Effective One Body Hamiltonians for spin-aligned coalescing binaries

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1. Introduction

The analysis of gravitational waves emitted from coalescing black hole binaries requires accurate waveform models based on General Relativity. Building such models stems from a synergy between analytical results, first obtained via Post Newtonian (PN) and Post Minkowskian (PM) perturbation theory, and Numerical Relaitivty (NR) simulations. On the analytical side, the most accurate description of the dynamics and radiation of coalescing binaries is achieved via the Effective One Body (EOB) method [1, 2], that properly compacts and resums PN (or PM) results to make them predictive also in the strong-field, fast-motion regime. The approach maps the relative dynamics of the two objects of masses m_1 and m_2 into the dynamics of an effective object of mass $\mu = m_1 m_2/M$, with $M = m_1 + m_2$ moving under the action of an effective potential. One identifies three building blocks: (i) a Hamiltonian that describes the conservative part of the relative dynamics; (ii) a flux, that accounts for the angular momentum losses through gravitational waves; (iii) a prescription for computing the waveform from this dynamics. In particular, the Hamiltonian can be see as a *deformation* of the Hamiltonian of a spinning particle around a Kerr black hole, where the deformation parameter is $\nu = \mu/M \in [0, 1/4]$. Once such analytical description is additionally *informed* by strong-field information obtained from Numerical Relativity simulations, the model can describe the full transition from early inspiral, through plunge, merger and ringdown. The two existing waveform models able to do so are called SEOBNR [3] and TEOBResumS [4] that proved to give compatible results on GW150914. Although they share the same EOB framework, they implement it in different ways. We focus here on discussing the corresponding Hamiltonians, highlighting in particular the different treatment of spin-orbit and spin-spin effects. We use units where $G = c = 1$.

2. Phase space variables

We use dimensionless phase space variables as: $r \equiv R/M$ the relative separation, $p_r \equiv P_r/\mu$ the radial momentum conjugate to r, φ the orbital phase, $p_\varphi \equiv P_\varphi/(\mu M)$ the angular momentum and $t = T/M$ the dimensionless time. It is convenient to replace p_r with $p_{r^*} = \sqrt{A/B}p_r$ where the functions (A, B) are defined next. Being (S_1, S_2) the dimensionful spin magnitudes of the two objects, we define $\tilde{a}_0 \equiv S_1/(m_1 M) + S_2/(m_2 M)$, $\hat{S} \equiv (S_1 + S_2)/M^2$, $\hat{S}^* \equiv (m_2 S_1/m_1 + m_1 S_2/m_2)/M^2$. We use conventions such $m_1 \geq m_2$ as well as dimensionless spins $\chi_i \equiv S_i/m_i^2$.

3. EOB Hamiltonians

The real EOB Hamiltonian is connected to the effective one $\hat{H}_{\rm eff} \equiv H_{\rm eff}/\mu$ as

$$H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\hat{H}_{\rm eff} - 1\right)}. \quad (1)$$

The effective Hamiltonian of TEOB [4, 5] is written as the sum of an "orbital" part (incorporating even powers of the spins) and a spin-orbit part (incorporating odd powers of the spins)

$$\hat{H}^{\rm TEOB}_{\rm eff} = \sqrt{A\left(1 + p_\varphi^2/r_c^2 + 2\nu(4-3\nu)p_{r^*}^4/r_c^2\right) + p_{r^*}^2} + \left(G_S\hat{S} + G_{S^*}\hat{S}^*\right)p_\varphi. \quad (2)$$

Starting from Eq. (54) of Ref. [6], one finds a similar structure

$$\hat{H}^{\rm SEOB}_{\rm eff} = \sqrt{\mathbb{A}\left(1 + p_\varphi^2/\bar{r}_c^2 + 2\nu(4-3\nu)p_{r^*}^4/r^2\right) + p_{r^*}^2} + \left(\bar{G}^0_S\hat{S} + \mathbb{G}_{\mathbb{S}^*}\hat{\mathbb{S}}^*\right)p_\varphi + \hat{H}_{ss}. \quad (3)$$

