GRAVITATIONAL WAVE EXPERIMENTS
2. DATA ANALYSIS

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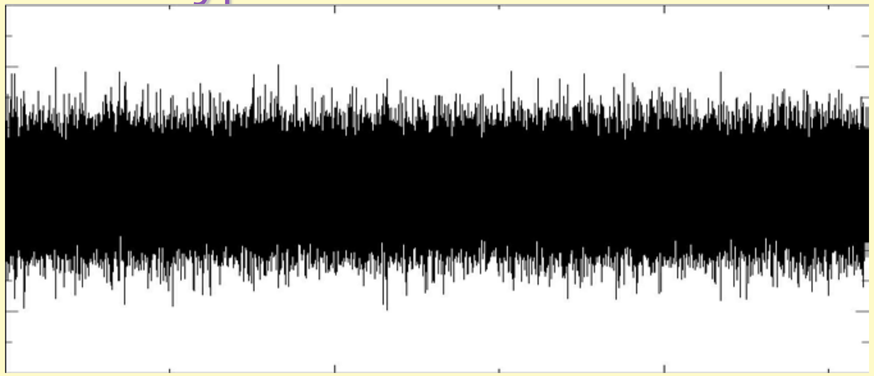
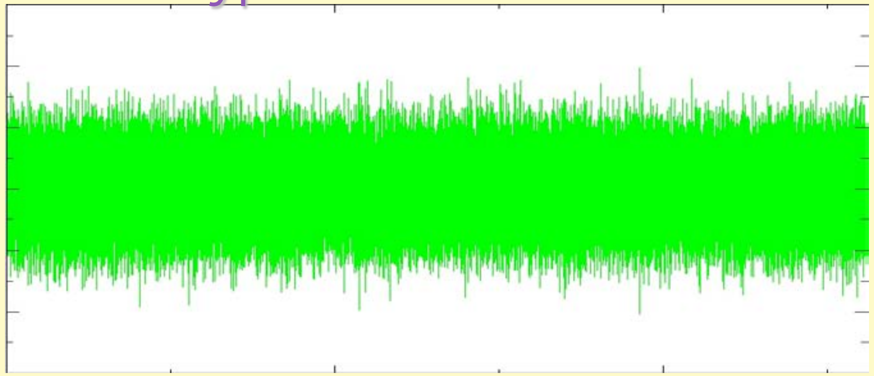
XIX FRASCATI SPRING SCHOOL "BRUNO TOUSCHEK" in Nuclear, Subnuclear and Astroparticle Physics

Example: two detection problems

"noise type A".



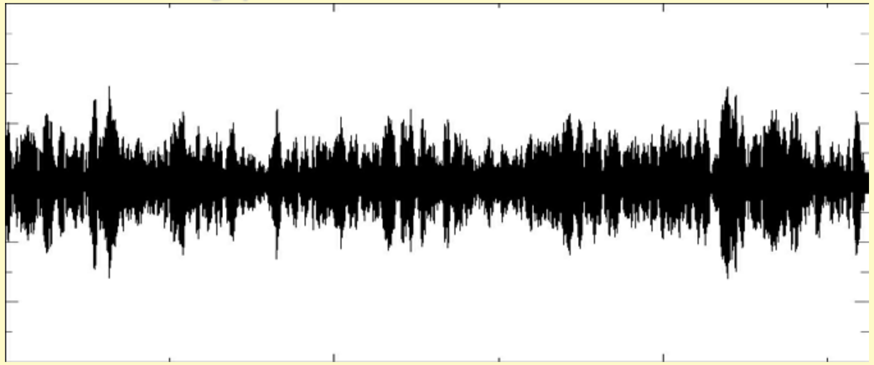
"noise type A" + sinusoid



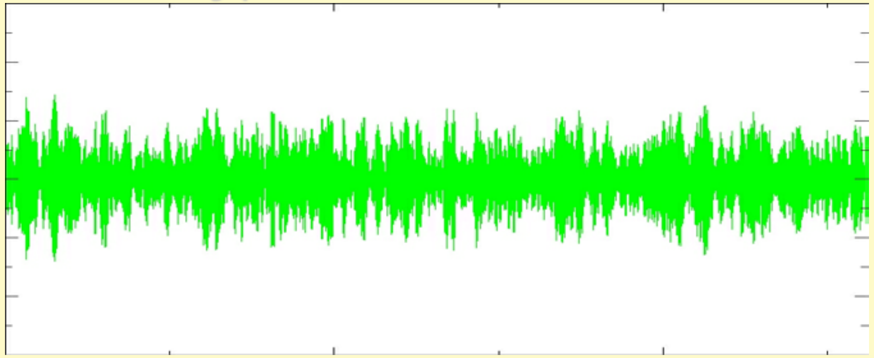
Is it easy to detect the presence of the sinusoid?

Very easy!

"noise type B".



"noise type B" + sinusoid

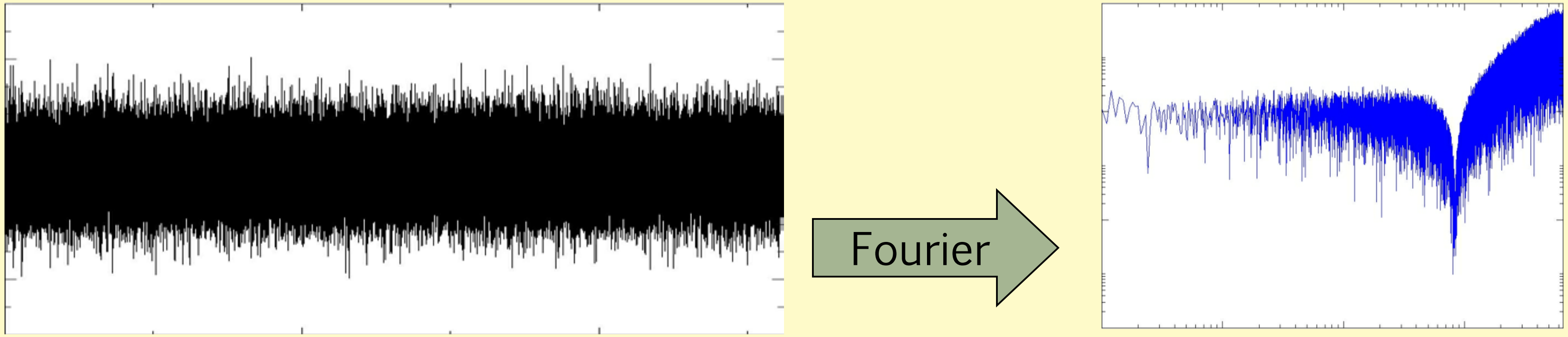


Is it easy to detect the presence of the sinusoid?

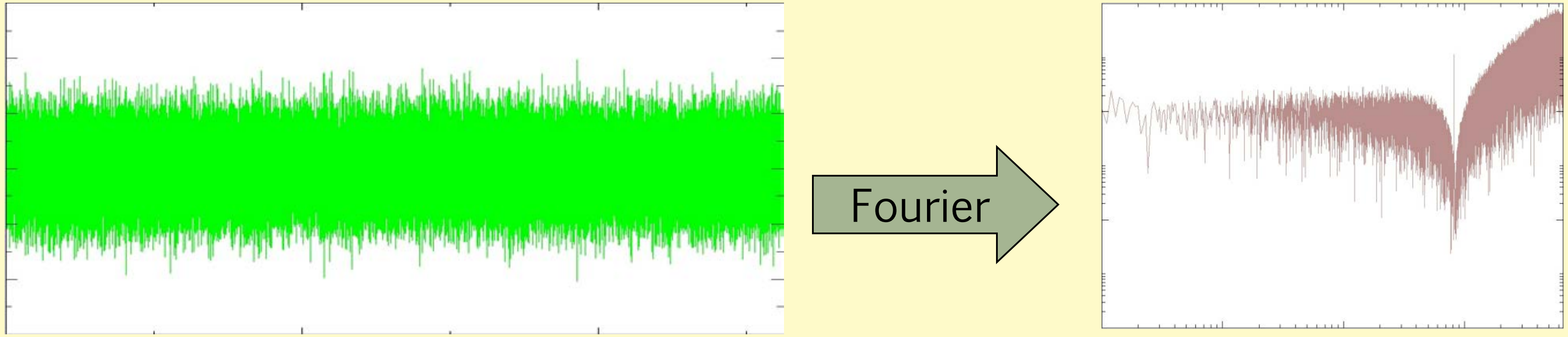
Very difficult!

Noise “A” + sinusoid

In the large N limit (if the noise is stationary) the eigenvalues of C_{ij} are trigonometric functions.

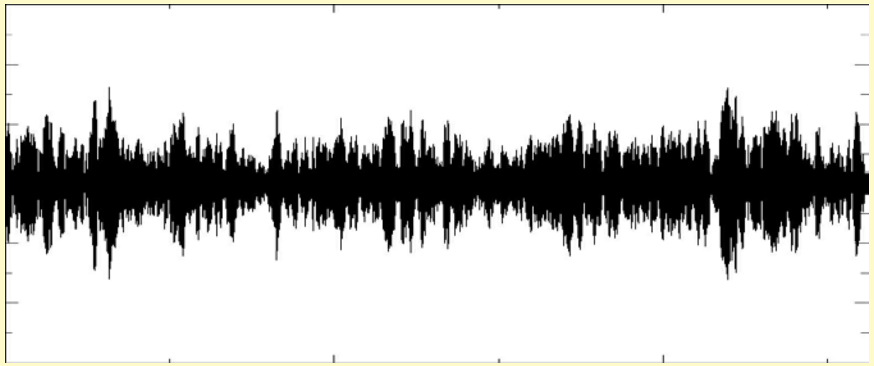


“noise type A”.

