Nonlocal quantum field theories

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OVERVIEW

- Motivations
- Nonlocal scalar fields
- Perturbative unitarity and Cutkosky rules
- From Stelle quadratic gravity to nonlocal gravity
- Power counting renormalization
- Inflation
- Degrees of freedom
- Stability
- Causality

Nonlocal interactions

Nonlocality can been introduced within the framework of different approaches

- Efimov (1970) introduced nonlocal interactions in order to obtain a finite theory of quantized fields
- Nonlocality of fields emerges from the stochastic nature of space-time (see K. Namsrai, Nonlocal QUF and stochastic quantum mechanics, 1986)
- String inspired nonlocal QFT, A. Sen,JHEP (2016) 2016: 87
- Nonlocal quantum gravity, L. Modesto, G. Calcagni, T. Biswas, G. Lambiase, L. Buoninfante and previously Krasnikov, Y. V. Kuz'min and T. Tomboulis.

Nonlocality can be introduced in agreement with the following basic requirements:

- The theory is invariant under diffeomorphism
- Unitarity is preserved

The introduction of nonlocal interactions improves the convergence of the loop integrals and makes possible to realize a finite quantum field theory
The nonlocality scale can be introduced as a new fundamental constant or be interpreted as an effective quantity

Nonlocal interactions (scalar field)

$$\mathcal{L}_\phi = -\frac{1}{2}\varphi\left(\Box + m^2\right)\varphi - \frac{\lambda_n}{n!}(e^{-\frac{1}{2}H(-\sigma\Box)}\varphi)^n .$$

- $E^{H(-\sigma\Box)}$ is the nonlocal form factor
- $\sigma = \ell^2$ sets the nonlocality scale
- λ_n couplings

The nonlocality is contained in the interaction terms

$$e^{-\frac{1}{2}H(-\sigma\Box)}\varphi(x) = \int e^{-\frac{1}{2}H(\sigma k^2)}\tilde{\varphi}(k)\, e^{ikx} d^4k = \int d^4y\, Q(x-y)\,\varphi(y)$$

with

$Q(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-\frac{1}{2}H(\sigma k^2)} e^{-ik(x-y)}$

Therefore the non-interacting theory is equivalent to the free local theory

Form factor

The form factor $\exp H(z)$ is assumed so verify the following conditions

- $\exp H(z)$ is an entire analytic function, with no poles nor zeros on the whole complex plane $|z| < +\infty$.
  This condition ensures the avoidance of ghosts and preserves the unitarity of the theory
- $\exp H(z) \to \infty$ sufficiently rapidly for $z \to +\infty$.
  This condition is necessary in order to have a super-renormalizable or a finite theory

For such a class of form factors one can perform a field redefinition $\phi = e^{\frac{1}{2}H(-\sigma\Box)}\varphi$ that maps the Lagrangian density into the following one

$$\mathcal{L}_\phi = -\frac{1}{2}\phi\, e^{H(-\sigma\Box)}\left(\Box + m^2\right)\phi - \frac{\lambda}{n!}\phi^n\,,$$

so that the nonlocality is contained only in the kinetic part of the Lagrangian density.

Feynman rules

The two lagrangian gives equivalent Feynman rules
I)

- Propagator $\frac{-i}{(k^2-m^2+i\epsilon)e^{H(\sigma\, k^2)}}$
- Vertex $-i\,\lambda$

II)

- Propagator $\frac{-i}{k^2-m^2+i\epsilon}$
- Vertex $-i\lambda\;\; e^{-\sum_{j=1}^{n} H(\sigma\, p_j^2)}$

As a consequence of nonlocality

- If $H(z) \to \infty$ for $z \to \infty$, the convergence of the propagator is improved.
- The resulting theory can be super-renormalizable or even finite
- Since $e^{H(\sigma\, k^2)} \neq 0$ in the finite complex plane, the propagator does not have extra poles. This ensures the avoidance of ghosts.

Note that a polynomial form factor, e. g. $e^H(z) = z - m^2$ would give extra poles, while we require entire form factor will be of the form $e^H(z) = \sum_k c_k z^k$, so that the theory contains derivatives of infinite order.

Examples of form factors and power counting

Asymptotically polinomial form factor

$$e^{H(k^2\sigma)} \sim k^{2(\gamma+1)} \quad \text{for} \quad \sigma k^2 \gg 1$$

In the ultraviolet the propagator scales as $k^{-2(1+\gamma)}$
In a theory with interaction $\lambda\phi^n$ in D dimensions, the superficial degree of divergence of a diagram with L loops, and I internal lines is

$$\omega_d = D\,L - 2(\gamma+1)I$$

Using the relation $I = L + V - 1$, where V is the number of vertices in the diagram, one has

$$\omega_d = (D - 2(1+\gamma))I - D(V-1)$$

Therefore, the theory is finite for $\gamma > \frac{D}{2} - 1$

Exponential form factor $e^{H(k^2\sigma)} = e^{k^2\sigma}$
The propagator scales as $k^{-2}e^{-k^2\sigma}$, and there is exponential convergence of all the diagrams

