Quantum Ricci Curvature in Discrete Models of Quantum Gravity

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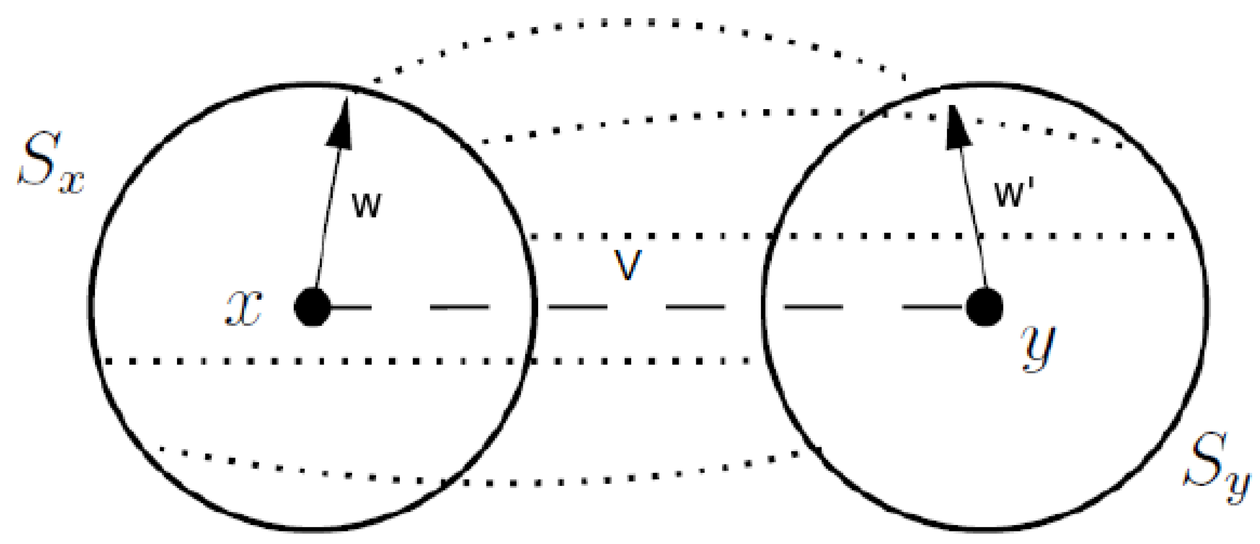
Motivation

An important unsolved problem in theoretical physics is how to combine the theories of general relativity (GR) and quantum mechanics. One of the issues of such a unification is that GR is a perturbatively non-renormalizable theory. One way to address this is to go beyond perturbation theory. Approaches such as CDT use a lattice regularization to achieve this. This implies that the continuous differentiable manifold of GR is replaced by a piecewise flat triangulation. We now have to redefine the classical observable, introducing a new set of observables suitable for this framework. This poster discusses such an observable, the recently introduced notion of quantum Ricci curvature [1, 2].

Curvature continuum definition

The following definition of curvature is used as an inspiration for our curvature construction. Consider a point $\mathbf{x}$ on a $\mathbf{D}$-dimensional Riemannian manifold and a vector $\mathbf{v}$ at $\mathbf{x}$. Move a distance δ along the geodesic in direction $\mathbf{v}$ to the point $\mathbf{y}$. Consider spheres of radius ϵ around $\mathbf{x}$ and $\mathbf{y}$. The points in the sphere around x can be defined through the initial vector $\mathbf{w}$. Calculate the average distance $\mathbf{d}(\mathbf{S}_x^\epsilon, \mathbf{S}_y^\epsilon)$ between pairs of points related by parallel transport of $\mathbf{w}$ along $\mathbf{v}$ to $\mathbf{w}'$. The Ricci curvature two-form along $\mathbf{v}$, $\mathbf{Ric}(\mathbf{v}, \mathbf{v})$, is related to $\mathbf{d}$ by the equation [3]

$$\mathrm{d}(\mathrm{S}_x^\epsilon, \mathrm{S}_y^\epsilon) = \delta\left(1 - \frac{\epsilon^2}{2\mathrm{D}}\mathrm{Ric}(\mathrm{v}, \mathrm{v}) + \mathrm{O}(\delta\epsilon^2 + \epsilon^3)\right). \tag{1}$$



