Casimir effect as a probe for extended theories of gravity

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Abstract

In a recent paper [1], the Casimir effect in a curved spacetime described by gravitational actions quadratic in the curvature is analyzed. In particular, we consider there the dynamics of a massless scalar field confined between two nearby plates and compute the corresponding mean vacuum energy density and pressure in the framework of quadratic theories of gravity. Since we are interested in the weak-field limit, as far as the gravitational sector is concerned we work in the linear regime. Remarkably, corrections to the flat spacetime result due to extended models of gravity (although very small) may appear at the first order of our perturbative analysis, whereas general relativity contributions start appearing at the second order. Future experiments on the Casimir effect might represent a useful tool to test and constrain extended theories of gravity.

Introduction

Einstein's general relativity (GR) has undergone many challenges in the last century, but it always proved its worth thanks to high precision experiments which have confirmed a plethora of its predictions. Despite its extraordinary achievements, however, there are still open questions which need to find an answer that is nowhere to be found in GR. In the past years, such fundamental issues stimulated a vivid investigation revolving around a plausible extension of Einstein's GR domain.

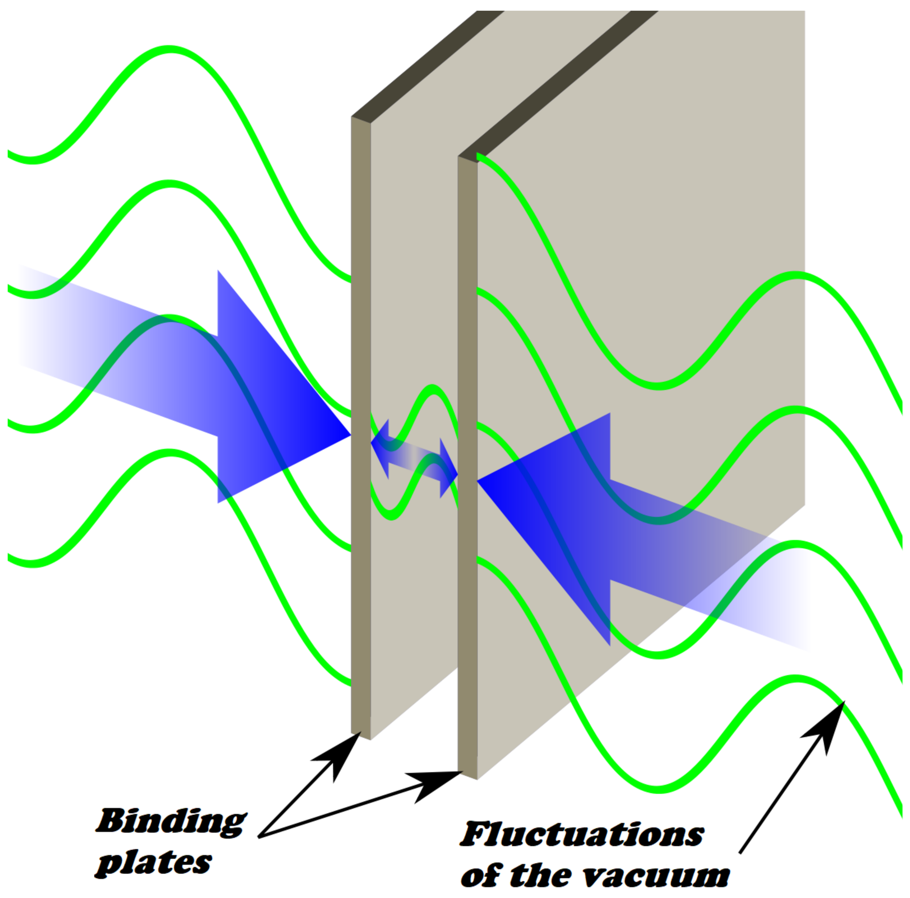


Figure 1: The simplest arrangement of the Casimir experiment. The discrepancy between the (external) continuous modes and the (internal) discrete ones arising from the absence of the quantum field on the plates allows for the emergence of a net attractive force.