Example of asymptotically polinomial form factor

Y.V. Kuzmin, Yad. Fiz. 50 (1989) 1630; E.T. Tomboulis,arXiv:hep-th/9702146; L. Modesto, Phys. Rev. D 86 (2012) 044005.

$$H(z) = \frac{1}{2}\left[\gamma_E + \Gamma\left(0, (p_{(\gamma+1)}(z))^2\right) + \log\left((p_{(\gamma+1)}(z))^2\right)\right]$$

where $\gamma_E = 0.577216$ is the Euler's constant,

$$p_{(\gamma+1)} = z^{(\gamma+1)} + c_1 z^\gamma + \ldots$$

is a polynomial of order $2(\gamma+1)$ and

$$\Gamma(a,z) = \int_z^{+\infty} t^{a-1} e^{-t} dt$$

One has

$$e^{H_i(z)} \sim e^{\gamma_E/2} p_{(\gamma+1)} \sim |z|^{\gamma+1} \quad \text{for} z \to +\infty$$

Perturbative unitarity

The unitarity condition $S^\dagger S = 1$ can be expressed in terms of the T-matrix as

$$T - T^\dagger = i\, T^\dagger T$$

where the T-matrix is defined by $S = 1 + iT$. For a given process $a \to b$, one has

$$T_{ba} - T^*_{ab} = i \sum_c T^*_{cb} T_{ca}$$

Recasting the T-matrix as $T_{ab} = (2\pi)^4 \, \mathcal{M}_{ab} \, \delta^{(4)}\left(\sum_i p_i - \sum_f p_f \right)$, the unitarity condition is finally expressed by

$$\mathcal{M}_{ba} - \mathcal{M}^*_{ab} = i \sum_c \mathcal{M}^*_{cb} \mathcal{M}_{ca} (2\pi)^4 \delta^{(4)}(p_c - p_a)\,,$$

where $\mathcal{M}_{ba} = \langle b|\mathcal{M}|a\rangle$ is the sum of all the connected amputated diagrams for the process $a \to b$.

Cutkosky rules and unitarity

Cutkosky rules gives a prescription for calculating the L. H. S. of the relation

$$\mathcal{M}_{ba} - \mathcal{M}^*_{ab} = i \sum_c \mathcal{M}^*_{cb} \mathcal{M}_{ca} (2\pi)^4 \delta^{(4)}(p_c - p_a) \, ,$$

for any diagram. Given such prescription, this relation is satisfied at any perturbative level.

Unitarity is proved in two steps: first one proves that Cutcosky rules are still valid in nonlocal theory, then that only normal thresholds contribute to the complex amplitudes.
The second step is more cumbersome, because one cannot use the largest time equation.

In what follows, the theory is defined in the Euclidean space, so that the loop and external energies are assumed to be purely imaginary. Then, the amplitudes are continued analytically to real values of the external energies.

Nonlocal propagator and largest time equation

Propagator

$$D_F(k) = \frac{-i}{(k^2 - m^2 + i\epsilon)e^{H(\sigma\, k^2)}}$$

Since one assumes that $e^{H(\sigma\, k^2)} \neq 0$ at finite momenta, the propagator has no extra poles, that is, there are no extra degrees of freedom, indeed no ghosts.

The propagator in coordinate spaces cannot be decomposed as

$$D_F(x-y) = \int \frac{d^4x}{(2\pi)^4} \frac{ie^{-i(x-y)k} e^{-H(\sigma\, k^2)}}{(k^2 - m^2 + i\epsilon)} \neq \Delta^+(x-y)\theta(x^0 - y^0) + \Delta^-(x-y)\theta(x^0 - y^0)$$

with

$$\Delta^+(x) = \frac{1}{(2\pi)^3} \int d^4 e^{ixk} \theta(k^0) \rho(k^2)$$

Therefore one cannot use the largest time equation to prove that anomalous thresholds do not contribute to the imaginary part of complex amplitudes (see t'Hooft and Veltman, Diagrammar).

Amplitudes in Euclidean spacetime for imaginary energies

$$\mathcal{M}(p_h,\epsilon) = -\frac{\lambda^V}{S_\#}\int_{(\mathcal{I}\times\mathbb{R}^3)^L}\prod_{i=1}^{L}\frac{i\,d^4k_i}{(2\pi)^4}\frac{1}{k_i^2-m^2+i\epsilon}\prod_{j=1}^{I-L}\frac{1}{q_j^2-m^2+i\epsilon}\,B(k_i,p_h)$$

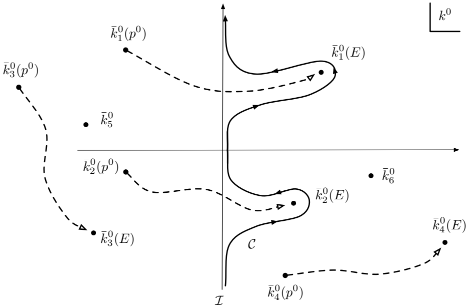
where q_j are linear combinations of k_1 and p_h and the nonlocal vertex function $\mathcal{V}(p^{(j)})$ are given by

$$\mathcal{V}(p^{(j)}) = \prod_{i=1}^{N}\exp\left[-\frac{H\left(\sigma\left(p_i^{(j)}\right)^2\right)}{2}\right], \qquad B(k_i,p_h) \equiv \prod_{j=1}^{V}\mathcal{V}(p_\ell^{(j)})$$