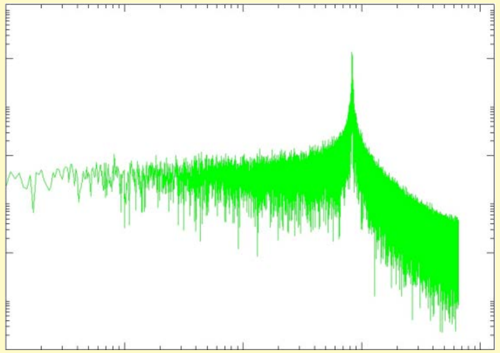


“noise type A” + sinusoid

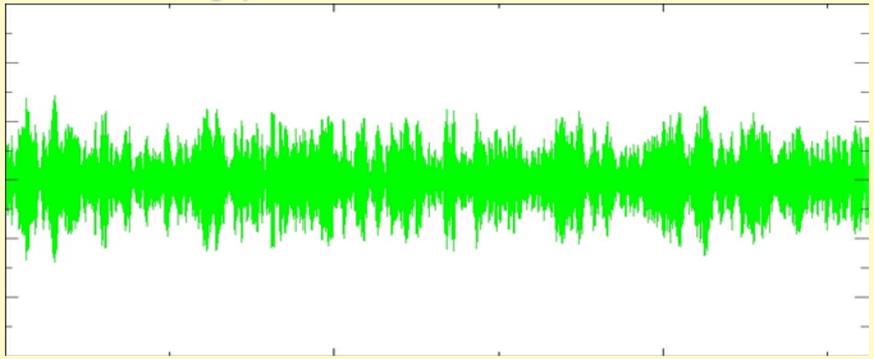
Noise "B" + sinusoid



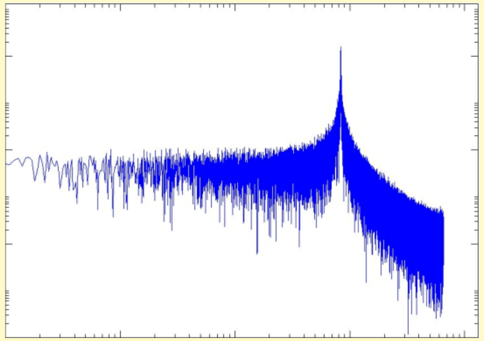
Fourier



"noise type B".



Fourier

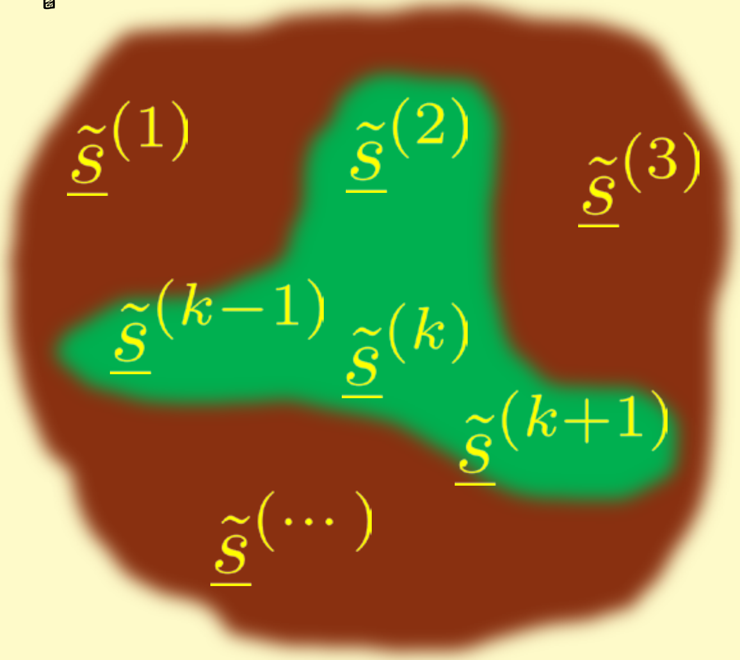


"noise type B" + sinusoid

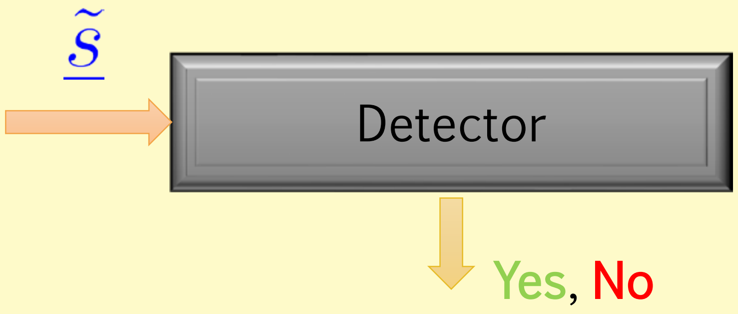
The amplitude of the signal must be compared with the amplitude of the "parallel" component of the noise.

The detection problem

- A detector is an arbitrary rule to partition the space of possible experimental results in two parts (yes or no answer to a well defined question)
- We characterize the detector with
  - Detection probability P_D
  - False alarm probability P_{FA}



Given answer \ True answer	Yes	No
No	$1-P_D$	$1-P_{FA}$
Yes	P_D	P_{FA}

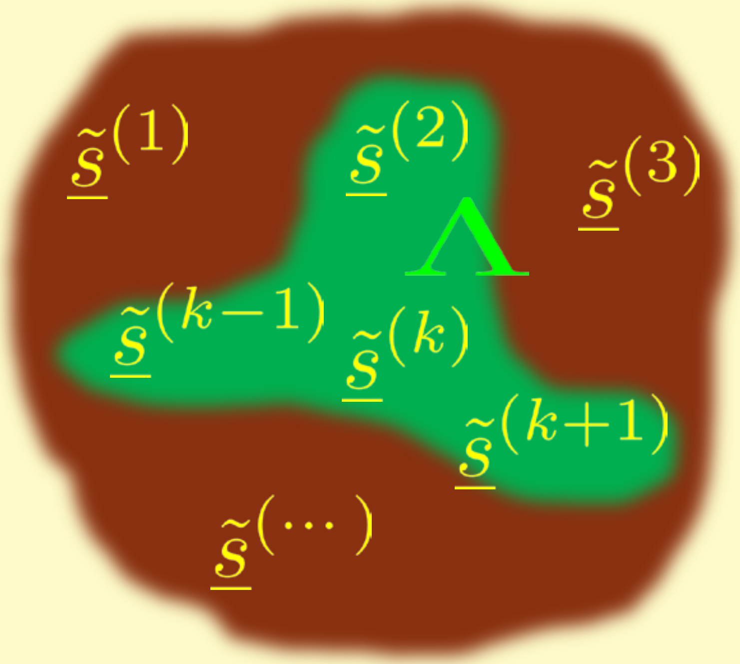


Example: random choice

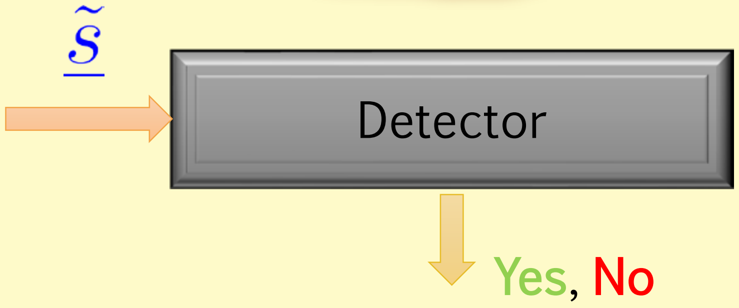
$$P_D = P_{FA} = 0.5$$

Best P_D at a given P_{FA}

Given answer \ True answer	Yes	No
No	1-P_D	1-P_{FA}
Yes	P_D	P_{FA}



- The detector decide Yes or No by throwing a coin. Is it a valid detector? What is P_D and P_{FA} in this case?



- Can you easily design a detector with $P_D = 1$? And one with $P_{FA} = 1$?

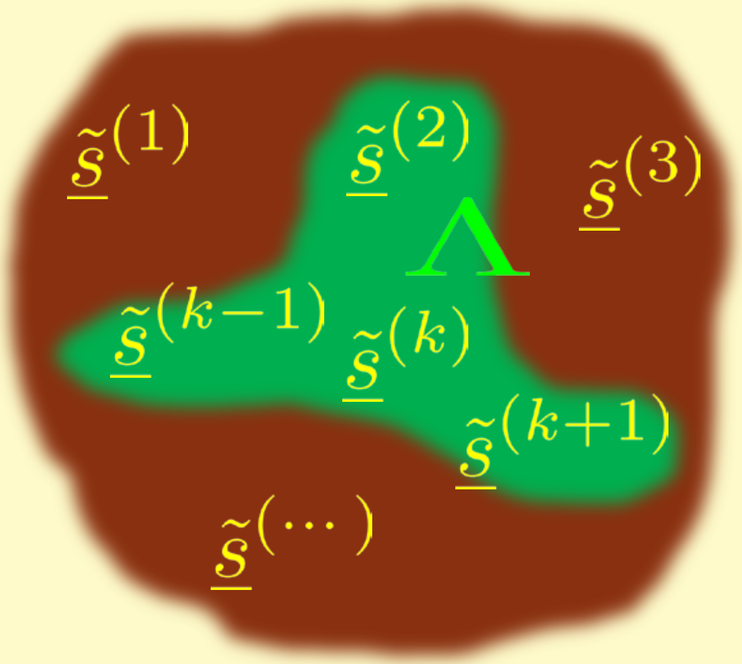
The detection problem

- We can try to maximize P_D keeping P_{FA} fixed

$$\max_{\Lambda} \int_{\Lambda} dP_1 - \lambda \int_{\Lambda} dP_0$$

- The solution: Λ_{opt} is the set of $\tilde{s}$ where

$$\frac{dP_1}{dP_0} > \lambda$$



? How we can choose the desired value of P_{FA}?

