**Abstract**

We show that (Eur Phys J C, 2018), the gravity can be considered as defined not by one but two fundamental constants which enables us to explain quantitatively both dark energy (the cosmological constant) in GR equations, and dark matter in weak-field limit simultaneously. Then, in order to throw more light on the nature of the constants appearing here, we generalize the Newton theorem on the sphere-point mass equivalency to arbitrary dimensions. We also turn into gravitational lensing, where this additional term predict a critical value for the involved weak-field parameter. If this value will be established at future observations, this will mark the first discrepancy with GR of the conventional weak-field limit, directly linked to the nature of the dark sector of the Universe.

**Introduction**

Newton in *Principia* had proved that at the universal gravitational law $r^{-2}$ a spherically symmetric body affects external objects gravitationally as though all of its mass is concentrated at its center.

The most general function for force satisfying Newton’s theorem is written as:$$F = \frac{G_1 m_1 m_2}{r^2} + C_2 r$$

**Identifying the Constants**

While $C_2$ is related to $G$, it is possible to find a correspondence between $C_2$ and $\Lambda$.

Considering the Einstein’s field equation (GR) together with cosmological constant:

$$\dot{g}_{\mu \nu} + Ag_{\mu \nu} = \frac{8\pi G}{c^4} \ T_{\mu \nu}$$

The weak filed limit is written as:$$g_{00} = 1 + \frac{2\Phi}{c^2} = 1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}$$

However, in order to identify $C_2$ as $\Lambda$, we need to fix its sign and numerical value.

**Dark Sector and Constants**

In the GR equations, $\Lambda$ is considered to describe the accelerated expansion of the Universe.

According to Planck data, it’s value is:$$\Lambda = 1.11 \times 10^{-57} \text{ m}^{-2}$$

The $\Lambda$-modified Newtonian gravity can be used to describe the Dark Matter.

In such case, the virial theorem will be:

$$\Lambda = \frac{3\sigma^2}{2} R d^2$$

For galaxy groups of the Hercules-Bootes region, the value of $\Lambda$ will be:

<table>
<thead>
<tr>
<th>Galaxy group</th>
<th>$\sigma$ (km/s)</th>
<th>$R_d$ (kpc)</th>
<th>$\Lambda$ (kms$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGC128</td>
<td>100</td>
<td>15</td>
<td>100</td>
</tr>
<tr>
<td>NGC106</td>
<td>150</td>
<td>10</td>
<td>150</td>
</tr>
<tr>
<td>NGC253</td>
<td>200</td>
<td>5</td>
<td>200</td>
</tr>
<tr>
<td>NGC311</td>
<td>250</td>
<td>2</td>
<td>250</td>
</tr>
<tr>
<td>NGC328</td>
<td>300</td>
<td>1</td>
<td>300</td>
</tr>
</tbody>
</table>

**Isometry Groups**

Since, $\Lambda$ was introduced not by Newton’s theorem but according to conservation of the energy-momentum tensor and General Covariance principle. So it seems quite reasonable that, to make as more powerful justification, we try to infer Newton’s theorem based on the above relativistic consideration.

Depending on $\Lambda$’s sign—positive, negative or zero—one has three different vacuum solutions (three different asymptotic limits) for the field equations as:

<table>
<thead>
<tr>
<th>Sign</th>
<th>Spacetime</th>
<th>Isometry Group</th>
<th>Curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda &gt; 0$</td>
<td>de Sitter (dS)</td>
<td>O(1, 3)</td>
<td>+</td>
</tr>
<tr>
<td>$\Lambda = 0$</td>
<td>Minkowski (M)</td>
<td>O(1, 3)</td>
<td>0</td>
</tr>
<tr>
<td>$\Lambda &lt; 0$</td>
<td>Anti de Sitter (AdS)</td>
<td>O(2, 3)</td>
<td>-</td>
</tr>
</tbody>
</table>

The interesting feature of all these 4D maximally symmetric Lorentzian geometries is that for all of them the stabilizer subgroup of isometry group is the Lorentz group $O(1, 3)$.

$$ds^2 = 0(1, 3), M = 10(1, 3), AdS = 0(2, 3)$$

In the non-relativistic limit we have:

$$0(1, 3) \rightarrow (0(3) \times 0(1, 1)) \rightarrow R^6$$

$$0(2, 3) \rightarrow (0(3) \times 0(2)) \rightarrow R^6$$

For all of them, the spatial part admits Euclidean algebra $E(3) = (0, 3) \times R^3$, which means that the group $O(3)$, is the stabilizer subgroup.

The same analysis can be generalized to arbitrary dimensions. In this case the most general form of the potential $\Phi$ satisfying the Newton’s theorem ($S^d$ point) is written as:

$$\Phi = \frac{G_2 C_2}{2d} \left( \frac{r}{d-2} \right)^{d-2} + \frac{\Lambda r^2}{2d}$$

Where the first constant can be identified as the Newton’s gravitational constant the second one is the cosmological constant. Consequently the group theoretical analysis justifies that at each point of spatial geometry we have the exact $O(d)$ symmetry.

**Observational Implications**

The use of the strong gravity lens ESO 325-G004 demonstrates the efficiency of the lens studies to constrain the weak-field GR in the intergalactic scales.

Based on the analysis of the data of ESO325-G004: $\gamma = 0.97 \pm 0.09$

Considering $\Lambda$ in our equations, the $\gamma$ parameter will be:

$$\gamma = 1 - \frac{\Lambda c^4 r^4}{12 G^2 M^2}$$

Where we have a critical parameter for ESO325-G004, $\gamma = 0.998$.

**Conclusion**

1. Gravity can be defined as not one but two constants, "G" and "$\Lambda$" which appear readily in the modified weak-field limit of General Relativity.
2. Both constants jointly are able to explain quantitatively both dark energy (the cosmological constant) in GR equations, and dark matter in weak-field limit simultaneously.
3. Considering the Einstein’s field equation (GR) together with cosmological constant:
   $$\dot{g}_{\mu \nu} + Ag_{\mu \nu} = \frac{8\pi G}{c^4} \ T_{\mu \nu}$$

4. The critical parameter, $\gamma$, if established at future observations, will mark the first discrepancy with General Relativity of conventional weak-field limit.

**Bibliography**