- External energies p_h^0 are assumed to be purely imaginary
- The k_i^0 are integrated on the imaginary axes
- The poles of the propagators are far from the integration contour
- The poles are functions of the external energies

Analytic continuation of the amplitudes for $p_h^0 \to E_h$

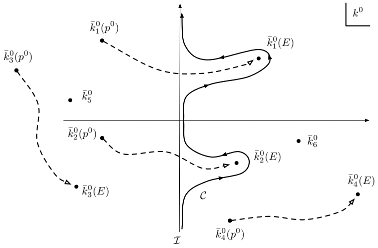
Since $B(k_i, p_h) \neq 0$, the poles are the same of the local theory
The integration contour is deformed around the poles



$$\mathcal{M}(E_h, \vec{p}_h, \epsilon) = -\frac{\lambda^V}{S_\#} \int_{(\mathcal{C}_L \times \mathbb{R}^3)} \cdots \int_{(\mathcal{C}_1 \times \mathbb{R}^3)} \prod_{i=1}^{L} \frac{i\, d^4 k_i}{(2\pi)^4} \frac{1}{k_i^2 - m^2 + i\epsilon} \prod_{j=1}^{I-L} \frac{B(k_i, E_h, \vec{p}_h)}{q_j^2 - m^2 + i\epsilon}$$

Singularity structure of the amplitudes

- After the limit $p_h^0 \to E_h$ is taken, the deformed integration contour is the same as in local theory after Wick rotation
- Singularities and branch cuts emerge in the limit $\epsilon \to 0$ when couples of poles merge, trapping the integration contour
- The poles are the same of the local theory (still given by Landau equations)
- At a Landau pole, two of more of the propagators are on-shell
- The discontinuity at a branch cut close to a pole is $(-2\pi i) \times Residue$ (see Cutkosky 1960)



Cutkosky rules

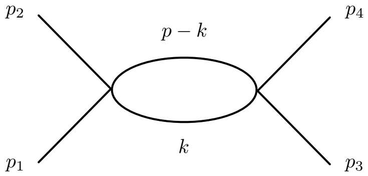
Cutkosky rules gives a prescription for calculating $\mathcal{M}_{ba} - \mathcal{M}^*_{ab}$ at a Landau pole, when some of the internal propagators are on shell.
Each propagator on-shell must be replaced with $\to (-2\pi i)\,\delta(p^2 - m^2)$
For instance, given the amplitude

$$\mathcal{M}(E_h, \epsilon) = -\frac{\lambda^2}{2} \int \frac{i\, d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(k-p)^2 - m^2 + i\epsilon}$$

at the Landau pole where the two propagators are on-shell

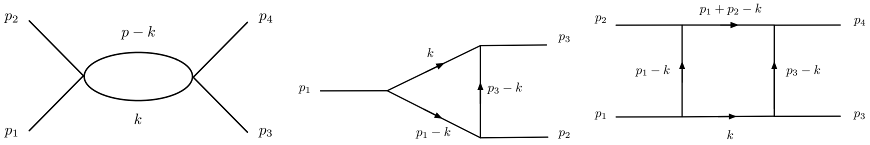
$$\mathcal{M} - \mathcal{M}^* = -\frac{\lambda^2}{2} \int_{(\mathbb{R}^4)} \frac{i\, d^4k}{(2\pi)^4} (-2\pi i)^2\, \sigma(k^0)\delta(k^2 - m^2)\sigma(p^0 - k^0)\delta((p-k)^2 - m^2)$$

$$\int_{(\mathbb{R}^4)} d^4k\, \sigma(k^0)\delta(k^2 - m^2) \to \int_{(\mathbb{R}^3)} \frac{d^4k}{2E_k} \Rightarrow \mathcal{M}_{ba} - \mathcal{M}^*_{ab} = i \sum_c \mathcal{M}^*_{cb}\mathcal{M}_{ca} (2\pi)^4 \delta^{(4)}(p_c - p_a)$$



Cutkosky rules and unitarity

- Normal thresholds: cut lines divide the diagram in two part
- Anomalous thresholds: cut lines do not divide the diagram in two part



Cutkosky rules ensure that at normal thresholds

$$\mathcal{M}_{ba} - \mathcal{M}^*_{ab} = i \sum_c \mathcal{M}^*_{cb} \mathcal{M}_{ca} (2\pi)^4 \delta^{(4)}(p_c - p_a) \,,$$

The unitarity of diagrams comes from Cutkoski rules and from the fact that anomalous thresholds do not contribute to $\mathcal{M}_{ba} - \mathcal{M}^*_{ab}$.