Among all the theories that popped out with the above intent, one of the straightforward approach consists of generalizing the Einstein-Hilbert action by including terms which are quadratic in the curvature, for example $\mathcal{R}^2$, $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ and $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$. All the models that can be introduced in this way are subject to further theoretical treatment. Indeed, they can be included in countless applications of the most disparate physical frameworks.

Here, we are mainly focused on the analysis of the Casimir effect when the spacetime background is bent by the effect of gravity. The Casimir effect is a concrete manifestation of quantum field theory (QFT), which occurs whenever a quantum field is bounded in a finite region of space. Since it was firstly introduced to the scientific community [2], it has risen a constant interest and investigative efforts, due to the possibility of extrapolating substantial pieces of information from experiments. In this context, we study the Casimir effect in a curved spacetime emerging from a pure gravitational action quadratic in the curvature invariants. For this purpose, we closely follow the approach introduced in Ref. [3].

1 Quadratic theories of gravity in the linearized regime

Let us consider the most general gravitational action which is quadratic in the curvature, parity-invariant and torsion-free [4]

$$S=\frac{1}{2\kappa^2}\int\sqrt{-g}\left[\mathcal{R}+\frac{1}{2}\left(\mathcal{R}\mathcal{F}_1(\Box)\mathcal{R}+\mathcal{R}_{\mu\nu}\mathcal{F}_2(\Box)\mathcal{R}^{\mu\nu}+\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{F}_3(\Box)\mathcal{R}^{\mu\nu\rho\sigma}\right)\right]d^4x, \tag{1}$$

where $\kappa:=\sqrt{8\pi G}$, $\Box=g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembert operator in curved spacetime and the form-factors $\mathcal{F}_i(\Box)$ are generic operators of $\Box$ that can be either local or non-local

$$\mathcal{F}_i(\Box)=\sum_{n=0}^{N}f_{i,n}\Box^n,\qquad i=1,2,3. \tag{2}$$

Our primary aim is to study the Casimir effect between two plates in a *slightly* curved background described by the action in Eq. (1). Thus, we can apply a weak-field approximation in order to derive the linearized version of Eq. (1) around the Minkowski background $\eta_{\mu\nu}$, so that $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$, where $h_{\mu\nu}$ is the metric perturbation. At the linearized level, the relevant contribution coming from the action is of the order $\mathcal{O}(h^2)$; in such a regime, the term $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{F}_3(\Box)\mathcal{R}^{\mu\nu\rho\sigma}$ in Eq. (1) can be safely neglected [1].

By expanding the spacetime metric around the Minkowski background, the quadratic gravitational action up to the order $\mathcal{O}(h^2)$ reads [4]

$$S=\frac{1}{4}\int\left[\frac{1}{2}h_{\mu\nu}a(\Box)\Box h^{\mu\nu}-h^\sigma_\mu a(\Box)\partial_\sigma\partial_\nu h^{\mu\nu}+hc(\Box)\partial_\mu\partial_\nu h^{\mu\nu}-\frac{1}{2}hc(\Box)\Box h+\frac{1}{2}h^{\lambda\sigma}f(\Box)\partial_\lambda\partial_\sigma\partial_\mu\partial_\nu h^{\mu\nu}\right]d^4x, \tag{3}$$

where $h\equiv\eta_{\mu\nu}h^{\mu\nu}$, $f(\Box)=[a(\Box)-c(\Box)]/\Box$ and

$$a(\Box)=1+\frac{1}{2}\mathcal{F}_2(\Box)\Box\,,\quad c(\Box)=1-2\mathcal{F}_1(\Box)\Box-\frac{1}{2}\mathcal{F}_2(\Box)\Box\,. \tag{4}$$

The related field equations are represented by

$$a(\Box)\left(\Box h_{\mu\nu}-\partial_\sigma\partial_\nu h^\sigma_\mu-\partial_\sigma\partial_\mu h^\sigma_\nu\right)+c(\Box)\left(\eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma}+\partial_\mu\partial_\nu h-\eta_{\mu\nu}\Box h\right)+f(\Box)\partial_\mu\partial_\nu\partial_\rho\partial_\sigma h^{\rho\sigma}=-2\kappa^2T_{\mu\nu}, \tag{5}$$

where $T_{\mu\nu}$ is the stress-energy tensor generating the gravitational field. We are now interested in finding the expression for the linearized metric generated by a static point-like source; in this case, the line element turns out to be

