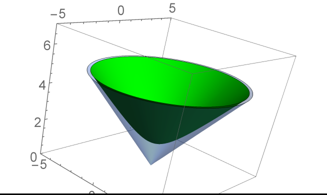
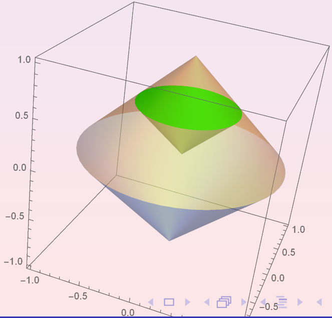
- Generic hyperboloidal initial data are known to be *non-trivially* polyhomogeneous





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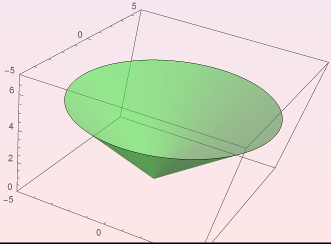

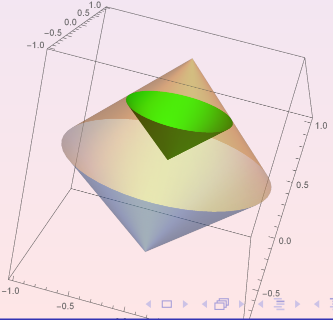

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(New proof, alternative to those of Christodoulou & Klainerman or of Lindblad & Rodnianski, of stability of Minkowski spacetime; cf. also Keir arXiv:1808.09982.)

Black hole Stability

Black hole Stability, $\Lambda = 0$: (on the 25th anniversary of the Christodoulou-Klainerman theorem, almost there...)

Theorem (Klainerman, Szeftel arXiv:1711.075)

Schwarzschild black holes are stable under axi-symmetric polarised non-linear perturbations

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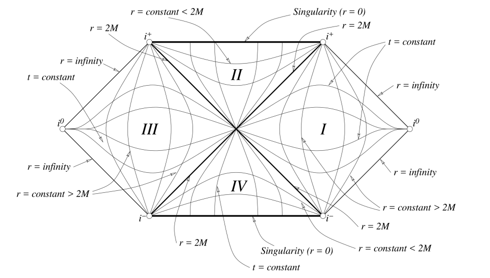

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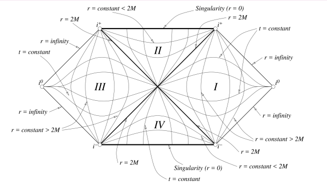

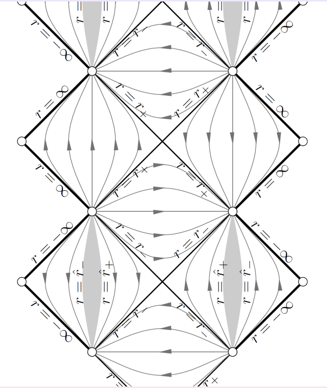

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Schwarzschild black holes are stable under non-linear perturbations

More precisely, the authors identify a subset of the set of initial data of *finite co-dimension* so that perturbations within this set evolve asymptotically to some Schwarzschild, while the remaining do not

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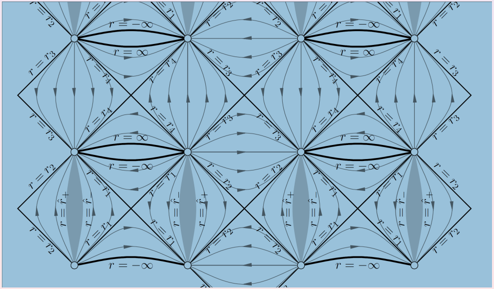

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Theorem (Hintz 2017)

Kerr-de Sitter black holes are the only stationary black holes near Kerr-de Sitter

Positive energy?

Positive energy for asymptotically flat manifolds
space-dimension n

Theorem (Lohkamp 2016; Schoen, Yau, 2017)

The ADM mass of n-dimensional asymptotically flat Riemannian manifolds, $n \geq 3$, is non-negative, and vanishes only for Euclidean space.