Cutkosky rules in nonlocal theory

The discontinuity in the imaginary part of the propagator, i. e. ,

$$\lim_{\epsilon \to 0} \{\mathcal{M}_{ba}(E_h, \epsilon) - \mathcal{M}^*_{ab}(E_h, \epsilon)\} \, .$$

corresponding to a specific pole, is obtained from the amplitude

$$\mathcal{M}(E_h, \vec{p}_h, \epsilon) = -\frac{\lambda^V}{S_\#} \int_{(\mathcal{C}_L \times \mathbb{R}^3)} \cdots \int_{(\mathcal{C}_1 \times \mathbb{R}^3)} \prod_{i=1}^{L} \frac{i\, d^4 k_i}{(2\pi)^4} \frac{1}{k_i^2 - m^2 + i\epsilon} \prod_{j=1}^{I-L} \frac{B(k_i, E_h, \vec{p}_h)}{q_j^2 - m^2 + i\epsilon}$$

by the following replacements:

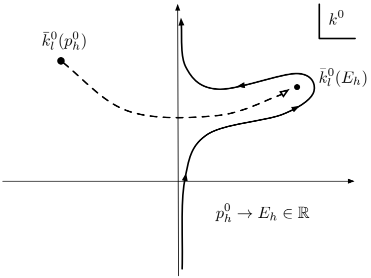
- Each propagator on-shell $\to (-2\pi i)\, \delta(p^2 - m^2)$
- For the corresponding loop momentum $(\mathcal{C} \times \mathbb{R}^3) \to \mathbb{R}^4$

This is due to the fact that $\mathcal{M}$ falls in the hypothesis of Cutkosky theorerm (Cutkosky 1960)

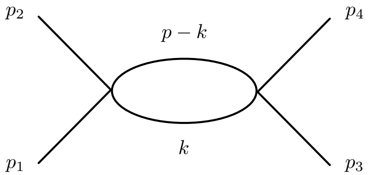
Integration technique

- We have to integrate $\mathcal{M}$ one loop at time
- For each integration, we have to study the positions of the poles and determine which pass through $\mathcal{I}$ for $p_h^0 \to E_h$
- For each integration we use the relation

$$\int_{\mathcal{C}\times\mathbb{R}^3} f(k,p_h)d^4k = \int_{\mathcal{I}\times\mathbb{R}^3} f(k,p_h)\,d^4k + (2\pi i)\int_{\mathbb{R}^3}\sum_l \mathrm{Res}\left\{f(k,p_h),\bar{k}_l^0(\vec{k},p_h)\right\}\,d^3k$$



One loop diagram



$$\mathcal{M}(p_h,\epsilon) = -\frac{\lambda^2}{2}\int_{(\mathcal{I}\times\mathbb{R}^3)} \frac{i\,d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2+i\epsilon}\,\frac{B(k_i,p_h)}{(k-p)^2-m^2+i\epsilon}$$

with $B(k_i,p_h) = \mathcal{V}(p_1,p_2,k,p-k)\,\mathcal{V}(p-k,k,p_3,p_4)$

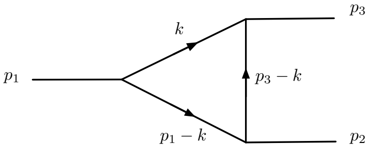
We have to find the analytic continuation to real energies

$$\mathcal{M}(E_h,\epsilon) = -\frac{\lambda^2}{2}\int_{(\mathcal{C}\times\mathbb{R}^3)} \frac{i\,d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2+i\epsilon}\,\frac{B(k_i,p_h)}{(k-p)^2-m^2+i\epsilon}$$

Cutkosky rules

Since $p_1^2 > p_2^2 + p_3^2$, the kinematics implies that the only propagators that can go on-shell together are those involving the momenta k and $p_1 - k$, so that one has

$$\mathcal{M}(p_h, \epsilon) - \mathcal{M}(p_h, \epsilon)^* = \int_{(\mathbb{R}^4)} \frac{i\, d^4k}{(2\pi)^4} (-2\pi i)^2 B(k, p_1, p_2, p_3)$$
$$\times \frac{\sigma(k^0)\, \sigma(p_1^0 - k^0)}{(k - p_3)^2 - m_3^2 + i\epsilon}\, \delta(k^2 - m_1^2)\, \delta((p_1 - k)^2 - m_2^2)$$



No contribution from the anomalous threshold!