DETECTION

Wiener Filter

An example: known signal in «coloured» noise

- We start from a model with two different alternatives for the observed time series:
  - H_1: Known signal with additive gaussian noise $O_i = S_i + N_i$
  - H_0: Only gaussian noise: $O_i = N_i$
- The probability distribution for the observations under H_0 is

$$dP_0\left[O_i\right] = \mathcal{N} \exp\left[-\frac{1}{2}\underline{\mathrm{O}}^T C^{-1}\underline{\mathrm{O}}\right] dO_i$$

- The probability distribution for the observations under H_1 is

$$dP_1\left[O_i\right] = \mathcal{N} \exp\left[-\frac{1}{2}\left(\underline{\mathrm{O}}-\underline{\mathrm{S}}\right)^T C^{-1}\left(\underline{\mathrm{O}}-\underline{\mathrm{S}}\right)\right] dO_i$$

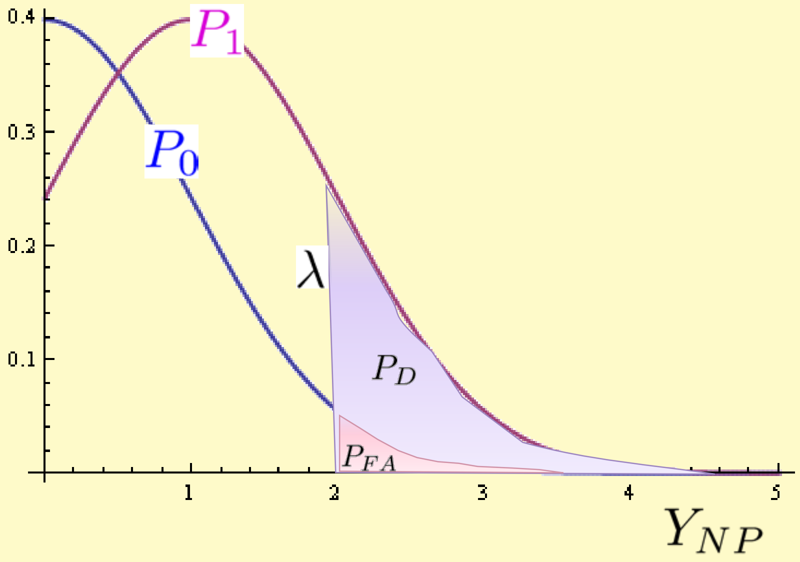
- The NP detector is given by

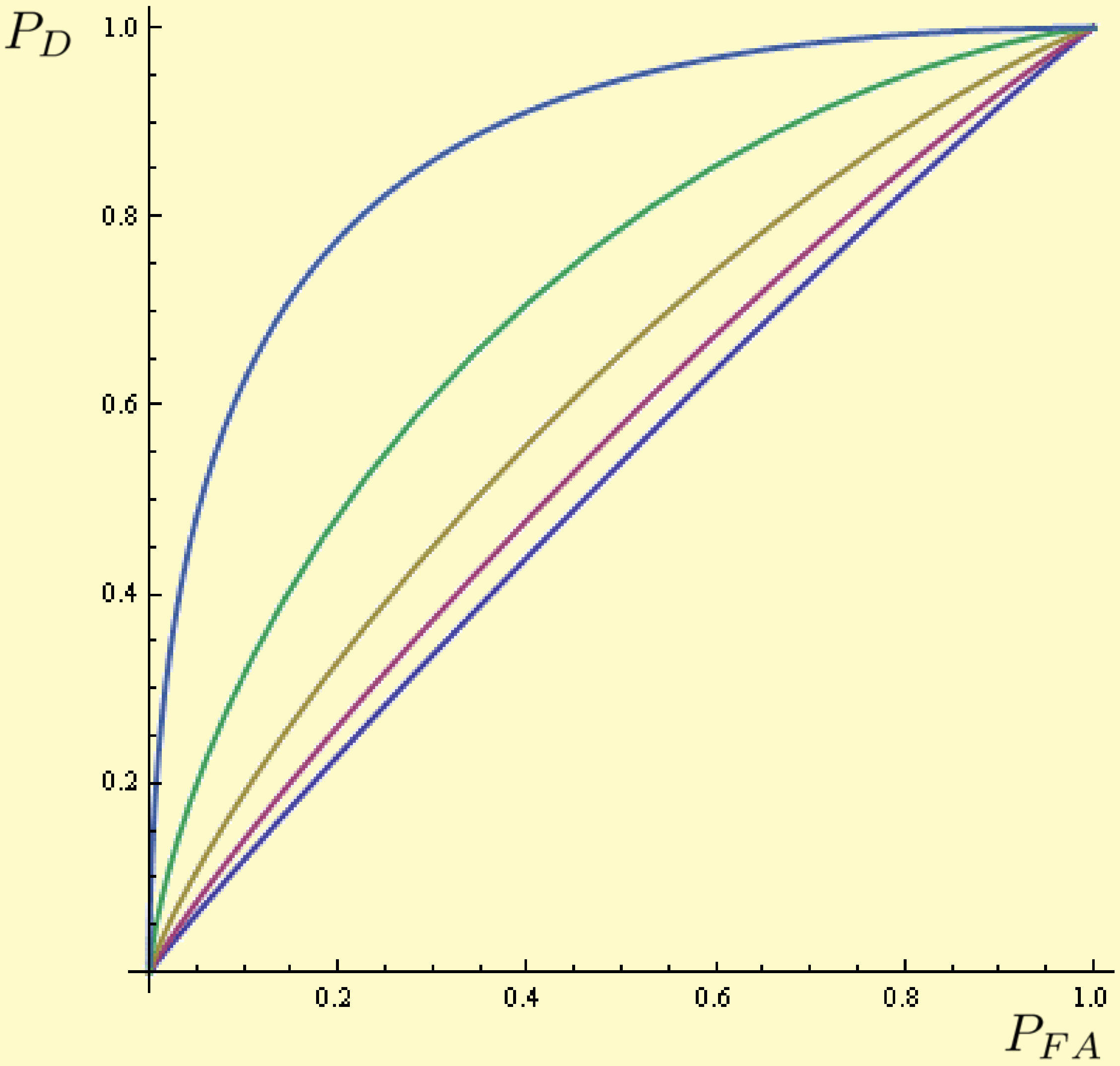
$$\exp\left[-\frac{1}{2}\left(\underline{\mathrm{O}}-\underline{\mathrm{S}}\right)^T C^{-1}\left(\underline{\mathrm{O}}-\underline{\mathrm{S}}\right)+\underline{\mathrm{O}}^T C^{-1}\underline{\mathrm{O}}\right] > \lambda$$

$$Y_{NP} = \underline{\mathrm{O}}^T C^{-1}\underline{\mathrm{S}} > \lambda'$$

Receiver Operating Characteristic

$$Y_{NP} = \underline{O}^T C^{-1} \underline{S}$$





The Wiener filter

$$\underline{\mathrm{O}}^T C^{-1} \underline{\mathrm{S}}$$

Frequency domain →

$$W\left(\vec{\alpha}, s\right] = \int_0^{\infty} \frac{\tilde{T}(\vec{\alpha}, f)^* \, \tilde{s}(f)}{S(f)} df$$

