# Test-particle dynamics in general spherically symmetric black hole spacetimes

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Event Horizon Telescope

BlackHoleCam

erc

Goethe Universität Frankfurt am Main


Credit: Ralph Eatough/MPIfR

# Requirements

- A model-independent, "agnostic", framework that parametrizes the most generic black-hole geometries through a finite number of adjustable parameters

- The parameters must be chosen in such a way that they can measure deviations from GR and, at the same time, can be estimated robustly from the observational data

- The space-time should be represents a spherically symmetric and slowly rotating black hole

- The space time has to recover the asymptotically flatness and contains a surface where the expansion of radially outgoing photons is zero (the event horizon)

# Parameterization of spherically-symmetric BH

The Lagrangian

$$2\mathcal{L} = -\mathcal{N}(r)\dot{t}^2 + \frac{\mathcal{B}(r)}{\mathcal{N}(r)}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2$$

x=0 Event Horizon

Compactify the radial cooridinate $x \equiv 1 - \frac{r_0}{r}$

x=1 Spatial infinity

Let us rewrite $\mathcal{N}(x) = xA(x)$

Where $A(x) > 0$ for $0 \leq x \leq 1$

L. Rezzolla, A. Zhidenko Phys. Rev. D 90, 084009 (2014)

# Parameterization of spherically-symmetric BH

We further express the functions

$$A(x) = 1 - \epsilon(1 - x) + (a_0 - \epsilon)(1 - x)^2 + \widetilde{A}(x)(1 - x)^3$$

$$\mathcal{B}(x) = 1 + b_0(1 - x) + \widetilde{B}(x)(1 - x)^2$$

Where

$$\epsilon = \frac{2M - r_0}{r_0} = -\left(1 - \frac{2M}{r_0}\right)$$

$$\widetilde{A}(x) = \frac{a_1}{1 + \frac{a_2 x}{1 + \frac{a_3 x}{1 + \dots}}} \qquad \widetilde{B}(x) = \frac{b_1}{1 + \frac{b_2 x}{1 + \frac{b_3 x}{1 + \dots}}}$$

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# Theory Models

- Einsteinian gravity: Reissner-Nordström

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{r_Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- Beyond Einstein: Brans-Dicke theory

$$ds^2 = -A(r)^{m+1} dt^2 + A(r)^{-1} dr^2 + r^2 d\Omega^2$$

- Beyond Einstein: f(R ) gravity theory

$$ds^2 = -\left[1 - \frac{2M}{r} + \frac{\beta - \gamma}{2}\left(\frac{2M}{r}\right)^2\right] dt^2 + \left(1 + \gamma\frac{2M}{r}\right) dr^2 + r^2 d\Omega^2$$

- Beyond Einstein: Einstein-Maxwell-Dilaton Axion gravity theory

$$ds^2 = -\left(1 - \frac{2M\rho}{r^2}\right) dt^2 + \left(\frac{r^2}{\hat{b}^2 + r^2}\right) dr^2 + r^2 d\Omega^2 \, ,$$

# Parameterize the theories

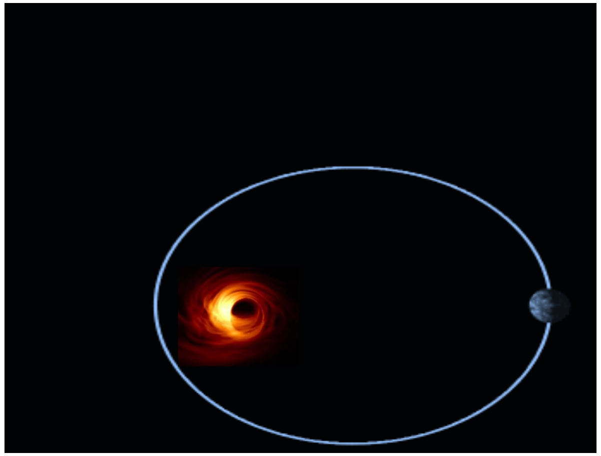
- First step: to compare the general expansions and the chosen one at spatial infinity, to find their representations at first order

- Second step: to compare the near-horizon expansions to find the other coefficients (i.e. higher order expansion)