Absence of anomalous thresholds

In the nonlocal theory one has

$$\mathcal{M}-\mathcal{M}^* = \sum \int_{\Omega_1} \cdots \int_{\Omega_L} \prod_{i=1}^{L} \frac{i\, d^4 k_i}{(2\pi)^4} \prod_{k=1}^{N} (-2\pi i)\, \delta(Q_k^2 - m^2)\sigma(Q_k^0) \prod_{j=1}^{I-N} \frac{B(k_i, p_h)}{Q_j^2 - m^2 + i\epsilon}$$

One has at most $L+1$ cuts (e. g., two cuts in the triangle and square diagrams)
Let us consider the local scalar field obtained assuming unitary form factor

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \lambda \sum_{n=4}^{N} \frac{c_n}{n!} \varphi^n \, .$$

In this case one has

$$\mathcal{M}-\mathcal{M}^* = \sum \int_{\Omega_1} \cdots \int_{\Omega_L} \prod_{i=1}^{L} \frac{i\, d^4 k_i}{(2\pi)^4} \prod_{k=1}^{N} (-2\pi i)\, \delta(Q_k^2 - m^2)\sigma(Q_k^0) \prod_{j=1}^{I-N} \frac{1}{Q_j^2 - m^2 + i\epsilon}$$

which contains the same terms with the same delta functions as in the nonlocal case.
Therefore anomalous diagrams cannot contribute in the nonlocal case!

Unitarity

- Cutkosky rules have been generalized to nonlocal theories
- Cutkosky rules ensure that the following relation is satisfied for normal thresholds

$$\mathcal{M}_{ba} - \mathcal{M}^*_{ab} = i \sum_c \mathcal{M}^*_{cb} \mathcal{M}_{ca} \, (2\pi)^4 \delta^{(4)}(p_c - p_a),$$

- Anomalous thresholds do not contribute to the imaginary part of the amplitudes
- Indeed the perturbative unitarity is proven!

Propagator

$$D_F(k) = \frac{-i}{(k^2 - m^2 + i\epsilon) e^{H(\sigma\, k^2)}}$$

See FB, L. Modesto, arXiv:1803.08827 [gr-qc] for more details and examples.
In agreement with similar results by T. Tomboulis (2018) and Pius & Sen (2018)

From Stelle theory to nonlocal gravity

It is a well established fact that the Stelle theory

$$S = -\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R + \alpha R_{\mu\nu}R^{\mu\nu} + \alpha R^2\right] \,,$$

is renormalizable, because it contains derivatives of fourth order, Stelle, PRD **16** (1977) 4.

However, Stelle gravity is plagued by ghosts. In facts the propagator contains new poles.

The occurrence of ghosts can be avoided introducing nonlocal form factors; see L. Modesto, L. Rachwa l, Nucl. Phys. B 889 (2014) 228 [arXiv:1407.8036].

$$S_{\rm g} = -\frac{2}{\kappa_D^2}\int d^4x\sqrt{-g}\left[R + R\,\gamma_0(-\Box)\,R + R_{\mu\nu}\,\gamma_2(-\Box)\,R^{\mu\nu})\right] \,.$$

where R, $R_{\mu\nu}$ and $G_{\mu\nu}$ are the Ricci scalar, Ricci curvature and the Einstein tensor respectively. Moreover, $V(\mathcal{R})$ is a generalized potential at least quadratic in $\mathcal{R}$ (scalar, Ricci or Riemann curvatures, and derivatives thereof). Finally, the form factors $h_{0,2}(-\sigma\Box)$ are defined by

$$\gamma_{0,2}(\Box) \equiv c_{0,2}\,\frac{e^{H_{0,2}(-\sigma\Box)} - 1}{\Box} \,,$$

Graviton Propagator

Consider the flat Minkowski background plus perturbations $g_{\mu\nu} = \eta_{\mu\nu} + \kappa_D \, h_{\mu\nu}$, the lagrangian of the gravitational field reads

$$\mathcal{L}_{\mathrm{g}} = \mathcal{L}_{\mathrm{g}} = \frac{1}{2} h^{\mu\nu} \mathcal{O}_{\mu\nu,\rho\sigma} h^{\rho\sigma}$$

where we have added a gauge fixing term

$$\mathcal{L}_{\mathrm{GF}} = \xi^{-1} \partial^{\nu} h_{\mu\nu} w(-\sigma\Box) \partial_{\rho} h^{\rho\mu}$$

The propagator is

$$\mathcal{O}^{-1} = \frac{\xi(2P^{(1)} + \bar{P}^{(0)})}{2k^2 \, w(\sigma k^2)} + \left(\frac{P^{(2)}}{k^2 e^{H_2(\sigma k^2)}} - \frac{P^{(0)}}{2k^2 e^{H_0(\sigma k^2)}} \right)$$

Projectors

$$P^{(2)}_{\mu\nu,\rho\sigma}(k) = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma}$$

$$P^{(1)}_{\mu\nu,\rho\sigma}(k) = \frac{1}{2}\left(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}\right)$$

$$P^{(0)}_{\mu\nu,\rho\sigma}(k) = \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma}$$

$$\bar{P}^{(0)}_{\mu\nu,\rho\sigma}(k) = \omega_{\mu\nu}\omega_{\rho\sigma}$$

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$$

$$\omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}$$

Power counting

Imposing the asymptotic behavior $e^{H_0(\sigma k^2)} \sim e^{H_2(\sigma k^2)} \sim k^{2(\gamma+1)}$, the propagator scales as

$$\mathcal{O}^{-1} \sim k^{-2} e^{-H(\sigma k^2)} \sim \frac{1}{k^{2(\gamma+2)}}$$