Quantum Ricci curvature

The definition of the previous section relies on parallel transport. The curvature singularities of simplicial geometries make it difficult to implement parallel transport beyond a small neighborhood. We therefore consider a slight variation with an easier implementation. We define the average transport distance $\bar{\mathbf{d}}$ as the distance between all pairs of points in $\mathbf{S}_x$ and $\mathbf{S}_y$. We replace the distance of the previous section with the average transport distance. The averaging procedure implies that orders linear in δ and ϵ will appear. In order to have a consistent expansion it is therefore necessary to assume a relation between δ and ϵ. We use $\epsilon = \delta$ and define

$$\frac{\bar{\mathrm{d}}(\delta)}{\delta} = (\mathrm{c_q}(1 - \mathrm{K_q}(\mathrm{x}, \mathrm{y}))\,, \tag{2}$$

where $\mathbf{c_q}$ is a positive constant and $\mathbf{K_q}(\mathbf{x}, \mathbf{y})$ is the quantum Ricci curvature. We implemented this formalism beyond the infinitesimal regime for three constant-curvature model spaces, flat space, the sphere and hyperbolic space. We see that flat space follows a straight line, with the sphere below this line and hyperbolic space above it. A qualitative determination of the sign of the curvature is therefore possible. Positive curvature leads to decreasing values and negative curvature to increasing values.