- Third step: to test the reliability of our continued-fraction expansions

- Fourth step: to compare a number of potentially observable quantities

Periastron advance and orbital period



# Particles motion

The momenta

$$p_t = \mathcal{N}(r)\frac{dt}{d\tau} = -E \qquad p_\varphi = r^2\frac{d\varphi}{d\tau} = L$$

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot r} = \frac{\mathcal{B}(r)}{\mathcal{N}(r)} \qquad p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = r^2\,\dot\theta$$

For time-like geodesics one obtains:

$$E^2 = \mathcal{N}(r)\left(\frac{L^2}{r^2}+1\right) + \mathcal{B}(r)\,\dot r^2 \qquad L = r^2\,\dot\varphi$$

In order derive the precession,

$$\left(\frac{dr}{d\varphi}\right)^2 = \mathcal{C}r^2\left[\left(\frac{E^2}{L^2}\right)r^2 - \left(1+\frac{r^2}{L^2}\right)\mathcal{N}(r)\right] \qquad \mathcal{C} := \frac{1}{\mathcal{B}(r)}$$

# The periastron advance

At zeroth order in the RZ parameterization

$$^{[0]}\Delta\widetilde{\varphi} := {}^{(0)}\Delta\widetilde{\varphi} - \Delta\widetilde{\varphi}_{\rm GR}$$

$$^{[0]}\Delta\widetilde{\varphi} = 2\sigma(2a_0 + 4b_0\epsilon + b_0 + 2\epsilon)$$

$$\Delta\widetilde{\varphi} \equiv \Delta\varphi/2\pi$$

$$\Delta\widetilde{\varphi}_{\rm GR} = 3\sigma$$

$$\sigma = \frac{r_g}{l}$$

The following notation was introduced

$$^{[n+1]}\Delta\widetilde{\varphi} := {}^{(n+1)}\Delta\widetilde{\varphi} - {}^{(n)}\Delta\widetilde{\varphi}$$

$$^{(n)}\Delta\widetilde{\varphi} = \sum_{i=0}^{n} {}^{[i]}\Delta\widetilde{\varphi}$$

Expanding to first order, one obtains:

Finally, at second order in the expansion

$$^{[1]}\Delta\widetilde{\varphi} := {}^{(1)}\Delta\widetilde{\varphi} - {}^{(0)}\Delta\widetilde{\varphi}$$

$$^{[2]}\Delta\widetilde{\varphi} := {}^{(2)}\Delta\widetilde{\varphi} - {}^{(1)}\Delta\widetilde{\varphi}$$

$$\begin{aligned} ^{[2]}\Delta\widetilde{\varphi} = {} & 4a_2\sigma(2a_0 + 4\epsilon\sigma + \epsilon + 5) - 4\sigma\left[6a_0a_2b_2\sigma \cdot 3b_0^2(1+2\epsilon) - 4b_0\epsilon + 4b_1\epsilon\right] \\ & -6a_0a_2\sigma + a_0a_2 - 4a_0b_2\sigma + 2a_1b_2\sigma \\ & +2a_2\epsilon\sigma - 8a_2\sigma - 3b_0^2b_2\sigma - 3b_0^2\sigma \\ & -6b_0b_2\sigma - 6b_0\sigma + 2b_2\epsilon\sigma]\,(1+b_2)^{-1}\,. \end{aligned}$$

# Orbital Period

The expression for the proper time :

$$ {}^{(n)}\tau = \frac{p^{\frac{3}{2}}\sqrt{2-(e^2+3)\,\sigma}}{\sqrt{r_0}} \int_{\chi}^{2\pi} d\chi' \; {}^{(n)}\!\left(\frac{d\varphi}{d\chi'}\right) \frac{1}{(e\cos\chi'+1)^2\left[1-\sigma\left(e\cos\chi'+1\right)\right]} $$

The observer time:

$$ {}^{(n)}t = \sqrt{2}\left(\frac{\sqrt{(\sigma-1)^2-\sigma^2 e^2}}{\sqrt{2-(e^2+3)\,\sigma}}\right) {}^{(n)}\tau $$