Interactions coming from the terms

$$\int d^4x\sqrt{-g}\,[\mathcal{R}\,\gamma(-\Box)\,\mathcal{R}] = \int d^4x\sqrt{-g}\left[\mathcal{R}\,\frac{H(-\sigma\Box)-1}{\Box}\,\mathcal{R}\right] \simeq -\int d^4x\,[h\Box h\,\Box^{\gamma+1}h + \ldots]$$

so that the leading contribution at each vertex scales as $k^{2(\gamma+2)}$

The superficial degree of divergence is

$$\delta = 4L - I\times 2(\gamma+2) + V\times 2(\gamma+2)$$

and using $I = V + L - 1$ one has

$$\delta = 4 - 2\gamma(L-1)$$

If $\gamma \geqslant 2$ one has only one loop divergences and the theory is super-renormalizable.

Finite theory

Since the theory has only one loop divergences, we can make it finite introducing extra terms that cancel such divergences.
L. Modesto, L. Rachwal, Jul 30, Nucl.Phys. B 889 (2014) 228-248
As we saw, quadratic terms has the following ultraviolet behaviour

$$\int d^4x\sqrt{-g}\,[\mathcal{R}\,\gamma(-\Box)\,\mathcal{R}] \sim k^{2(\gamma+2)}$$

Therefore one can introduce a class of extra terms in the action

$$S_{\rm g} = -\frac{2}{\kappa_D^2}\int d^4x\sqrt{-g}\,[R + R\,\gamma_0(-\Box)\,R + R_{\mu\nu}\,\gamma_2(-\Box)\,R^{\mu\nu} + V(\mathcal{R})]\ .$$

where

$$V(\mathcal{R}) = \sum_{k=3}^{\gamma+2}\sum_i \left(\Box^{\gamma+2-k}\,\mathcal{R}^k\right) c_{k,i} = RR_{\mu\nu}\Box^{\gamma-1}R^{\mu\nu} + R^3\Box^{\gamma-2}R + \ldots \sim k^{2(\gamma+2)}$$

in such a way that these terms cancel the divergences in loop diagrams.

Emergence of R^2 gravity in SRQG

Consider the SRQG Lagrangian in 4 dimensions

$$\mathcal{L} = R + R\,\gamma_0(\Box)R + R_{\mu\nu}\gamma_2(\Box)R^{\mu\nu}$$

At lowest order, using

$$\gamma(\Box) = \frac{e^{H(\sigma\Box)} - 1}{\Box} \simeq \sigma$$

SRQG reduces to Starobinsky theory

$$\mathcal{L} = R + \frac{\sigma}{6}R^2 + O\left(\sigma^2 R\,\Box R\right)$$

when $R\Box R \ll \sigma R^2$ the model reduces to Starobinsky $f(R)$ gravity.

FB, A. Marciano, L. Modesto, E. N. Saridakis, PRD 87 (2013) no.8, 083507
FB, L. Mosesto, S. Tsujikawa, PRD 89 (2014) no.2, 024029
S. Koshelev, L. Modesto, L. Rachwal, A. A. Starobinsky, JHEP 1611 (2016) 067

SRQG mimics R^2 gravity

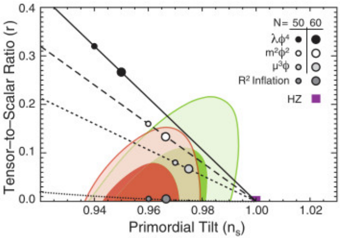
$$\frac{1}{\ell M_{\rm pl}} = 1.3 \times 10^{-5} \left(\frac{55}{N}\right)$$

The scalar spectral index $n_s - 1 \equiv d\ln \mathcal{P}_{\mathcal{R}}/d\ln k|_{k=aH}$ is

$$n_s - 1 = -\frac{2}{N} = -3.6 \times 10^{-2} \left(\frac{55}{N}\right)$$

the tensor-to-scalar ratio $r \equiv \mathcal{P}_h/\mathcal{P}_{\mathcal{R}}$ is

$$r = \frac{12}{N^2} = 4.0 \times 10^{-3} \left(\frac{55}{N}\right)^2$$



Maximal content on Minkowski spacetime

The Starobinsky model is an $f(R)$ gravity model, indeed it contains an extra scalar degree of freedom.

To reproduce Starobinsky inflation, the nonlocal model must contain (and in facts it contains at most) one extra scalar degree of freedom without breaking unitarity. This is achieved introducing a form factor with one zero by the replacement

$$e^{H_0} \to e^{H_0(\Box)} \frac{k^2 - m^2}{m^2}$$

$$\mathcal{O}^{-1} = \frac{P^{(2)}}{k^2 e^{H_2}} - \frac{m^2 P^{(0)}}{2k^2 e^{H_0}(k^2 - m^2)} = \frac{P^{(2)}}{k^2 e^{H_2}} - \frac{P^{(0)}}{2k^2 e^{H_0}} + \frac{P^{(0)}}{2e^{H_0}(k^2 - m^2)}$$

Classical solutions

We consider the following minimal nonlocal action for the gravitational field,

$$S_{\rm g} = -\frac{2}{\kappa_D^2} \int d^4x \sqrt{-g} \left[R + G_{\mu\nu} \gamma(\Box) R^{\mu\nu} + V(\mathcal{R}) \right] .$$

where $V(\mathcal{R})$ is a generalized potential at least cubic in the Ricci tensor and scalar.