$$ds^2=-(1+2\Phi)dt^2+(1-2\Psi)(dr^2+r^2d\Omega^2), \tag{6}$$

where Φ and Ψ are the metric potentials generated by $T_{\mu\nu}=m\delta^0_\mu\delta^0_\nu\delta^{(3)}(\vec{r})$. By using $\kappa h_{00}=-2\Phi$, $\kappa h_{ij}=-2\Psi\delta_{ij}$, $\kappa h=2(\Phi-3\Psi)$ and assuming the source to be static, that is $\Box\simeq\nabla^2$, $T=\eta_{\rho\sigma}T^{\rho\sigma}\simeq-T_{00}=-\rho$, the field equations for the two metric potentials read

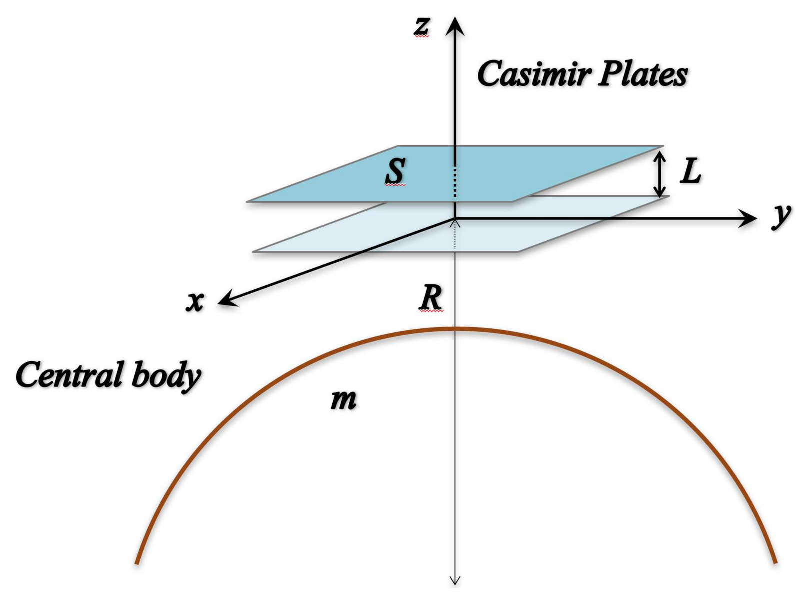
$$\frac{a(a-3c)}{a-2c}\nabla^2\Phi(r)=8\pi G\rho(r),\qquad\frac{a(a-3c)}{c}\nabla^2\Psi(r)=-8\pi G\rho(r). \tag{7}$$

We can solve the two differential equations in Eq. (7) by employing Fourier transform and then anti-transform to coordinate space. Thus, recalling that here $a\equiv a(k^2)$ and $c\equiv c(k^2)$, we obtain

$$\Phi(r)=-\frac{4Gm}{\pi r}\int_0^\infty dk\frac{a-2c}{a(a-3c)}\frac{\sin(kr)}{k}\,,\qquad\Psi(r)=\frac{4Gm}{\pi r}\int_0^\infty dk\frac{c}{a(a-3c)}\frac{\sin(kr)}{k}\,. \tag{8}$$

2 Massless scalar field dynamics and the Casimir effect

We now want to study the behavior of a massless scalar field $\psi(t,\vec{r})$ confined between two plates and embedded in a gravitational field; to this aim, we basically follow the procedure presented in Ref. [3].



In the configuration of the above picture, $\psi(t,\vec{r})$ obeys the following field equation [5]:

$$(\Box+\xi\,\mathcal{R})\psi(t,\vec{r})\,=\,\frac{1}{\sqrt{-g}}\partial_\alpha\left[\sqrt{-g}\,g^{\alpha\beta}\,\partial_\beta\psi(t,\vec{r})\right]+\xi\,\mathcal{R}\,\psi(t,\vec{r})\,=\,0\,, \tag{9}$$

where ξ is the coupling parameter between geometry and matter. We consider $\psi(t,\vec{r})$ confined between two parallel plates separated by a distance L and with extension S, placed at a distance R from the source ($R\gg L,\sqrt{S}$).

