Linear perturbations of an anisotropic Bianchi I model with a uniform magnetic field

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Theory of cosmological perturbations with magnetic fields

- FRW models [4]: no GR effects
- GR models in 3+1 covariant formalism: they usually work in isotropic limit for simplicity [3]. Some works consider anisotropy, but don't solve the equations [2]
- Our approach [5]: we start from GR 3+1 equations and solve them in synchronous gauge:
  - No obvious physical association between variables and algebric modes
  - Easy to maniplate equations
  - Gauge modes must be taken care of
  - **Anisotropy is considered in a correct way**, as required by the presence of the magnetic field

Background model with barotropic fluid: anisotropic evolution

Magnetic field along x_3 implies an anisotropic evolution:

Metric tensor

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^2(t) & & \\ & & a^2(t) & \\ & & & c^2(t) \end{pmatrix}$$

Energy coservation relations

Fluid energy: $\dot\rho + \left(2\frac{\dot a}{a} + \frac{\dot c}{c}\right)(\rho + p) = 0$

Magnetic field energy: $(\dot{B^2}) + 4\frac{\dot a}{a}B^2 = 0$

Fluid pressure $p = w\rho$

Einstein equations

$$2\frac{\ddot a}{a} + \frac{\ddot c}{c} = -\frac{1}{2}\left(\rho + 3p + B^2\right)$$

$$\frac{\ddot a}{a} + \frac{\dot a}{a}\left(\frac{\dot a}{a} + \frac{\dot c}{c}\right) = \frac{1}{2}\left(\rho - p + B^2\right)$$

$$\frac{\ddot c}{c} + 2\frac{\dot a \dot c}{a c} = \frac{1}{2}\left(\rho - p - B^2\right)$$

Defining

$$v_A^2 = \frac{B^2/2}{\rho} \qquad 3H = 2\frac{\dot a}{a} + \frac{\dot c}{c} \qquad S = \frac{1}{H}\left(\frac{\dot a}{a} - \frac{\dot c}{c}\right)$$

we have

Radiation dominated universe $w = 1/3$**:**

$$v_A^2 = v_{A0}^2 = \text{const}$$
$$S = 6v_A^2$$
$$H = \frac{1}{2t}$$
$$\rho = \frac{3}{4t^2}(1 - v_{A0}^2)$$

Matter dominated universe $w = 0$**:**

$$v_A^2 = v_{A0}^2\left(\frac{t}{t_0}\right)^{-2/3}, \quad v_{A0}^2, t_0 = \text{const}$$
$$S = 12v_A^2(t)$$
$$H = \frac{2}{3t}(1 - v_A^2(t))$$
$$\rho = \frac{4}{t^2}\left(\frac{1}{3} - v_A^2(t)\right)$$

This is an improvement over the results of [1].

Perturbations in synchronous gauge

We require the gauge to be synchronous at perturbative level:

Fluid energy: $\delta\rho$
Fluid pressure: $\delta p = v_S^2\delta\rho$
Fluid velocity: $\delta u^\mu, \quad \delta u^0 = 0$

Metric: $\delta g_{\mu\nu}, \quad \delta g_{0\mu} = \delta g_{\mu 0} = 0$
Magnetic field: $\delta(B^2)$

We require the magnetic field to be purely spatial with respect to the fluid:

Spatial projector: $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$
$\delta(h^\nu_\mu \nabla_\nu B^\mu) = 0$

Gauge modes

With a generic coordinate transformation

$$x^\mu \to x^\mu + \epsilon^\mu$$

imposing the synchronous gauge we find

$$\epsilon^0 = \epsilon^0(x^j)$$

$$\epsilon^P = \tilde\epsilon^P(x^j) + \partial^P\epsilon^0(x^j)\, a^2 \int \frac{\mathrm{d}t}{a^2}, \qquad P = 1, 2$$

$$\epsilon^3 = \tilde\epsilon^3(x^j) + \partial^3\epsilon^0(x^j)\, c^2 \int \frac{\mathrm{d}t}{c^2}$$