The equations of motion for the action read

$$E_{\mu\nu} \equiv \left(1 + \Box\,\gamma\left(\Box\right)\right) G_{\mu\nu} + \left(g_{\mu\nu} \nabla_\alpha \nabla_\beta - g_{\alpha\mu} \nabla_\beta \nabla_\nu\right) \gamma\left(\Box\right) G^{\alpha\beta} + Q_{2\mu\nu}\left(\mathrm{Ric}\right) = 8\pi G_{\rm N} T_{\mu\nu}$$

where $Q_{2\mu\nu}\left(\mathrm{Ric}\right)$ is at least quadratic in the Ricci tensor and scalar, e.g.

$$\sigma \left((\sigma\Box)^n R_{\mu\alpha} \right) \left((\sigma\Box)^m R^\alpha{}_\nu \right) \quad \text{or} \quad \sigma^2 \left((\sigma\Box)^n R_{\mu\alpha} \right) \left((\sigma\Box)^m R^\alpha{}_\nu \right) \left((\sigma\Box)^l R \right) ,$$

At classical level, all the solutions of Einstein's gravity in vacuum are solutions of NLG too

$$G_{\mu\alpha} = 0 \Rightarrow E_{\mu\alpha} = 0$$

However, NLG has more solutions, e. g., $G_{\mu\alpha} = -\frac{1}{c}\, g_{\mu\alpha}$ for $Q_{2\mu\nu}\left(\mathrm{Ric}\right) = c\, R_{\mu\alpha} R^\alpha{}_\nu$

Dynamics of small perturbations in NLG

FB, L. Modesto, arXiv:1811.05117 [gr-qc]
Let us consider small perturbations of the Minkowski metric, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, h_{\mu\nu}, \qquad \text{with} \qquad |\epsilon\, h_{\mu\nu}| \ll 1 \tag{0}$$

and expand all the terms in the EoM in powers of ϵ

$$h_{\mu\nu} = \sum_{n=0}^{\infty} \epsilon^n h_{\mu\nu}^{(n)} \quad \text{and} \quad G_{\mu\nu}(g_{\mu\nu}) = \sum_{n=0}^{\infty} \epsilon^n G_{\mu\nu}^{(n)}\,, \text{with } G_{\mu\nu}^{(0)} \equiv G_{\mu\nu}(\eta) = 0$$

$$\nabla_\alpha = \sum_{n=0}^{\infty} \epsilon^n \nabla_\alpha^{(n)} = \partial_\alpha + \sum_{n=1}^{\infty} \epsilon^n \nabla_\alpha^{(n)}\,,$$

$$\left(g_{\mu\nu}\nabla_\alpha\nabla_\beta - g_{\alpha\mu}\nabla_\beta\nabla_\nu\right)\gamma\left(\Box\right) = \left(\eta_{\mu\nu}\partial_\alpha\partial_\beta - \eta_{\alpha\mu}\partial_\beta\partial_\nu\right)\gamma\left(\Box^{(0)}\right) + \sum_{n=1}^{\infty}\epsilon^n A_{\mu\nu\alpha\beta}^{(n)}\,,$$

$$\left(1 + \Box\,\gamma\left(\Box\right)\right) = 1 + \Box^{(0)}\,\gamma\left(\Box^0\right) + \sum_{n=1}^{\infty}\epsilon^n f^{(n)}$$

Inserting the previous expansions in the EoM one has the chain of implications

$$G^{(0)}_{\mu\nu} = 0 \quad \Rightarrow \left(1 + \Box^{(0)}\,\gamma\left(\Box^{(0)}\right)\right) G^{(1)}_{\mu\nu} = 0 \quad \Rightarrow G^{(1)}_{\mu\nu} = 0$$

$$G^{(0)}_{\mu\nu} = 0 \quad \text{and} \quad G^{(1)}_{\mu\nu} = 0 \quad \Rightarrow \left(1 + \Box^{(0)}\,\gamma\left(\Box^{(0)}\right)\right) G^{(2)}_{\mu\nu} = 0 \quad \Rightarrow G^{(2)}_{\mu\nu} = 0$$

and in general

$$G^{(k)}_{\mu\nu} = 0 \quad \text{for} \quad k \leq n \quad \Rightarrow \left(1 + \Box^{(0)}\,\gamma\left(\Box^{(0)}\right)\right) G^{(n+1)}_{\mu\nu} = 0 \quad \Rightarrow G^{(n+1)}_{\mu\nu} = 0$$