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1. Known previously in dimensions $n \leq 7$ (Schoen & Yau 1981, 1989)
2. or assuming that the manifold admits a spin structure (Witten, 1981)
3. *Lohkamp 2017: the energy-momentum vector is timelike future pointing*
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Mass or energy?

What is it good for anyway? some good news in the asymptotically flat case

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2. $m \geq 0$ for AF metrics $\Longrightarrow$ existence (*Schoen* 1984, all dim) and compactness (*Khuri, Marques, Schoen* 2018, dim $n \leq 24$, sharp) for the Yamabe problem
3. $m \geq 0$ for AF metrics $\Longrightarrow$ suitably regular static black holes are Schwarzschild in all dimensions
4. *Hollands and Wald* (2016): variational identities involving total mass for AF metrics can be used to prove existence of instabilities in "black strings"

Asymptotically Anti-de Sitter metrics

Geometric formulae for total energy (Ashtekar Romano 1992; Herzlich 2015; PTC, Barzegar, Höerzinger 2017), space-dimension n

$$\mathbf{g} \to_{r\to\infty} \overline{\mathbf{g}} = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2 \,, \qquad V = r^2 + 1\,.$$

- For any Killing vector X of $\overline{\mathbf{g}}$ we have

$$H_b(X, \mathscr{S}) = \frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{t=0, r=R} X^\nu Z^\xi W^{\alpha\beta}{}_{\nu\xi} dS_{\alpha\beta} \,,$$

where $W^{\alpha\beta}{}_{\nu\xi}$ is the Weyl tensor of $\mathbf{g}$ and $Z = r\partial_r$ is the dilation vector field

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- Riemannian version, asymptotically hyperbolic Riemannian metrics g, $\mathrm{R}^i{}_j$ is the Ricci tensor of g:

$$H_b(X, \mathscr{S}) = -\frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{r=R} X^0 V Z^j (\mathrm{R}^i{}_j - \frac{\mathrm{R}}{n} \delta^i_j) dS_i\,.$$

Theorem (PTC, Delay, arXiv:1901.05263)

The mass of n-dimensional asymptotically hyperbolic Riemannian manifolds, $n \geq 3$, is non-negative.

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Given two asymptotically hyperbolic vacuum initial data sets one can construct a new one by making a connected sum at the conformal boundary at infinity. The construction can be localised by a Carlotto-Schoen type hyperbolic gluing.

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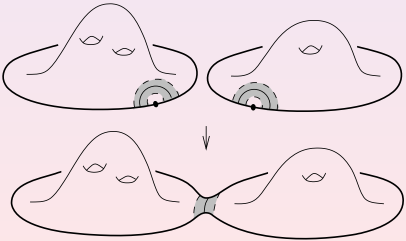

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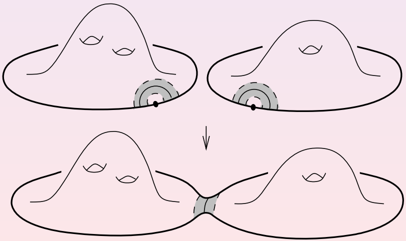

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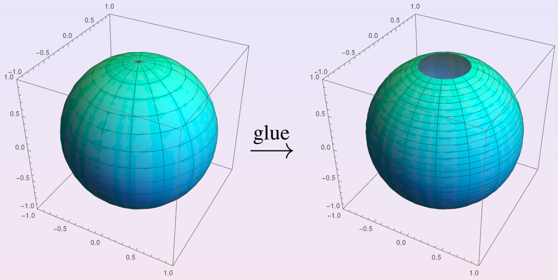
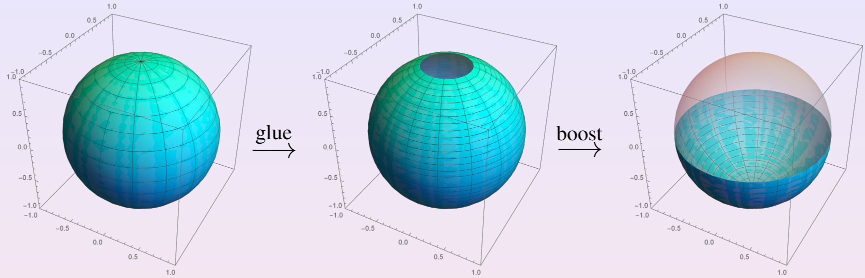