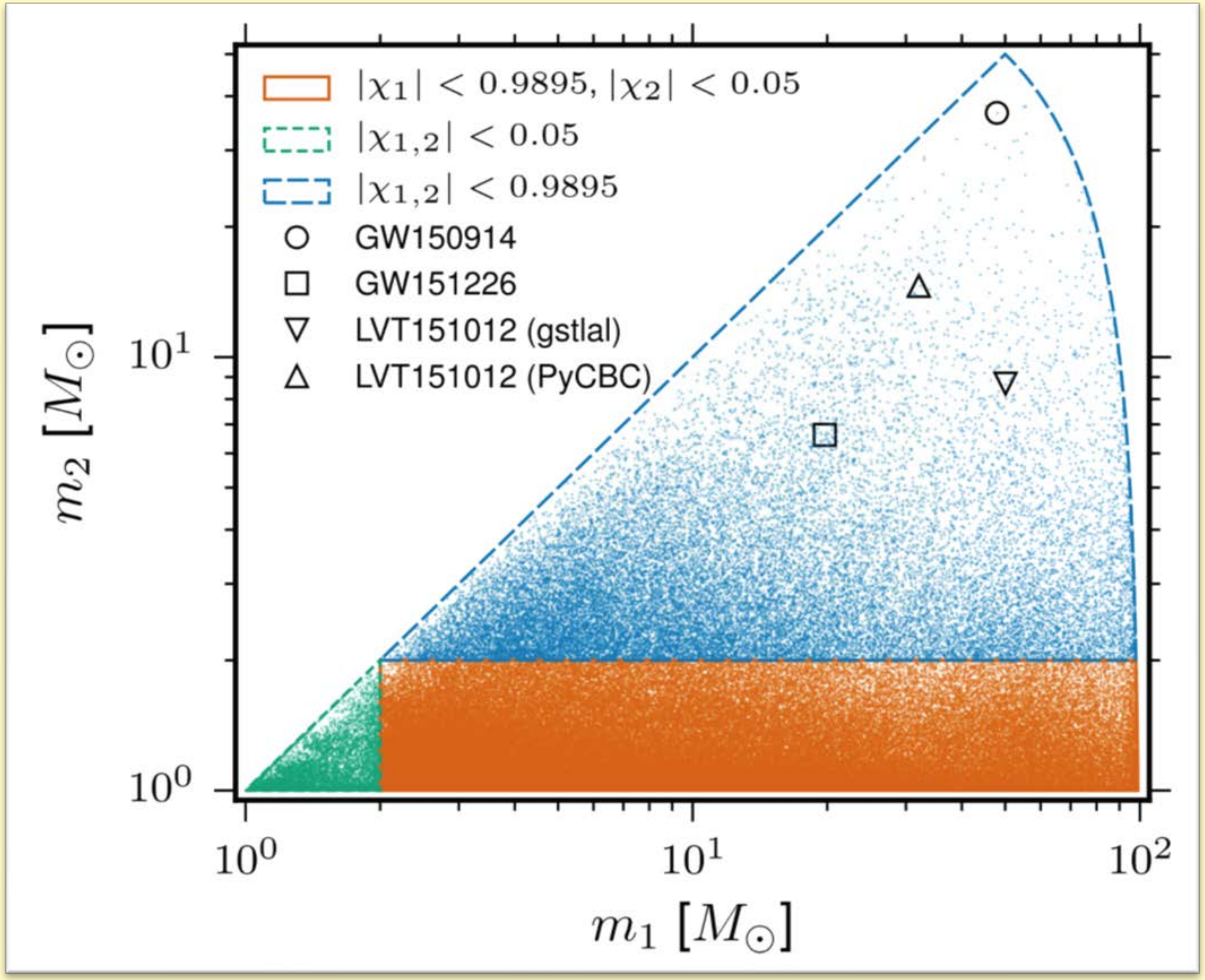
- This is a scalar product between the expected signal and the observed one
- Weighted with the noise.
- Fourier domain is convenient when the noise is stationary
- If the signal is not completely known? Try all of them and use

$$\max_{\vec{\alpha}} W\left(\vec{\alpha}, s\right] > \lambda$$

DETECTION

Binary systems

Unknown parameters



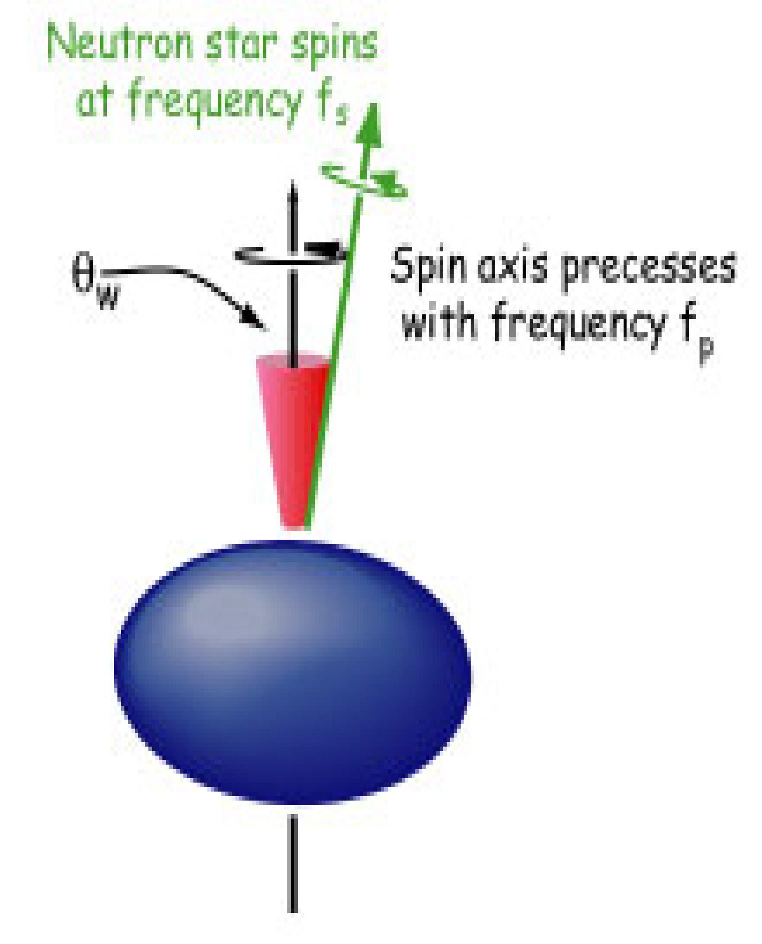
- ~250000 templates
- Parameter estimate (fast, rough, single detector)

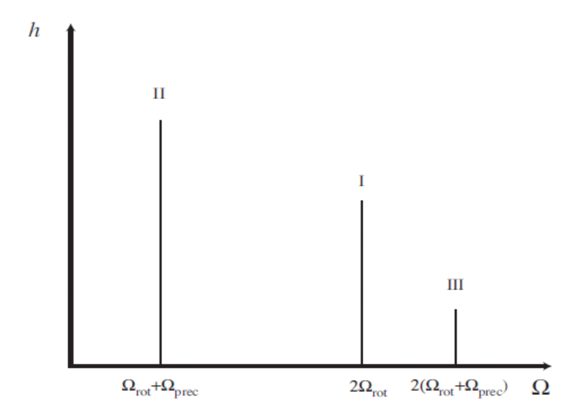
DETECTION

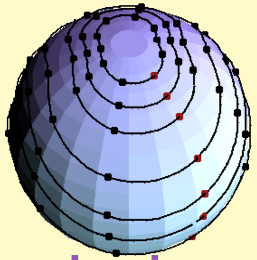
Continuous sources

The signal from a NS

- General case: free rigid body rotation.
  - The rotation can be described in term of Elliptic functions
  - Two periodicities T and T^0: discrete spectrum
- Small deviations from axisymmetry:
  - Deviation from axisymmetry: $2\Omega_{rot}$
  - Precession: $\Omega_{rot} + \Omega_{prec} \simeq \Omega_{rot}$
  - excited oscillatory modes such as the r-mode







$$\frac{4}{3}\Omega_{rot}$$

R-modes in accreting stars

The expected signal at the detector

A gravitational wave signal we detect from a NS will be:

- Frequency modulated by relative motion of detector and source
- Amplitude modulated by the motion of the non-uniform antenna sensitivity pattern of the detector