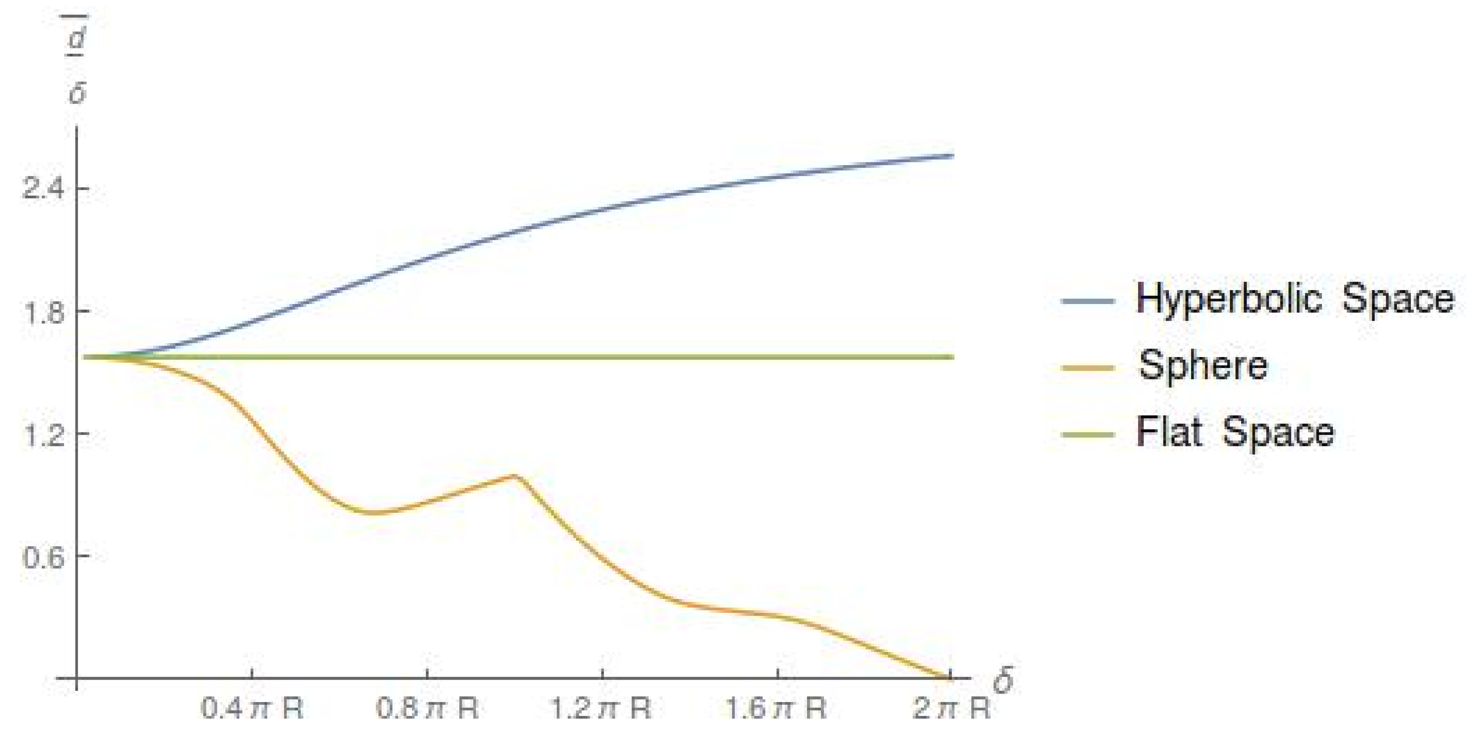


Figure : Plot of $\frac{\bar{d}}{\delta}$ as a function of δ for the three model spaces, hyperbolic space in blue, sphere in yellow and flat space in green.

Criteria for quantum gravity observables

- Well defined under averaging
- Finite
- Going beyond global properties
- Scalable (Allows for a well-defined continuum limit)
- Continuum interpretation

Euclidean Dynamical Triangulations (EDT) results

We tested the construction of average transport curvature by implementing it in 2-d EDT with spherical topology. 2-d EDT is a two-dimensional toy model of quantum gravity. Earlier measurements of the Hausdorff dimension found $\mathbf{d_H} = \mathbf{4}$ implying that this space is very far from a classical geometry. [4] Plotting $\frac{\bar{\mathbf{d}}}{\delta}$ we find that the values are decreasing, which indicates that the quantum Ricci curvature is positive. This makes it natural to fit these measurements to continuum expectations of a sphere. Disregarding small values of δ that are dominated by discretization effects, we find that the data is quite close to this classical fit. The fit has a free parameter corresponding to the radius of the sphere. We can therefore determine an effective radius of curvature ρ based on the fits.