Therefore one has that

$$E_{\mu\nu} = 0 \quad \Rightarrow \quad G^{(n)}_{\mu\nu} = 0 \quad \forall n \quad \Rightarrow \quad G_{\mu\nu}(g_{\mu\nu}) = G_{\mu\nu}(\eta_{\mu\nu} + \epsilon h_{\mu\nu}) = \sum_{n=1}^{\infty} \epsilon^n G^{(n)}_{\mu\nu} = 0\,.$$

In the harmonic gauge and for asymptotically at initial data satisfying the global smallness assumption, the solutions of vacuum Einstein equations converges asymptotically in time to Minkowski spacetime, i. e., $h_{\mu\nu} \sim t^{-1}\ln(t)$ for $t \gg t_0$. That implies that asymptotic series are valid at any time. See H. Lindblad, I. Rodnianski, Ann. of Math. 171 (2010), 14011477. H. Lindblad, Commun. Math. Phys. (2017) 353: 135.

Stability of Minkowski spacetime and GW

In NLG the dynamics of small perturbations of Minkowski spacetime is the same as in Einstein's gravity

$$E_{\mu\nu} = 0 \quad \Rightarrow \quad G_{\mu\nu}(\eta_{\mu\nu} + \epsilon h_{\mu\nu}) = 0$$

Therefore

- any Strongly Asymptotically Flat initial data set satisfying a Global Smallness Assumption entails a smooth, geodesically complete, and asymptotically at solution of NLG EoM in the vacuum, which is in facts a solution of the Einstein's equations in the vacuum.
- the dynamics of GW is the same as in Einstein's theory
- This result can be generalized to Ricci-flat and maximally symmetric metrics
  FB, G. Calcagni, L. Modesto, arXiv:1901.03267 [gr-qc]

Degrees of freedom

G. Calcagni, L. Modesto, G. Nardelli, JHEP 1805 (2018) 087
The EoM of nonlocal gravity can be rearranged as a second order integro-differential system.

$$S = \int d^4x\sqrt{-g}\,R + \int d^4x d^4y d^4z \sqrt{-g(x)}\sqrt{-g(y)}\sqrt{-g(z)}$$
$$G_{\mu\nu}(x)\left[\mathcal{G}(x,y;\sigma) - [-g(y)]^{-1/2}\delta^D(x-y)\right]\tilde{G}(y,z)\,R^{\mu\nu}(z)\,,$$

where $\Box\tilde{G}(y,z) = [-g(z)]^{-1/2}\delta^D(y-z)$.
In the case of the following form factors

- Kuz'min form factor with $H^{\rm pol}(\Box) := \alpha\{\ln(\sigma\Box) + \Gamma[0,\sigma\Box] + \gamma_{\rm E}\}$
- String-related form factor with $H^{\rm exp}(\Box) := -l^2\Box$

the function $\mathcal{G}(x,y;\sigma)$ is determined by a second order diffusion equations, and the dynamics is that a second order dynamical system with 8 degrees of freedom.

Causality violation at nonlocality scale

C. D. Carone, PRD 95, 045009 (2017).
Buoninfante, Lambiase, Mazumdar, arXiv:1805.03559 [hep-th]

Causality is violated at the nonlocality scale: particles production by classical source

$$\mathcal{L}_\phi = -\int \frac{1}{2}\varphi\left(\Box + m^2\right)\varphi + e^{-H(-\sigma\Box)}\varphi\, J\,.$$

Integrating by part the nonlocal factor can be transferred on the current J, so that the EoM is

$$\left(\Box + m^2\right)\varphi(x) = e^{-H(-\sigma\Box)}\, J(x) \equiv \tilde{J}(x)$$

The solution is expressed in terms of the local retarded green function

$$G_R(x-y) = \theta(x^0 - y^0)\int \frac{d^3k}{(2\pi)^2}\frac{1}{2E_k}\left(e^{-ik(x-y)} - e^{ik(x-y)}\right)$$

as

$$\phi(x) = \phi_0(x) + \int d^4y\, G_R(x-y)\,\tilde{J}(y)$$

Causality violation at nonlocality scale

If one consider an impulsive current $J(x) = \delta^4(x-a)$ centered at some spacetime point a, the effective current

$$\tilde{J}(x) = e^{-H(-\sigma\Box)}\delta^4(x-x_0) = \int \frac{d^4k}{(2\pi)^4} e^{-H(\sigma k^2)} e^{ikx}$$

will have an extended support, indeed there will be a contribution to the ϕ field for $x^0 < a^0$, therefore causality violation.

$$\phi(x) = \phi_0(x) + \int d^4y \, G_R(x-y)\, \tilde{J}(y)$$

- Causality violation is an open issue
- It is belived that causality violations are confided to the nonlocality scales
- In nonlocal quantum gravity a Shapiro time advance never occurs
  S. Giaccari, L. Modesto, arXiv:1803.08748 [hep-th]

Conclusions

- Nonlocality can be introduced without breaking diff. invariance and unitarity
- It makes possible to construct super-renormalizable or even finite gravitational models with nice propoerties (e. g. stability of classical solutions, nice inflationary behaviour etc.)
- It implies causality violations that seems to be cofined at the nonlocality scale

Thank you!