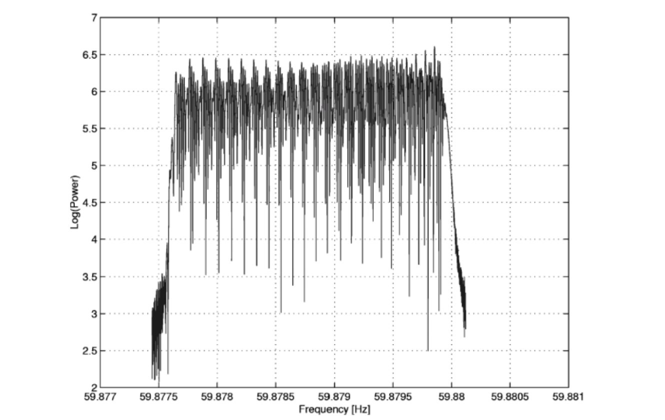
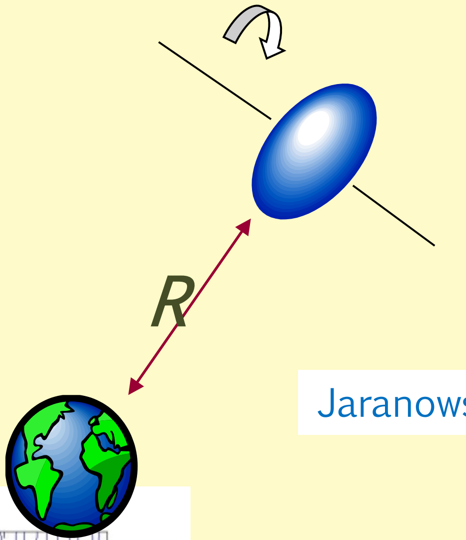
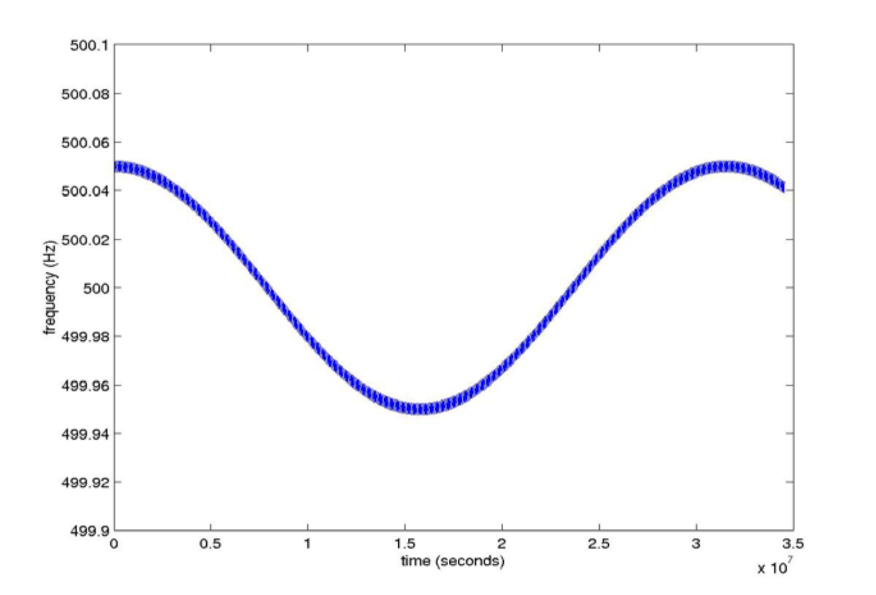
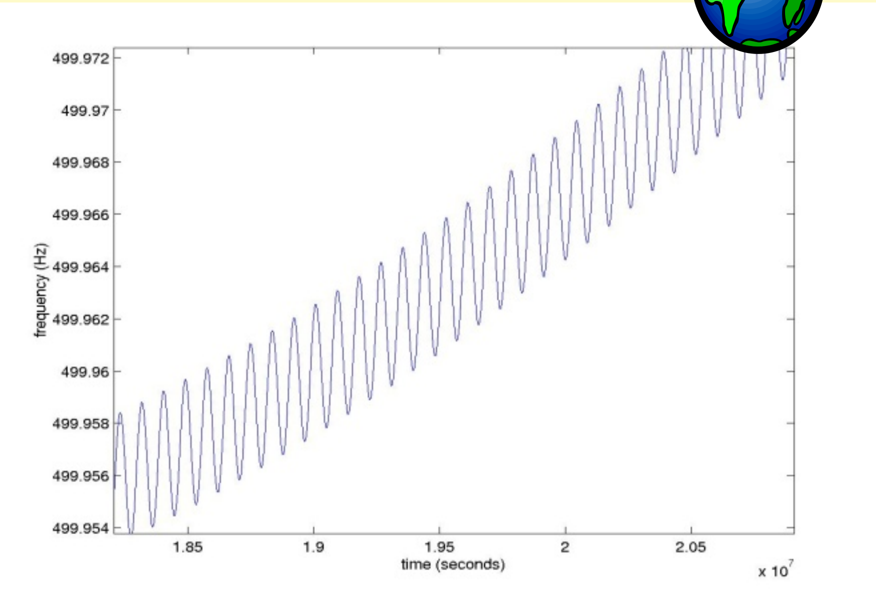


FIG. 1. Power spectrum of the noise-free response of an interferometer located near Hannover to gravitational-wave signal from the Crab

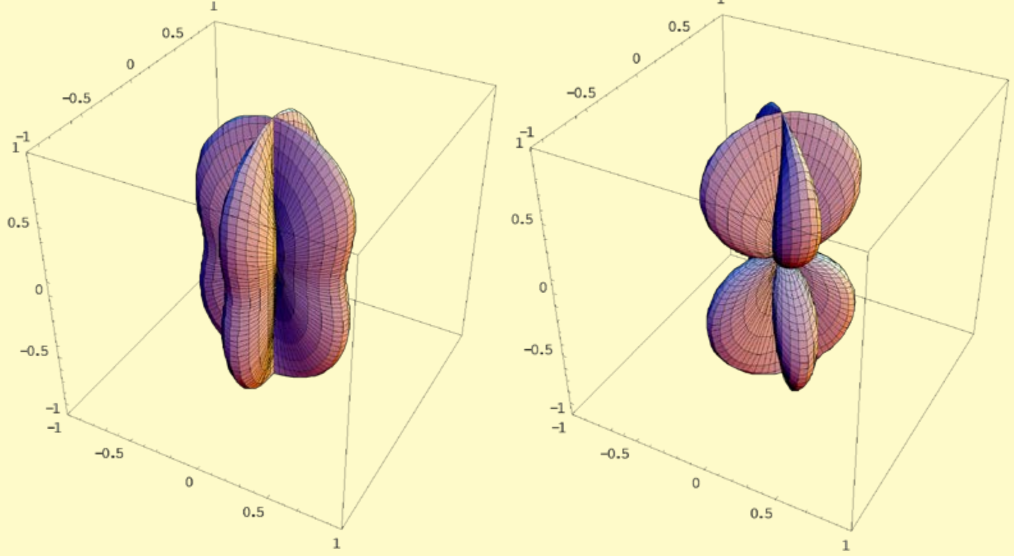


Jaranowski et al. 1998





Signal received from *an isolated* NS



$$h(t) = F_+ A_+ \cos\Phi(t) + F_\times A_\times \sin\Phi(t)$$

$$\left.\begin{array}{l} F_+(t,\psi) \\ F_\times(t,\psi) \end{array}\right\}$$ strain antenna patterns. They depend on the orientation of the detector and source and on the polarization of the waves.

$$\Phi(t) = \Phi_0 + 2\pi \sum_{n=0}^{\infty} \frac{f^{(n)}}{(n+1)!} \left[T(t) - T(t_0)\right]^{n+1}$$

T(t): time of arrival of a signal at the solar system barycenter, *t* the time at the detector.

- The phase depends on
  - Initial conditions
  - Frequency evolution
  - Relative velocity between source and detector

In the case of an isolated tri-axial neutron star emitting at twice its rotational frequency

$$A_+ = \frac{1}{2} h_0 (1 + \cos^2 \iota)$$

$$A_\times = h_0 \cos \iota$$

$$h_0 = \frac{4\pi^2 G}{c^4} \frac{I_{zz} \varepsilon f_{gw}^2}{d}$$

h_0 - amplitude of the gravitational wave signal

ι - angle between the pulsar spin axis and line of sight

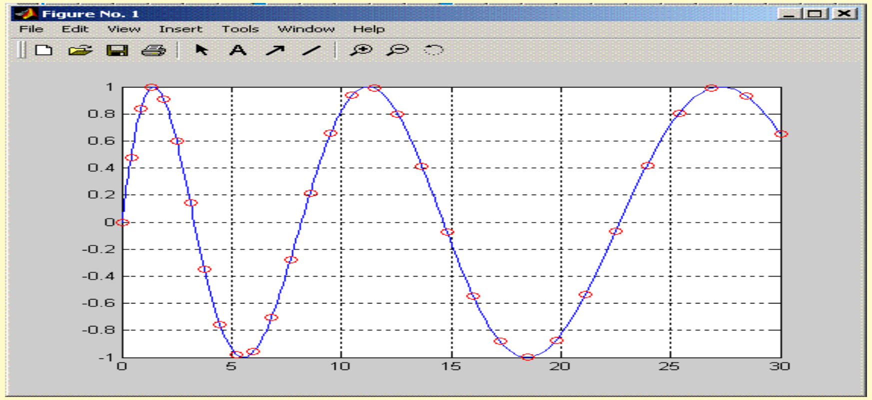
$$\varepsilon = \frac{I_{xx} - I_{yy}}{I_{zz}}$$ - equatorial ellipticity

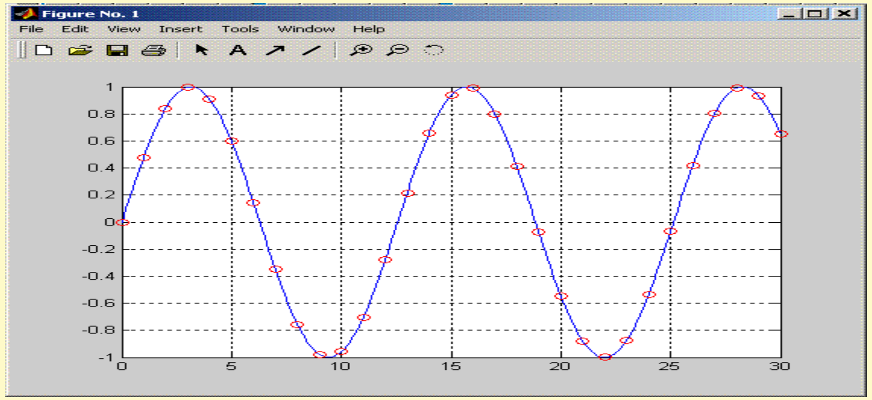
Four neutron star populations

- Known pulsars
  - Position & frequency evolution known (including derivatives, timing noise, glitches, orbit)
- Unknown neutron stars
  - Nothing known, search over position, frequency & its derivatives
- Accreting neutron stars in low-mass x-ray binaries
  - Position known, sometimes orbit & frequency
- Known, isolated, non-pulsing neutron stars
  - Position known, search over frequency & derivatives

Targeted coherent search

- Applicable when we know all (or many) parameters of the source
  - Conceptually: a Wiener filtering. In time domain: resampling + Fourier Transform