Expressing t and $\tau$ in units of the Newtonian period

$$ P_{\mathrm{Newton}} = \left(\frac{8\pi^2 a^3}{G_{\mathrm{N}} r_0}\right)^{\frac{1}{2}} $$

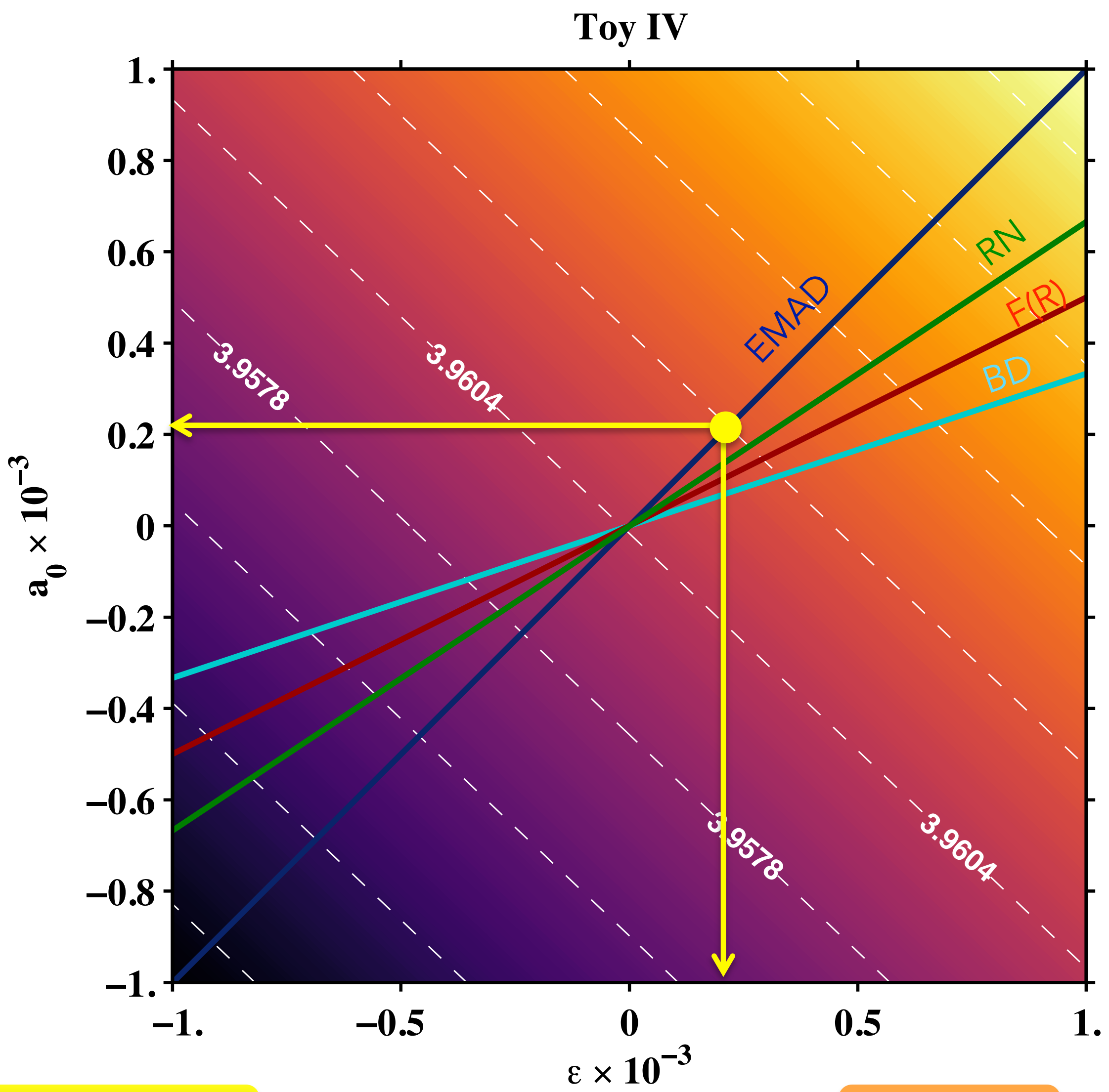
$$ \frac{P_{\mathrm{Newton}}}{2\pi}(1-e^2)^{\frac{3}{2}} \frac{\sqrt{2-(e^2+3)\,\sigma}}{\sqrt{2}} $$