Each Hamiltonian implements a different deformation of the Hamiltonian of a spinning particle on a Kerr background. More precisely: (i) both functions can be written in terms of a "centrifugal radius" [4] $r_c \neq \bar{r}_c$, each reducing to the Kerr one in the $\nu \to 0$ limit so to incorporate part of the spin-spin interactions in resummed form; (ii) $\hat{H}_{ss}$ is an additional spin-spin Hamiltonian that takes into account terms not included in $\bar{r}_c$; (iii) the spin-orbit coupling part is encoded into the phase-space functions $(G_S, G_{S^*}, \bar{G}^0_S, \mathbb{G}_{\mathbb{S}^*}, \hat{\mathbb{S}}^*)$ that, in the case of SEOB, exactly incorporate the linear-in-spin coupling of a spinning particle with a Kerr black hole.

4. Centrifugal Radius

To clarify the meaning of $(r_c, \bar{r}_c)$ appearing in Eqs. (2) (3), let us first recall the definition of the centrifugal radius r_c^K in the Hamiltonian of a test-particle on a Kerr black hole of dimensionful spin S^K and mass M_K

$$\left[r_c^K\right]^2 = \frac{(r^2+\hat{a}^2)^2 - \hat{a}^2\Delta^K}{r^2} = r^2 + \hat{a}^2\left(1 + \frac{2}{r}\right), \quad (4)$$

where $\hat{a} \equiv S^K/M_K^2$ and

$$\Delta^K = r^2\left(1 - \frac{2}{r} + \frac{\hat{a}^2}{r^2}\right). \quad (5)$$

In TEOBresumS, r_c^2 is obtained by first replacing $\hat{a} \to \tilde{a}_0$, which takes into account, in resummed form, of spin-spin interaction at leading order, and then by adding a correction, $r_c^2 \to r_c^2 + \delta a^2/r$, to incorporate next-to-leading-order (or even higher) spin-spin terms through the function δa^2 [4, 7]. In SEOB, one replaces in Eq. (4) $\hat{a} \to \hat{S}$ which yields

$$\bar{r}_c^2 = \frac{(r^2+\hat{S}^2)^2 - \hat{S}^2\Delta(r)}{r^2} = r^2 + 2\hat{S}^2 + \frac{\hat{S}^2}{r^2} - \hat{S}^2\Delta_u, \quad (6)$$

where $\Delta_u \equiv \Delta/r^2$ and Δ is obtained from Δ^K by replacing $1 - 2/r$ with the full orbital EOB potential $A_{\rm orb}$, in some resummed form, as detailed below. In the following we also use $u_c \equiv 1/r_c$ and $\bar{u}_c \equiv 1/\bar{r}_c$.

5. A Function

For a Kerr black hole, the A^K function is written as

$$A^K = \left(1 - 2u_c^K\right)\frac{1 + 2u_c^K}{1 + 2u}. \quad (7)$$

In the EOB formalism, this function is modified by ν-dependent higher PN corrections that are then suitably resummed. The PN-expansion of the orbital A function is analytically fully known at 4PN order $A^{\rm PN}_{\rm orb} = 1 - 2u + 2\nu u^3 + \nu a_4 u^4 + \nu\left(a_5^c + a_5^{\ln}\ln u\right)u^5$. In TEOBResumS, the 4PN $A^{\rm PN}_{\rm orb}$ is first augmented by an effective 5PN corrections $(a_6^c + a_6^{\ln}\ln u)u^6$ where a_6^c is a parameter informed by NR simulations after it is resummed by a $(1, 5)$ Padé approximant that yields

$$A = P^1_5\left[A^{\rm PN}_{\rm orb}(u_c)\right]\frac{1 + 2u_c}{1 + 2u}. \quad (8)$$

For SEOB, one starts by observing that Eq. (7) can be expanded as

$$A^K = \frac{1 - 2u + \hat{a}^2u^2}{(1 + \hat{a}^2u^2)^2 - \hat{a}^2u^2(1 - 2u + \hat{a}^2u^2)}. \quad (9)$$