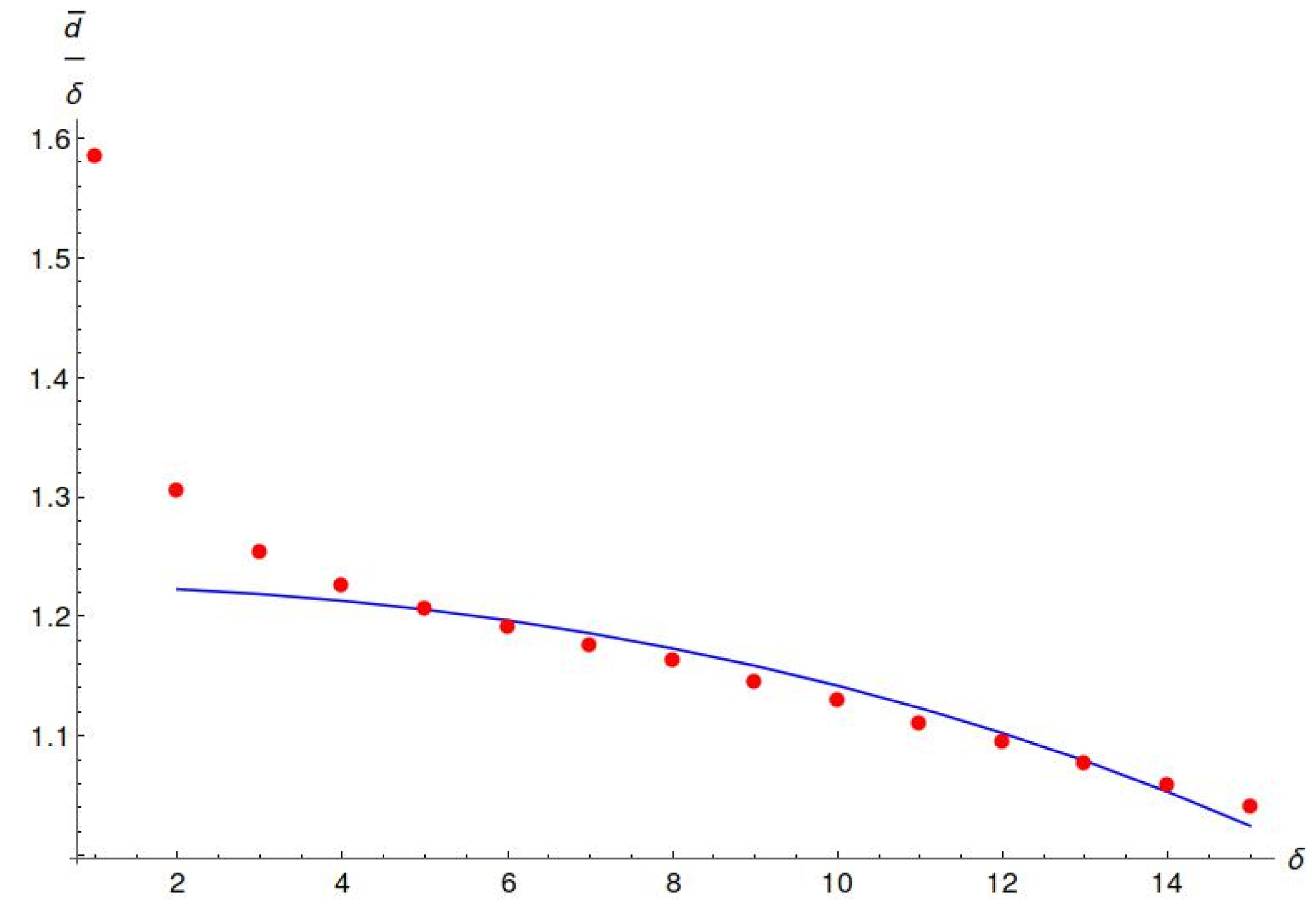


Figure : Plot of the value of $\frac{\bar{d}}{\delta}$ as a function of δ for EDT triangulations with $\mathbf{N} = \mathbf{40K}$ fitted to a 5-sphere with $\rho = \mathbf{16.7}$.

We measured ρ as a function of the total system size. From classical geometry we would expect a power series $\rho \propto \mathbf{N}^{1/\mathbf{D}}$, with $\mathbf{D}$ the dimension of the space. We do find such dependence with $\mathbf{D} = \mathbf{4.85(16)}$. This is consistent with the fits to a 5-sphere and shows that our results can be successfully extrapolated to infinite system size.

Conclusion

We introduced the notion of quantum Ricci curvature. This definition of Ricci curvature extends to the regularizations found in models of quantum gravity such as Causal Dynamical Triangulations (CDT). The definition uses the normalized average distance $\frac{\bar{\mathbf{d}}}{\delta}$ between spheres. Decreasing values indicate positive curvature and increasing values indicate negative curvature. We extended the definition beyond the infinitesimal regime by comparing to constant-curvature model spaces. This allowed us to determine the value of curvature at finite values of δ.
We measured the average transport curvature for 2-d EDT, finding that its Ricci could be modeled by a 5-sphere. It is known that the geometries of 2-d EDT are fractal and dominated by highly non-classical configurations. The ability to make consistent predictions for these configurations indicates the robustness of the curvature definition.

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