glue



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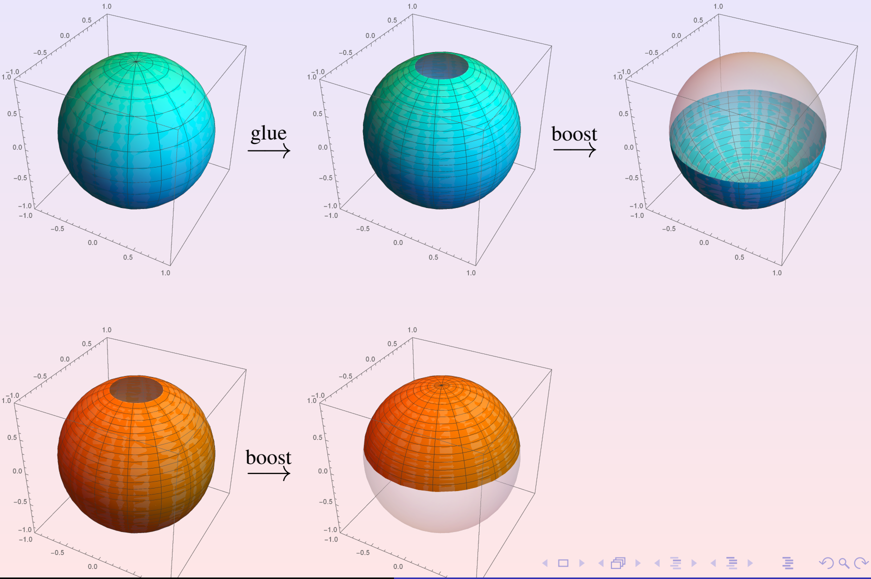
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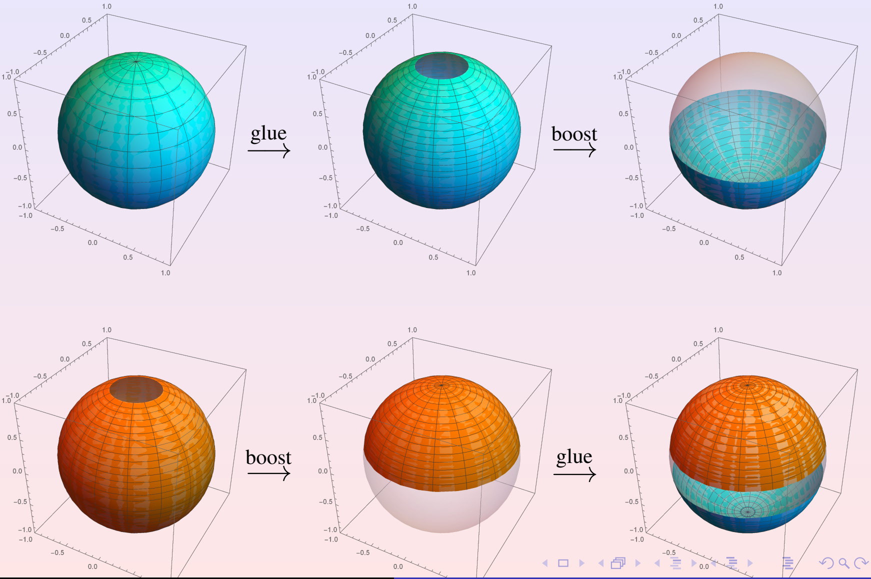
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$$m = \int_{S^{n-1}} \mu \, d^{n-1} y ,$$

where the mass aspect function is defined as

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1. Uses a deformation argument independent of dimension, and a positivity theorem valid for $3 \le n \le 7$
2. Different story if conformal infinity is not spherical
3. One can use the Lohkamp – Schoen-Yau theorem to remove the dimension assumption
4. This (equivalent) version uses another deformation argument for $n \ge 4$
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