Then, the function $\mathbb{A}$ is obtained by first replacing $\hat{a} \to \hat{S}$ and then substituting the block $1 - 2u + \hat{S}^2u^2$ with the resummed potential Δ_u given by [3]

$$\Delta_u = \hat{S}^2(u - u_+)(u - u_-) \times \left[1 + \nu\Delta_0 + \nu\ln\left(1 + \sum_{i=1}^{5}\Delta_i u^i\right)\right], \quad (10)$$

that is $\Delta_u \to u^2\hat{S}^2 + A^{\rm PN}_{\rm orb}$ when $u \to 0$. We finally have

$$\mathbb{A} = \frac{\Delta_u}{(1 + \hat{S}^2u^2)^2 - \hat{S}^2u^2\Delta_u}. \quad (11)$$

6. B function

The B potential for TEOBResumS [4] is defined as $AB = (u_c/u)^2 D$, where $D^{-1}_{\rm TEOB} = 1 + 6\nu u_c^2 + 2(26 - 3\nu)\nu u_c^3$ is taken at 3PN accuracy and resummed with its inverse Taylor representation. For SEOB [3], the same equation holds, provided that u_c is replaced by $\bar{u}_c$ and the 3PN-accurate D reads $D^{-1}_{\rm SEOB} = 1 + \ln\left[1 + 6\nu u^2 + 2(26 - 3\nu)\nu u^3\right]$.

7. Spin sector

In TEOBResumS the gyro-gravitomagnetic functions are written in factorized form as

$$G_S = G^0_S\hat{G}_S, \quad (12)$$

$$G_{S^*} = G^0_{S^*}\hat{G}_{S^*} \quad (13)$$

where $G^0_S = 2uu_c^2$ mimicks the corresponding Kerr coupling $2u[u_c^K]^2$ [4] while $G^0_{S^*} \equiv 3/2u_c^3$ is the leading order spinning-particle limit [8, 9] where $u \to u_c$. The functions $\hat{G}_S$ and $\hat{G}_{S^*}$ encode high-PN spin-orbit corrections up to next-to-next-to-leading order in Damour-Jaranowski-Schäfer (DJS) gauge [10], that is chosen so as to remove the dependence on the total momentum square $p^2 = p_\varphi^2 u^2 + p_r^2$.
In SEOB, we have

$$\bar{G}^0_S = 2u\bar{u}_c^2. \quad (14)$$

Then, recalling that the spin-orbit coupling function of a spinning particle on Kerr [8] can be rewritten as [9]

$$G^K_{S^*} = \frac{\sqrt{A^K}}{\sqrt{Q}}[u_c^K]^2\left(1 - \frac{u^2}{[u_c^K]^2}\frac{\partial_u u_c^K}{\sqrt{B^K}}\right) - \frac{u_c^K u^2}{1 + \sqrt{Q}}\frac{\partial_u\sqrt{A^K}}{\sqrt{B^K}} \quad (15)$$

where the functions (u_c, A^K, B^K) and $Q \equiv 1 + p_\varphi^2u_c^2 + p_{r^*}^2/A^K$ are replaced by $(\bar{u}_c, \mathbb{A}, \mathbb{B})$ of above

$$\mathbb{G}_{\mathbb{S}^*} = \frac{\sqrt{\mathbb{A}}}{\sqrt{\mathbb{Q}}}\bar{u}_c^2\left(1 - \frac{u^2}{\bar{u}_c^2}\frac{\partial_u\bar{u}_c}{\sqrt{\mathbb{B}}}\right) - \frac{\bar{u}_c u^2}{1 + \sqrt{\mathbb{Q}}}\frac{\partial_u\sqrt{\mathbb{A}}}{\sqrt{\mathbb{B}}}, \quad (16)$$