We select a reference frame with the origin in the point-like source of gravity and the z-axis along the radial direction, perpendicular to the surface of the plates. We can then expand the metric tensor components and $\mathcal{R}(\vec{r})$ around the distance R along the z direction by using the gravitational potentials $\Phi(\vec{r})$ and $\Psi(\vec{r})$ given in Eq. (8)

$$g_{00}(\vec{r})\simeq-1-2\,\Phi_0-2\,\Phi_1\,z\,,\quad g_{ij}(\vec{r})\simeq1-2\,\Psi_0-2\,\Psi_1\,z\,,\quad\mathcal{R}(\vec{r})\simeq\mathcal{R}_0+\mathcal{R}_1\,z\,, \tag{10}$$

where

$$\Phi_0=\Phi(R),\;\;\Phi_1=\frac{d\Phi(r)}{dr}\Big|_{r=R},\;\;\Psi_0=\Psi(R),\;\;\Psi_1=\frac{d\Psi(r)}{dr}\Big|_{r=R},\;\;\mathcal{R}_0=\mathcal{R}(R),\;\;\mathcal{R}_1=\frac{d\mathcal{R}(r)}{dr}\Big|_{r=R}, \tag{11}$$

and the variable z is free to range in the interval $[0,L]$.

By adopting the metric Eq. (10), the field equation for $\psi(t,\vec{r})$ of Eq. (9) becomes

$$\ddot{\psi}(t,\vec{r})-\Big[1+4\eta+4\gamma z\Big]\nabla^2\psi(t,\vec{r})+\xi\,\Big[\mathcal{R}_0+\mathcal{R}_1z\Big]\psi(t,\vec{r})\,=\,0\,, \tag{12}$$

where the dot indicates a derivative with respect to t, $\eta\equiv\Phi_0+\Psi_0$ and $\gamma\equiv\Phi_1+\Psi_1$.

In order to calculate the mean vacuum energy density $\mathcal{E}$ between the plates after the evaluation of the field modes solution of Eq. (12), we use the general relation [5]

$$\mathcal{E}\,=\,\frac{1}{V_P}\sum_n\int dk_xdk_y\int dx\,dy\,dz\,\sqrt{g_\Sigma}\,(g_{00})^{-1}\,T_{00}(\psi_n,\psi_n^*)\,, \tag{13}$$

where V_P is the proper volume and g_Σ is the induced metric on a spacelike Cauchy hypersurface Σ.

Using the Schwinger proper-time representation and the ζ-function regularization, we find [3]

$$\mathcal{E}=-\Big[1+3\big(\Phi_0-\Psi_0\big)-\big(2\Psi_1-\Phi_1\big)L_P\Big]\frac{\pi^2}{1440\,L_P^4}+\frac{\xi\,\mathcal{R}_1}{192\,L_P}\,, \tag{14}$$

where L_P is the proper length of the cavity. The relevant physical observable of the Casimir effect is the attractive force between the plates, defined as $\mathcal{F}=-\partial E/\partial L_P$, where $E=\mathcal{E}\,V_P$ is the Casimir vacuum energy. If then we introduce the pressure as $\mathcal{P}=\mathcal{F}/S_P$, with S_P being the proper area, we finally obtain

$$\mathcal{P}=\mathcal{P}_0+\mathcal{P}_G\,,\quad\mathcal{P}_0=-\frac{\pi^2}{480\,L_P^4}\,,\quad\mathcal{P}_G=\Big[3\big(\Phi_0-\Psi_0\big)-\frac{2}{3}\big(2\Psi_1-\Phi_1\big)\,L_P\Big]\mathcal{P}_0\,, \tag{15}$$

where $\mathcal{P}_0$ is the pressure in the flat case, whereas $\mathcal{P}_G$ is the correction induced by gravity.

It must be emphasized that the quantity $\mathcal{P}_G$ is the sum of two distinct contributions, namely $\mathcal{P}_G=\mathcal{P}_{GR}+\mathcal{P}_Q$, with $\mathcal{P}_{GR}$ being the contribution of Einstein's GR and $\mathcal{P}_Q$ the one arising from the quadratic part of Eq. (1).