# Results: test on stars and pulsars

| Object | $e$ | $a$ [AU] | $\sigma$ | $\Delta\varphi_{GR}$ | ${}^{(2)}\Delta\varphi$ |
|---|---|---|---|---|---|
| $S1$ | 0.358 | 3.29951[+3] | 1.55089[−5] | 2.91131[−4] | 2.92199[−4] |
| $S2$ | 0.876 | 9.79960[+2] | 2.01195[−4] | 3.77681[−3] | 3.79065[−3] |
| $S9$ | 0.825 | 2.33559[+3] | 6.01199[−5] | 1.12273[−3] | 1.12909[−3] |
| $S13$ | 0.395 | 9.53220[+2] | 5.54552[−5] | 1.04410[−3] | 1.04481[−3] |
| Toy I | 0.800 | 1.75400[+2] | 7.03608[−4] | 1.32627[−2] | 1.33114[−2] |
| Toy II | 0.800 | 4.38500[+1] | 2.81443[−3] | 5.30508[−2] | 5.32456[−2] |
| Toy III | 0.786 | 5.00000[+0] | 2.32488[−2] | 4.38229[−1] | 4.39850[−1] |
| Toy IV | 0.888 | 1.00000[+0] | 2.10110[−1] | 3.96047[+0] | 3.97695[+0] |

| Object | $P_{Newton}$ [s] | $t_{GR}$ [s] | ${}^{(2)}t$ [s] |
|---|---|---|---|
| $S1$ | 2.81918[+9] | 2.81924[+9] | 2.82202[+9] |
| $S2$ | 4.37764[+8] | 4.37794[+8] | 4.38220[+8] |
| $S9$ | 1.67898[+9] | 1.67903[+9] | 1.68065[+9] |
| $S13$ | 4.37764[+8] | 4.37794[+8] | 4.38225[+8] |
| Toy I | 3.45546[+7] | 3.45678[+7] | 3.46012[+7] |
| Toy II | 4.31933[+6] | 4.32593[+6] | 4.33012[+6] |
| Toy III | 1.66308[+5] | 1.68621[+5] | 1.68789[+5] |
| Toy IV | 1.48750[+4] | 1.65539[+4] | 1.65751[+4] |