The gauge mode is then

$$\delta\rho_{\mathrm{G}} = -\epsilon^0\dot\rho = 3H(1+w)\rho\epsilon^0$$

Results and perspectives: small scale evolution

We define $\Delta = \dfrac{\delta\rho}{(1+w)\rho}$

We study matter dominated universe at small scales through Fourier expansion $\exp\left(\mathrm{i}k_i x^i\right)$

- Sound speed $v_S^2\big|_{z>z_{\text{rec}}} = \frac{1}{3}\frac{k_B T_b \sigma}{m_p + k_B T_b \sigma} \approx \frac{1}{3}, \qquad v_S^2\big|_{z<z_{\text{rec}}} = \frac{5}{3}\frac{k_B T_b}{m_p}$
- Baryon temperature $T_b\big|_{z>100} \propto 1 + z, \qquad T_b\big|_{z<100} \propto (1+z)^2$
- We define $\Lambda_S^2 = v_S^2 k_i k^i t^{2\gamma - 2/3} = \text{const}, \ \Lambda_A^2 = v_A^2 k_i k^i t^2 = \text{const}, \ \gamma = 4/3 + \nu$
- Small scales: $k_i k^i \gg H^2$, we discard v_S^2 and v_A^2 and we keep $k_i k^i v_S^2$ and $k_i k^i v_A^2$
- Sound speed $v_S/c \approx 10^{-7} - 10^{-5}$ for $10 < 1 + z < 1100$
- Alfvén speed $v_A/c \approx 10^{-5} - 10^{-4}$ for $10 < 1 + z < 1100$

Small anisotropies

$$\text{Hypothesis} \begin{cases} k_3 k^3 = k_i k^i/3 + \mathcal{O}\left(v_A^2\right) \\ k_3 \delta u^3 = k_i \delta u^i/3 + \mathcal{O}\left(v_A^2\right) \\ \delta g_3^3 = \delta g_i^i/3 + \mathcal{O}\left(v_A^2\right) \end{cases}$$

Case $\nu = 0$

$$\Delta(t) = \Delta_\pm \cdot (t/t_0)^{-\frac{1}{6} \pm \sqrt{\frac{25}{36} - \Lambda_S^2 - \frac{4}{3}\Lambda_A^2}}$$

We have a growing mode if $\Lambda_S^2 + \frac{4}{3}\Lambda_A^2 < 2/3$

Case $\nu = 1/3$

$$\Delta(t) \approx \Delta_\pm t^{-1/6} J_{\frac{\mp\sqrt{25 - 48\Lambda_A^2}}{6\nu}}\left(\frac{\Lambda_S}{\nu t^\nu}\right)$$

We have a growing mode if both $\frac{\Lambda_S}{\nu t^\nu} \ll 1$ and $\Lambda_A^2 < \frac{1}{2}$ hold

General case

We recover the Newtonian approximation of [4]:

$$\Delta(t) = A_n t^{x_n} {}_2\mathcal{F}_3\left[(a_{1n}, a_{2n}); (b_{1n}, b_{2n}, b_{3n}); -\frac{\Lambda_S^2}{4\nu^2 t^{2\nu}}\right], \quad n = 1, \dots, 4$$

$$x_n = (-1 \pm \sqrt{\Delta_\pm})/6, \quad \Delta_\pm = 13 - 36\Lambda_A^2 \pm 6\sqrt{(6\Lambda_A^2 - 2)^2 + 48\mu\Lambda_A^2}, \quad \mu = k_3 k^3 / k_i k^i$$

Perturbations orthogonal to $\vec B$:
Growing mode if both $\frac{\Lambda_S^2}{4\nu^2 t^{2\nu}} \ll 1$ and $\Lambda_A^2 < \frac{1}{3}$ hold

Perturbations in other directions:
Growing mode if $\frac{\Lambda_S^2}{4\nu^2 t^{2\nu}} \ll 1$

Summary of main results

- Full relativistic anisotropic model
- Perfect agreement with the Newtonian limit of [4]
- Improvement over [3]: anisotropy
- Improvement over [2]: exact solution

Perspectives

- Next perturbative order: full coupling with primordial GWs (shown in [2])
- Numerical integration of equations for arbitrary scales

References

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