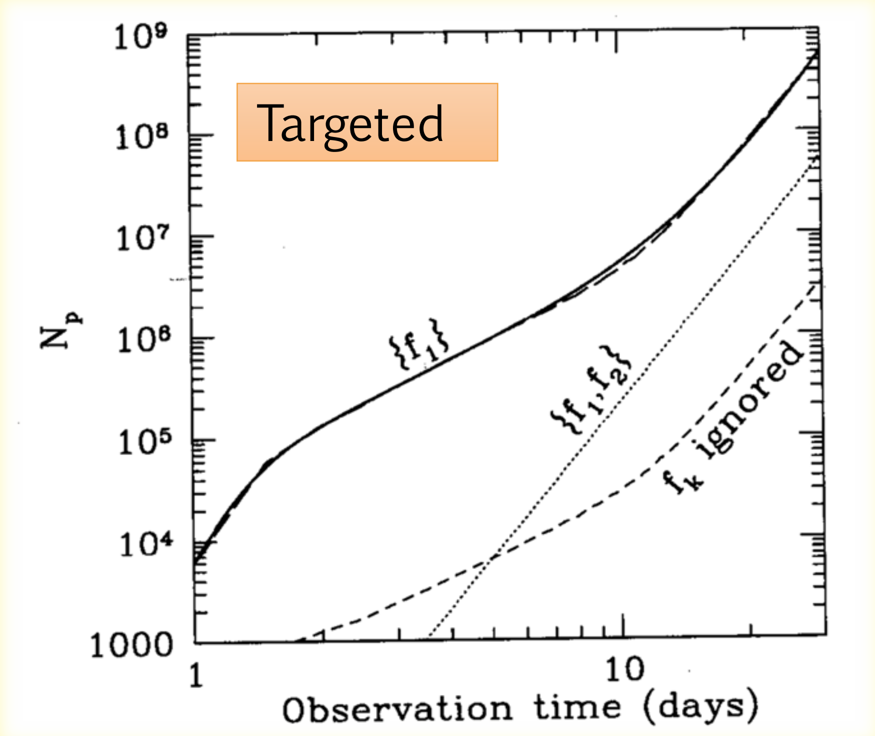
- What if some parameters are missing?
  - In some cases they are irrelevant (e.g. amplitude of the signal). They can be adsorbed in the threshold of the detector.
  - In some cases they give a modified detector. Example: if initial phase is unknown, the optimal NP filter is a quadrature sum of two Wiener filters
  - General case: try all the possibilities, choose the largest answer (GLRT detector). This requires a bank of templates. This is no more a targeted search.

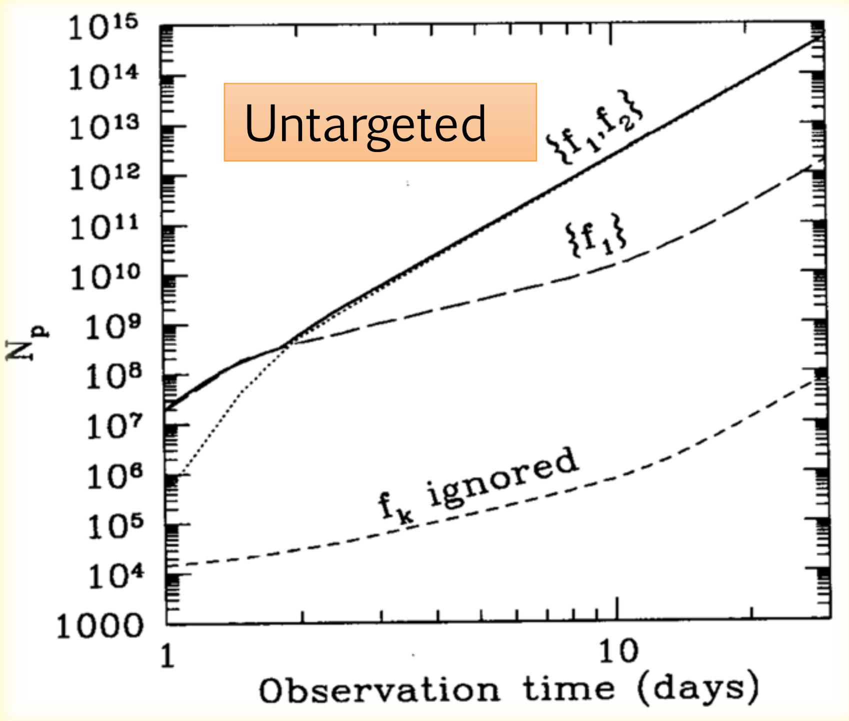
Blind searches and coherent detection methods

- Coherent methods are the most sensitive methods (amplitude SNR increases with square root of observation time) but they are the most computationally expensive.
  - Our templates are constructed based on different values of the signal parameters (e.g. position, frequency and spindown)
  - The parameter resolution increases with longer observations
  - Sensitivity also increases with longer observations
  - As one increases the sensitivity of the search, one also increases the number of templates one needs to use.

Number of templates

The number of templates grows dramatically with the coherent integration time baseline and the computational requirements become prohibitive



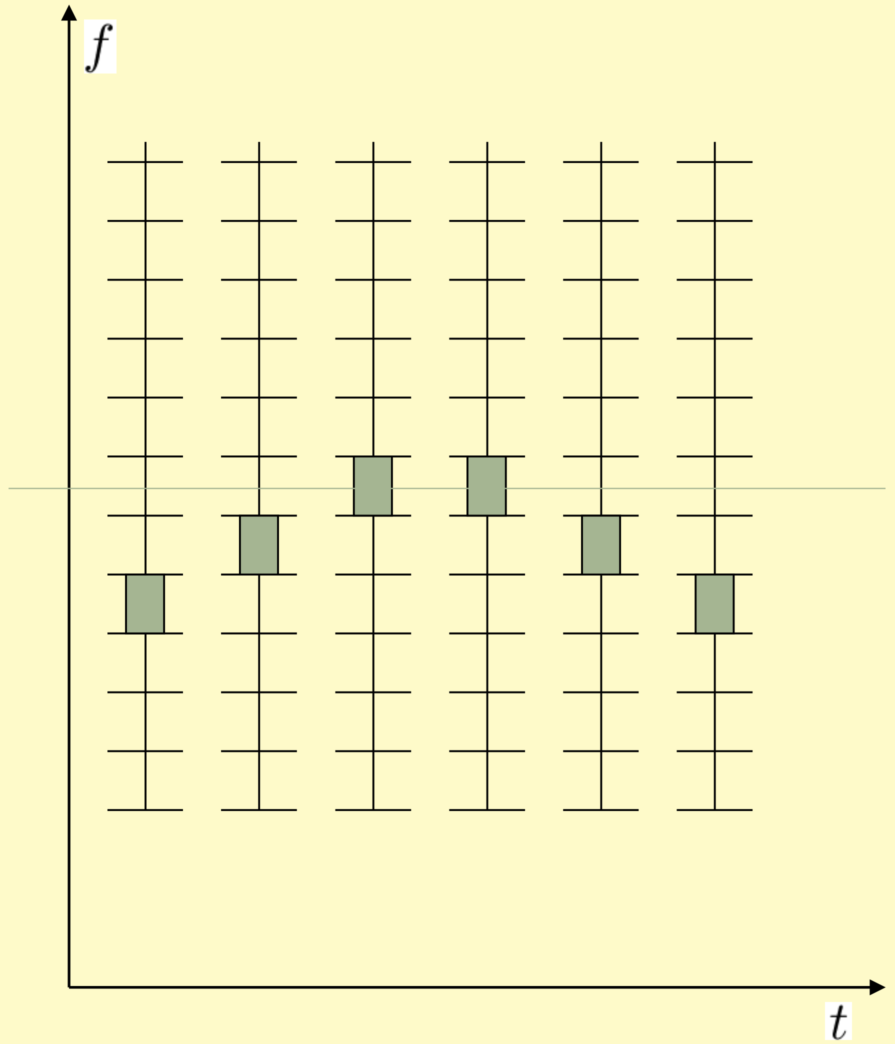


[*Brady et al., Phys.Rev.D57 (1998)2101*]