# Results: periastron advance Toy models




$$a_{0\mathrm{EMAD}} = \frac{b}{2\mu}$$

$$a_{0\mathrm{RN}} = \frac{r_Q^2}{\left[\sqrt{M^2 - r_Q^2} + M\right]^2}$$

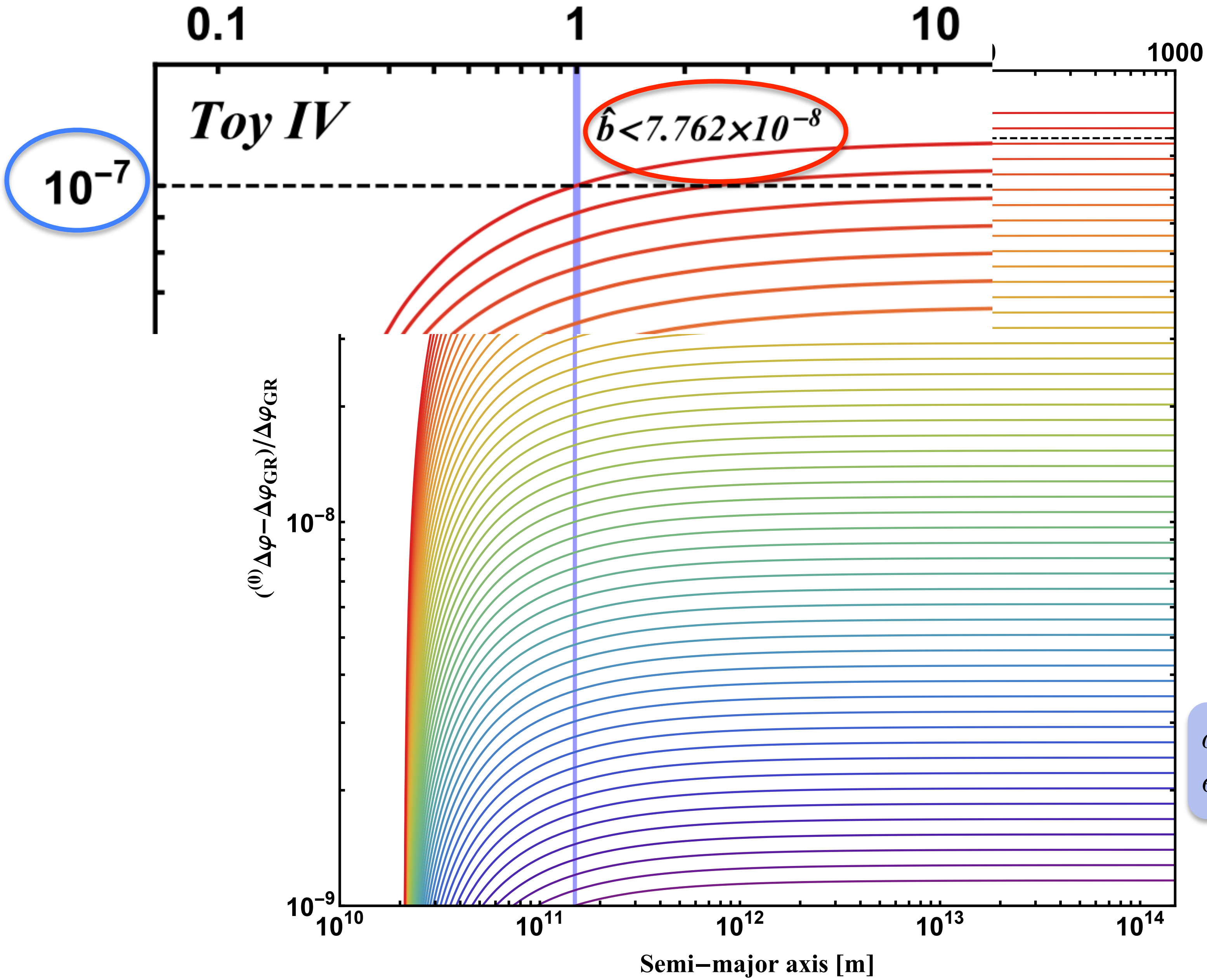
$$a_{0\mathrm{f(R)}} = \frac{(\beta_{\mathrm{f(R)}} - \gamma_{\mathrm{f(R)}})(1 + \epsilon_{\mathrm{f(R)}})^2}{2}$$

$$a_{0\mathrm{BD}} = \frac{\omega + 3(\omega + 2)^2 \epsilon_{\mathrm{BD}}}{2(\omega + 2)^2}$$