and $\mathbb{Q} = 1 + p_\varphi^2\bar{u}_c^2 + p_{r^*}^2/\mathbb{A}$. Finally, note that the "effective spin" $\hat{\mathbb{S}}_*$ is actually a linear combination of $(\hat{S}, \hat{S}^*)$ with coefficient that are, in general, functions of $\left(\mathbb{Q} - 1, p_{r^*}^2, u\right)$. More precisely [6]

$$\mathbb{S}^* = \hat{S}^* + \Delta^{(1)}_{\sigma^*} + \Delta^{(2)}_{\sigma^*}, \quad (17)$$

where $\Delta^{(i)}_{\sigma^*} = c^{(i)}_S\hat{S} + c^{(i)}_{S^*}\hat{S}^*$, so that the spin-orbit effective Hamiltonian of SEOB can be formally written as the one of TEOB, i.e.

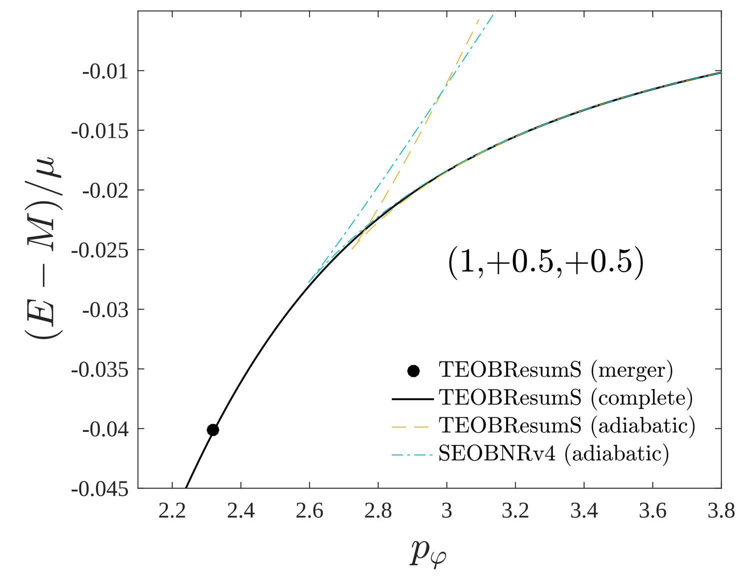
$$\left\{\bar{G}^0_S + \left[c^{(1)}_S + c^{(2)}_S\right]\mathbb{G}_{\mathbb{S}^*}\right\}\hat{S}p_\varphi + \left[1 + c^{(1)}_{S^*} + c^{(2)}_{S^*}\right]\mathbb{G}_{\mathbb{S}^*}\hat{S}^*p_\varphi. \quad (18)$$

Finally, note that the functions $\{c^{(i)}_S, c^{(i)}_{S^*}\}$ *are not* written in DJS gauge but in another gauge that puts to zero any explicit dependence on p_r^2, so that the dependence on the momenta is accomplished only through the function $\mathbb{Q}$.
In SEOB the higher spin orders are introduced in a new parts of the Hamiltonian [6], $\hat{H}_{ss}$,

$$\hat{H}_{ss} = \left[\bar{G}^0_s + \frac{u^2}{2\bar{u}_c}\frac{\mathbb{A}}{\sqrt{\mathbb{B}}}\left(\frac{1}{\sqrt{\mathbb{Q}}(1 + \sqrt{\mathbb{Q}})}\left(\frac{p_{r^*}^2}{\mathbb{A}} - p_\varphi^2\bar{u}_c^2\right) - 1\right) \times \partial_u\bar{G}^0_s\right]\mathbb{S}^*\hat{S} - \frac{u^3}{2}(\mathbb{S}^*)^2. \quad (19)$$

8. Binding energy

We compare the binding energies along the adiabatic circular dynamics for the illustrative binary $(q, \chi_1, \chi_2) \equiv (1, +0.5, +0.5)$. All the NR-informed coefficients of Ref. [11] are included in the SEOB Hamiltonian (SEOBNRv4). The complete TEOBResumS energetics is also added for completeness. The cusp corresponds to the Last Stable Orbit.



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