We need suboptimal methods

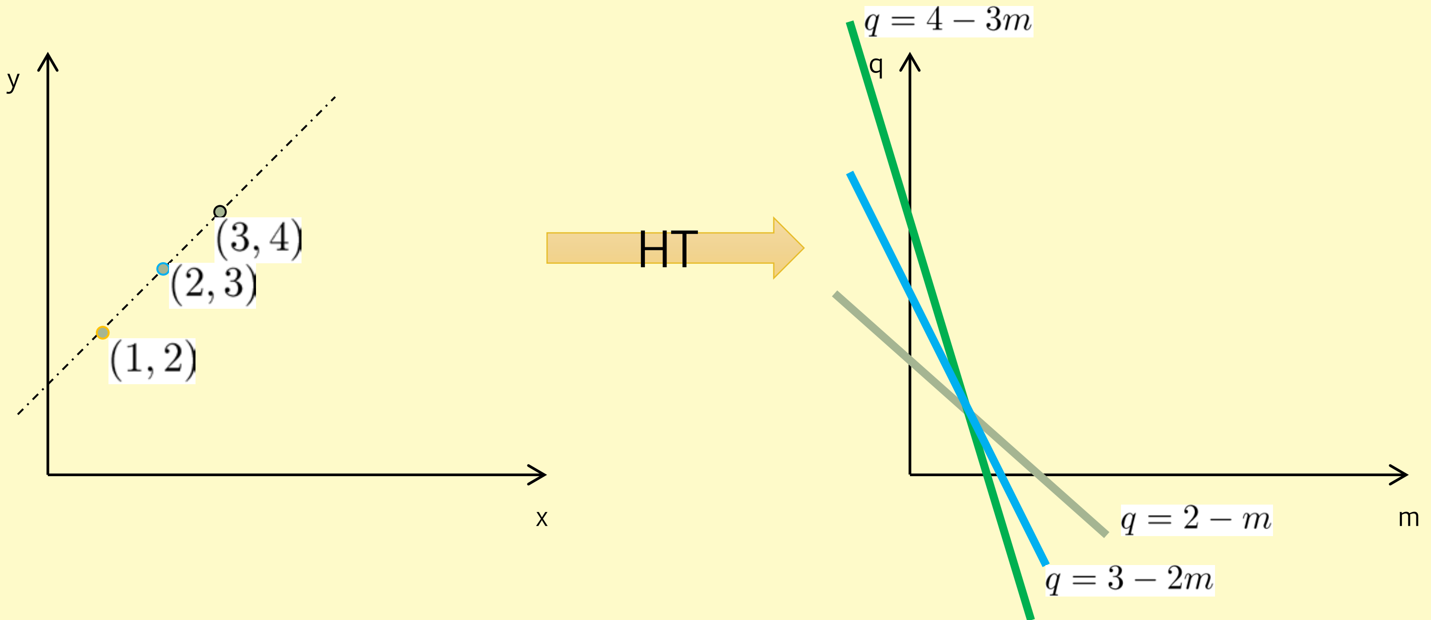
Radon transform

- Compute periodograms from short periods
- Shift (slide) periodograms accordingly to the frequency evolution
- Sum (stack) the periodograms



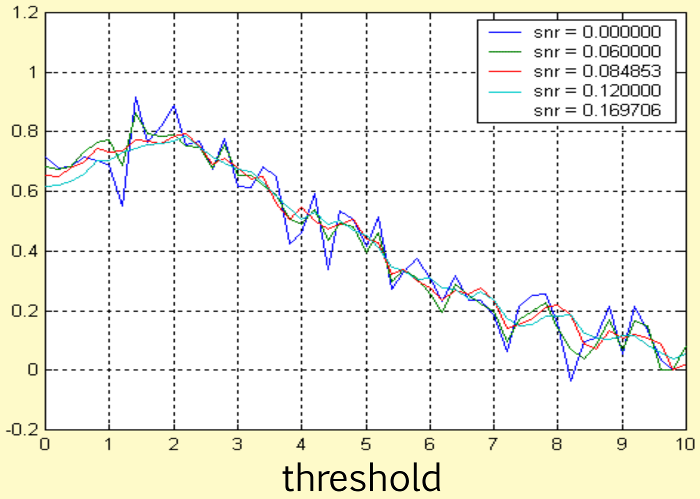
Hough transform

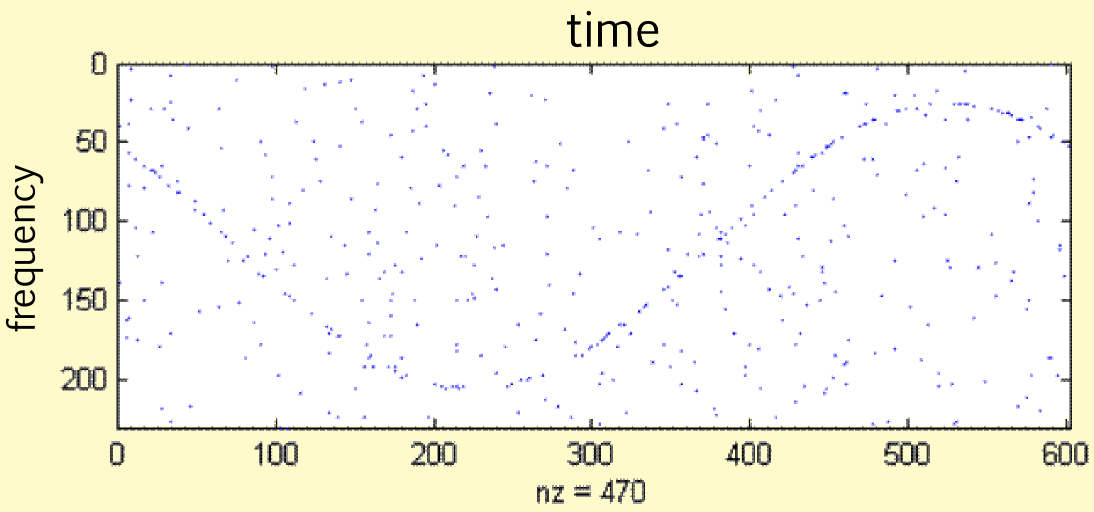
- Each point (xi, y_i) corresponds to the set of lines $y=mx+q$ with $q=y_i-m\ x_i$
- The set is represented by a line in the (q,m) plane
- The transform maps (xi, y_i) to this line

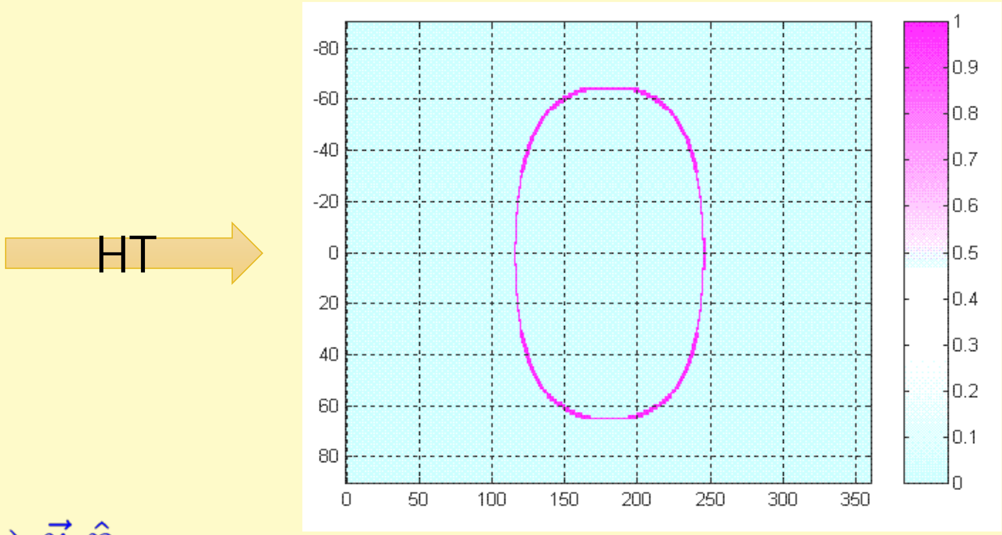


Hough transform: application

- Threshold on the periodograms
- Point mapped by HT to a circle in the sky
- Slightly less sensitive than Radon
- More robust
- Computing power reduced by a factor 10