$a = 1.496 \times 10^{11} \mathrm{m}$

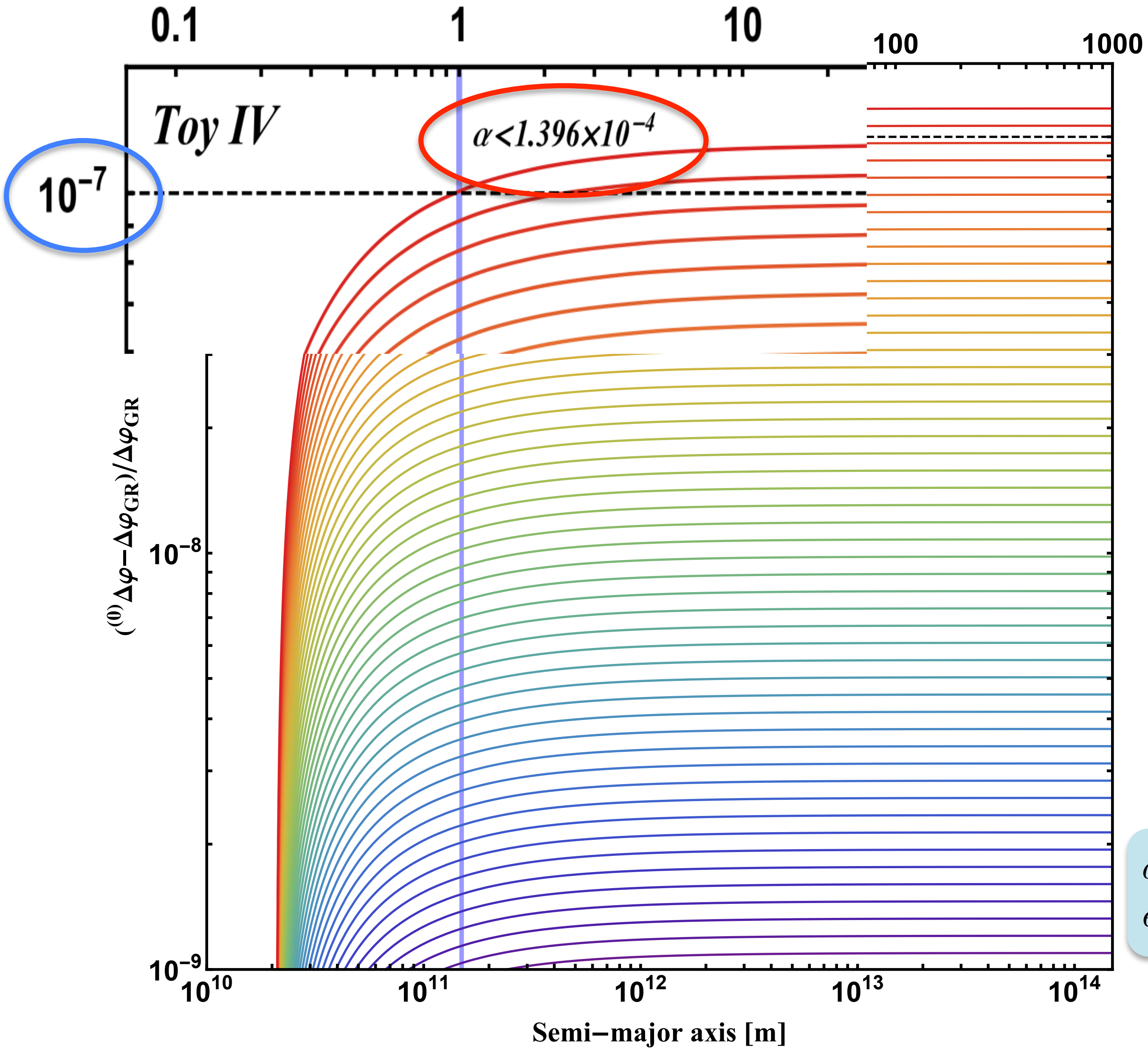
$e = 0.888$

# Constraining EMAD theories




$a = 1.496 \times 10^{11}$m
$e = 0.888$

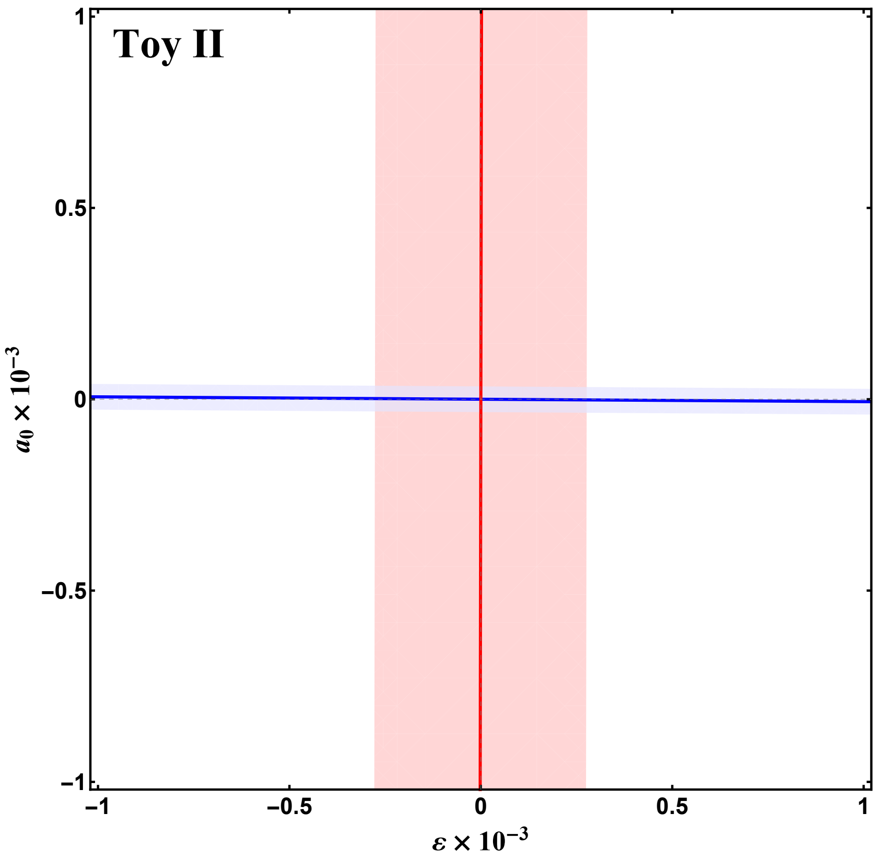
# Constraining f(R) theory



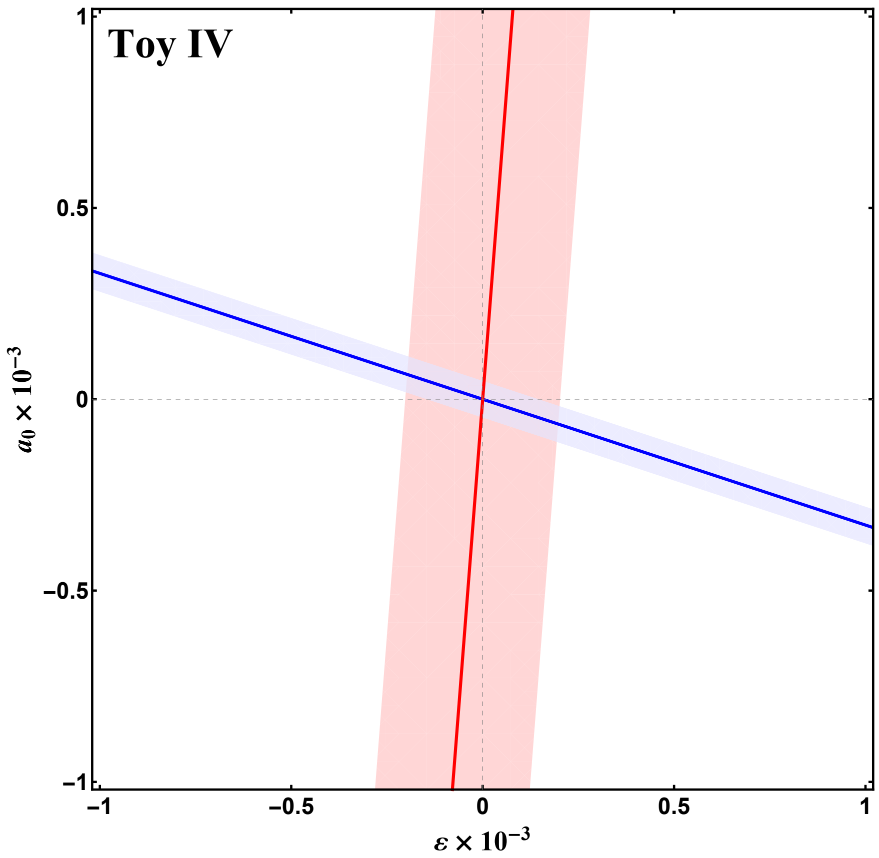

$a = 1.496 \times 10^{11}\mathrm{m}$

$e = 0.888$

# An example: constraining EMAD theory




$a = 6.56 \times 10^{12}$m
$e = 0.800$




$a = 1.496 \times 10^{11}$m
$e = 0.888$

# Conclusions

- We also should have found a pulsar around Sgr A*

- A spherically symmetric metric parameterized in view of possible deviation from GR is considered

- The body motion of a companion orbiting around the SMBH that could be S-star around the SMBH or a generic pulsar in a close binary system is studied

- Maybe GR will turn out to be the ultimate theory of gravity, or maybe a new theory starts to emerge – we will see!!!