The final step of our analysis consists in constraining quadratic theories of gravity by means of the present experimental sensitivity. Since we can impose $|\mathcal{P}_G|\lesssim\delta\mathcal{P}$, where $\delta\mathcal{P}$ is the experimental error, we obtain

$$\left|3\big(\Phi_0-\Psi_0\big)-\frac{2}{3}\big(2\Psi_1-\Phi_1\big)\,L_P\right|\lesssim\frac{\delta\mathcal{P}}{\mathcal{P}_0}\,. \tag{16}$$

3 Conclusions

The bounds that can be derived by means of Eq. (16) are summarized in the table below. It is worth underlining that the first-order corrections to the pressure cannot be attributable to GR, for which $\Phi_0=\Psi_0$. Hence, any gravitational contribution arising at this order is intimately connected to an extended theory of gravity. Finally, in order to enhance the attained outcomes, we note that there is the necessity either to significantly improve the sensitivity of the experimental instruments or to considerably lower the ratio R/Gm (i.e. the case of black holes), but our formalism based on a linear regime would no longer be valid in similar conditions.

	Form-factors	Bound
Simplest f(R) model	$\mathcal{F}_1=\alpha\;\;\mathcal{F}_2=0$	$e^{-\frac{R}{\sqrt{3\alpha}}}\lesssim\frac{\delta\mathcal{P}}{\mathcal{P}_0}\frac{R}{2Gm}$
Fourth order gravity	$\mathcal{F}_1=\alpha\;\;\mathcal{F}_2=\beta$	$\left\vert e^{-\frac{2R}{\sqrt{12\alpha+\beta}}}-e^{-\frac{\sqrt{2}R}{\sqrt{-\beta}}}\right\vert\lesssim\frac{\delta\mathcal{P}}{\mathcal{P}_0}\frac{R}{2Gm}$
Sixth order gravity	$\mathcal{F}_1=\alpha\Box\;\;\mathcal{F}_2=\beta\Box$	$\left\vert e^{-\frac{2^{-1/2}R}{\sqrt[4]{-3\alpha-\beta}}}\cos\left(\frac{2^{-1/2}R}{\sqrt[4]{-3\alpha-\beta}}\right)-e^{-\frac{R}{\sqrt[4]{2\beta}}}\cos\left(\frac{R}{\sqrt[4]{2\beta}}\right)\right\vert\lesssim\frac{\delta\mathcal{P}}{\mathcal{P}_0}\frac{R}{2Gm}$
Infinite derivative gravity	$\mathcal{F}_1=-\frac{1}{2}\mathcal{F}_2=\frac{1-e^{-\Box/M_s^2}}{2\Box}$	$\left\vert\frac{e^{-\frac{M_s^2R^2}{4}}M_s}{\sqrt{\pi}}-\frac{1}{R}\mathrm{Erf}\left(\frac{M_sR}{2}\right)\right\vert\lesssim\frac{\delta\mathcal{P}}{L_P\mathcal{P}_0}\frac{3R}{2Gm}\,.$
Non-local gravity (1)	$\mathcal{F}_1=\frac{\alpha}{\Box}\;\;\mathcal{F}_2=0$	$\left\vert\frac{\alpha}{3\alpha-1}\right\vert\lesssim\frac{\delta\mathcal{P}}{\mathcal{P}_0}\frac{R}{6Gm}$
Non-local gravity (2)	$\mathcal{F}_1=\frac{\beta}{\Box^2}\;\;\mathcal{F}_2=0$	$\left\vert1-e^{-\sqrt{3\beta}R}\right\vert\lesssim\frac{\delta\mathcal{P}}{\mathcal{P}_0}\frac{R}{2Gm}$

References

[1] L. Buoninfante, G. Lambiase, L. Petruzziello and A. Stabile, Eur. Phys. J. C **79**, 41 (2019).
[2] H. Casimir, Proc. K. Ned. Akad. Wet. **51** 793 (1948).
[3] G. Lambiase, A. Stabile and An. Stabile, Phys. Rev. D **95**, 084019 (2017).
[4] T. Biswas, E. Gerwick, T. Koivisto and A. Mazumdar, Phys. Rev. Lett. **108**, 031101 (2012).
[5] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, 1982.