# Conclusions

- We also should have found a pulsar around Sgr A*

- Our approach can contribute to fix strict ranges on the parameters of a given class of metrics and then select reliable gravity models without fixing a priori the theory

- Maybe GR will turn out to be the ultimate theory of gravity, or maybe a new theory starts to emerge – we will see!!!

# Overview

- Probing strong gravity regime with pulsars?
- Constraining the theory of gravity?
- Generic parameterization for spherically BH
- Test with observables

# f(R)-gravity

Assuming a static spherically symmetric metric, a generic PN-approximation can be written as:

$$ds^2 = -\left[1 - \frac{2M}{r} + \frac{\beta - \gamma}{2}\left(\frac{2M}{r}\right)^2\right] dt^2 + \left(1 + \gamma\frac{2M}{r}\right) dr^2 + r^2 d\Omega^2$$

Generalized PPN-parameters, can be expressed in term of f(R) as:

$$\gamma_{\mathrm{f(R)}} - 1 = -\frac{f''(R)^2}{f'(R) + 2f''(R)^2} \qquad \beta_{\mathrm{f(R)}} - 1 = \frac{1}{4}\frac{f'(R) \cdot f''(R)}{2f'(R) + 3f''(R)^2}\frac{d\gamma_{f(R)}}{dR}$$

In particular, let us consider a quadratic correction:

$$f(R) = R + \alpha R^2 + \ldots$$

# Einstein-Maxwell-Axion-Dilaton

The line element reads:

$$ds^2 = -\left(\frac{r-2\mu}{r+2\hat{b}}\right)dt^2 + \left(\frac{r+2\hat{b}}{r-2\mu}\right)d\rho^2 + (r^2+2\hat{b}r)d\Omega^2$$

Furthermore: $\rho^2 = r^2 + 2\hat{b}r$ and $\mu := M - \hat{b}$

Rewriting the line element in a more convenient way:

$$ds^2 = -\left(1-\frac{2M\rho}{r^2}\right)dt^2 + \left(\frac{r^2}{\hat{b}^2+r^2}\right)dr^2 + r^2 d\Omega^2\,,$$

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D. Garfinkle, G. Horowitz, A. Strominger, Phys. Rev. D 43, 3140 (1991)
G. Horowitz, A. Strominger, Nucl. Phys. 8360, 197 (1991)

# Einstein-Maxwell-Axion-Dilaton

Comparing with the RZ-parameterization:

$$r_{0_{\rm EMAD}} = 2\sqrt{M(M-b)} = 2\sqrt{M\mu} \qquad \epsilon_{\rm EMAD} = \sqrt{1+\frac{b}{\mu}} - 1$$

Expanding the metric coefficient at spatial infinity:

$$a_{0_{\rm EMAD}} = \frac{b}{2\mu} \qquad b_{0_{\rm EMAD}} = 0$$

and near the horizon:

$$a_{1_{\rm EMAD}} = 2(\epsilon_{\rm EMAD} + 1) + (1 + a_{0_{\rm EMAD}})^{-1} - 3 - a_{0_{\rm EMAD}}$$

$$b_{1_{\rm EMAD}} = \frac{\epsilon_{\rm EMAD} + 1}{1 + a_{0_{\rm EMAD}}} - 1$$

$$a_{2_{\rm EMAD}} = \frac{2\left(a_{0_{\rm EMAD}}\epsilon_{\rm EMAD} - a_{0_{\rm EMAD}}^2 + \epsilon_{\rm EMAD}\right) + 1}{2(a_{0_{\rm EMAD}} + 1)^2}$$

$$b_{2_{\rm EMAD}} = b_{1_{\rm EMAD}} - \frac{b^2}{(1 + a_{0_{\rm EMAD}})^2}$$