$$f(t) - f_0(t) = f_0(t)\frac{\vec{v}\cdot\hat{n}}{c}$$

DETECTION

Stochastic signals

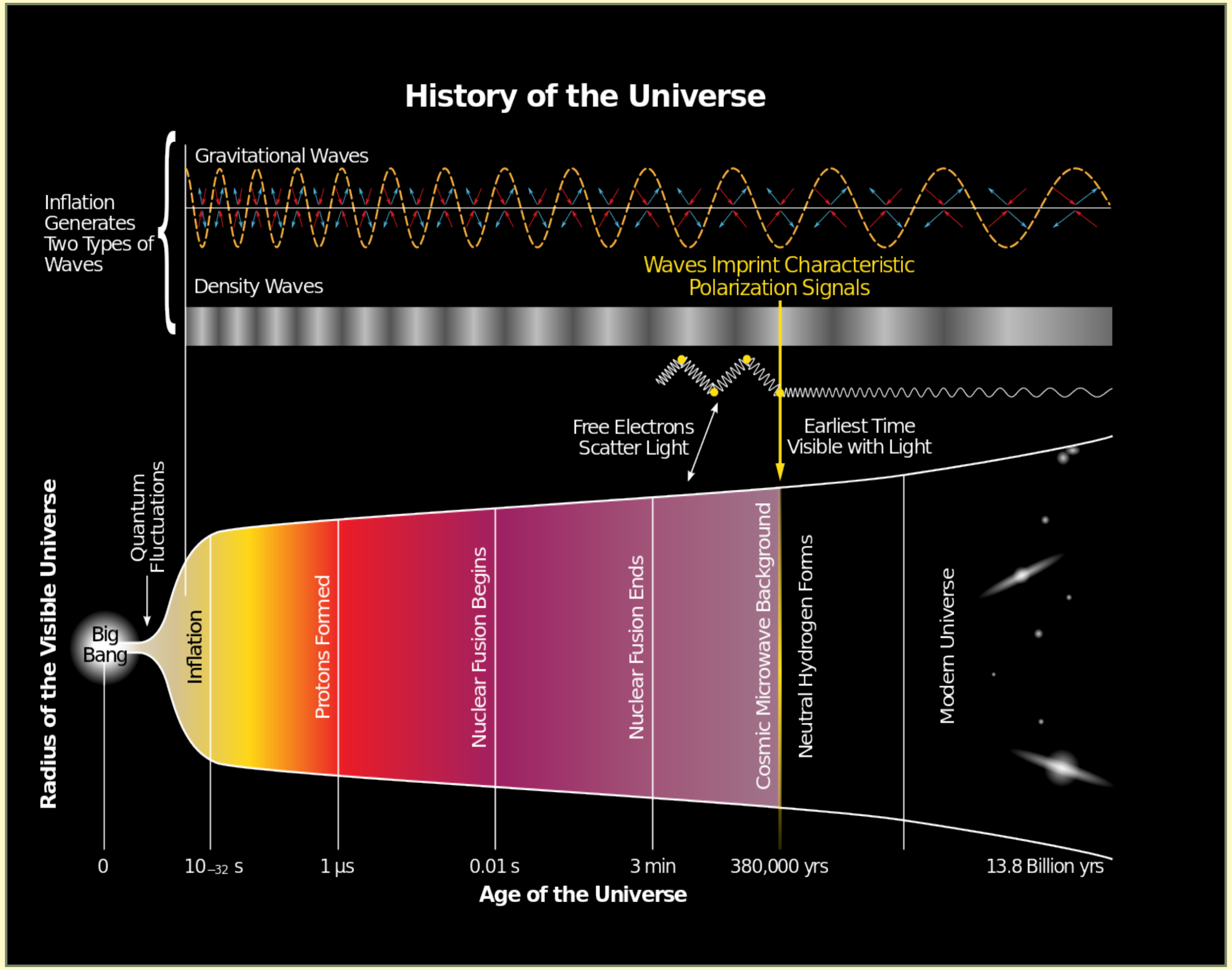
Stochastic background in a nutshell

A stochastic background can be seen as

- a GW field which evolves from an initially random configuration
- the result of a superposition of many uncorrelated and unresolved sources

Two different kinds:

- Cosmological:
  - signature of the early Universe: coupling of gravitational field is small!
  - *inflation, cosmic strings, phase transitions…*
- Astrophysical:
  - sources since the beginning of stellar activity
  - *compact binaries, supernovae, rotating NSs, core-collapse to NSs or BHs, supermassive BHs…*



Typical «first approximations» :

1) Gaussian, because sum of many contributions
2) Stationary, because physical time scales much larger than observational ones
3) Isotropic (at least for cosmological backgrounds)
4) Unpolarized

How we can observe a GW stochastic background?

If we accept these working hypothesis:

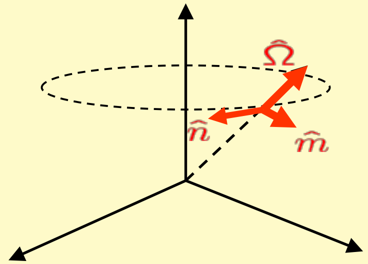
"A stochastic background is completely described by its power spectrum"

$$h_{ij}(t,\vec{r}) = \sum_{P=+,\times} \int_{S^2} d\hat{\Omega}\, \varepsilon^P_{ij}(\hat{\Omega}) \int_{-\infty}^{\infty} df\, \tilde{h}_P(f,\hat{\Omega}) e^{i2\pi f(t-\hat{\Omega}\cdot\vec{r})}$$

Polarizations — Directions — Frequencies

$$\varepsilon^+_{ij} = m_i m_j - n_i n_j$$

$$\varepsilon^\times_{ij} = m_i n_j + n_i m_j$$



For a given mode decomposition, we can consider the amplitudes $h_{+,\times}(f,\hat{\Omega})$ as stochastic variables. What can be said about their statistical properties?

$$< h^\star_A(f,\hat{\Omega},\psi) h_B(f',\hat{\Omega}',\psi') >= \delta_{AB}\delta(f-f') \frac{\delta^2(\hat{\Omega},\hat{\Omega}')}{4\pi} \frac{\delta(\psi-\psi')}{2\pi} \frac{1}{2} S_{gw}(f)$$

Detection: the basic idea

- Output of the first interferometer: $s_1(t) = h_1(t) + n_1(t)$
- Output of the second interferometer: $s_2(t) = h_2(t) + n_2(t)$
- This is because we suppose a model with additive noise

Now:

$$\langle s_1 s_2 \rangle = \langle h_1 h_2 \rangle + \langle h_1 n_2 \rangle + \langle n_1 h_2 \rangle + \langle n_1 n_2 \rangle$$

This is because:

- Noise and signal are uncorrelated
- Noise between two detectors are uncorrelated

Can you figure out a mechanism which makes signal and noise correlated?
Can you figure out a mechanism which makes noises of two detectors correlated?

What does this really means?

- We have 2 stochastic processes, the detectors' output

$$\underline{\tilde{s}}(f) = (\tilde{s}_1(f), \tilde{s}_2(f))$$

$$\underline{\tilde{s}}(f) = \underline{\tilde{h}}(f) + \underline{\tilde{n}}(f)$$

◈ If we suppose both signal and noise to be Gaussian

$$dP = \mathcal{N} \exp\left(-\int \frac{1}{2} \underline{\tilde{s}}(f)^{+} \mathcal{C}^{-1}(f)\, \underline{\tilde{s}}(f) df\right) \prod_f d\underline{\tilde{s}}(f)$$

This is completely specified by the spectral covariance array

Stationarity implies that signals at different frequencies are statistically independent

What does this really means?

- The problem is now clearly defined: we must discriminate between two different hypothesis:

$\mathcal{H}_0$ There is only instrumental noise: $dP \equiv dP_0$

$$\mathcal{C} = \mathcal{C}_N = \begin{pmatrix} S_n^{(1)} & 0 \\ 0 & S_n^{(2)} \end{pmatrix}$$

$\mathcal{H}_1$ There is instrumental noise and a stochastic background $dP \equiv dP_1$

$$\mathcal{C} = \mathcal{C}_N + \mathcal{C}_h = \begin{pmatrix} S_n^{(1)} + S_h & \gamma S_h \\ \gamma S_h & S_n^{(2)} + S_h \end{pmatrix}$$

? Is it possible to search for a stochastic background with a single detector?

$$h_0^2 \Omega_{gw}(f) = \frac{1}{\rho_c} \frac{d\rho_{gw}}{d \log f} = \frac{4\pi^2 h_0^2}{3 H_0^2} f^3 S_h(f)$$

In our case:

$$\frac{dP_1}{dP_0} = \exp\left\{-\frac{1}{2}\int df\, \underline{\tilde{s}}^+ \left[(\mathcal{C}_N + \mathcal{C}_h)^{-1} - \mathcal{C}_N^{-1}\right] \underline{\tilde{s}}\right\}$$

- Note that this is equivalent to the detector

$$\int df\, \underline{\tilde{s}}(f)^+ \left[\mathcal{C}_N^{-1} - (\mathcal{C}_N + \mathcal{C}_h)^{-1}\right] \underline{\tilde{s}}(f) > \lambda'$$

which can be written in the suggestive form

$$\mathrm{Tr}_{f,D}\left\{\left[\underline{\tilde{s}\tilde{s}}^+\right]\left[\mathcal{C}_N^{-1} - (\mathcal{C}_N + \mathcal{C}_h)^{-1}\right]\right\} > \lambda'$$

In the applications, $S_N \gg S_h$, and we can simplify further this expression

$$\mathcal{C}_N^{-1} - (\mathcal{C}_N + \mathcal{C}_h)^{-1} \simeq \mathcal{C}_N^{-1} \mathcal{C}_h \mathcal{C}_N^{-1}$$

and we get the optimal detector

$$Y \equiv \int df \, S_h \, \underline{\tilde{s}}^{+} \begin{pmatrix} \left(\frac{1}{S_n^{(1)}}\right)^2 & \frac{\gamma}{S_n^{(1)} S_n^{(2)}} \\ \frac{\gamma}{S_n^{(1)} S_n^{(2)}} & \left(\frac{1}{S_n^{(2)}}\right)^2 \end{pmatrix} \underline{\tilde{s}} > \lambda'$$

We can expect the contributions from the out of diagonal elements of the array to be dominant? Why? When?

Y is approximately a Gaussian variable (why?). We can evaluate its mean and its variance.

$$Y \equiv \int df \left[S_h \frac{\gamma}{S_n^{(1)} S_n^{(2)}} \left(\tilde{s}_1^* \tilde{s}_2 + \tilde{s}_2^* \tilde{s}_1 \right) \right]$$

$$\langle Y \rangle_{H_0} = 0 \qquad \langle Y \rangle_{H_1} = T \int df \frac{S_h^2 \gamma^2}{S_n^{(1)} S_n^{(2)}}$$

$$\mathrm{var}_{H0}(Y) \simeq \mathrm{var}_{H1}(Y) \simeq T \int df \frac{S_h^2 \gamma^2}{S_n^{(1)} S_n^{(2)}}$$

So, we have two gaussian distributions with the same variance and different means.

? Explain the T factors. How the detection probability is expected to improve with the measurement time (Discussion)?

Overlap reduction function

The signal is a linear combination of the elements of the strain tensor.

- **Gaussian**
- **Stationary, at least if D^{ij} is time independent**

$$\left\langle \tilde{h}_A^\star(f)\tilde{h}_B(f')\right\rangle = \left\langle \left(\tilde{D}_A^{ij}\star\tilde{h}_{ij}\right)^\star(f)\left(\tilde{D}_B^{kl}\star\tilde{h}_{kl}\right)(f')\right\rangle$$

This can be written as

$$\left\langle \tilde{h}_A^\star(f)\tilde{h}_B(f')\right\rangle \propto \delta(f-f')S_h(f)\gamma_{AB}(f)$$

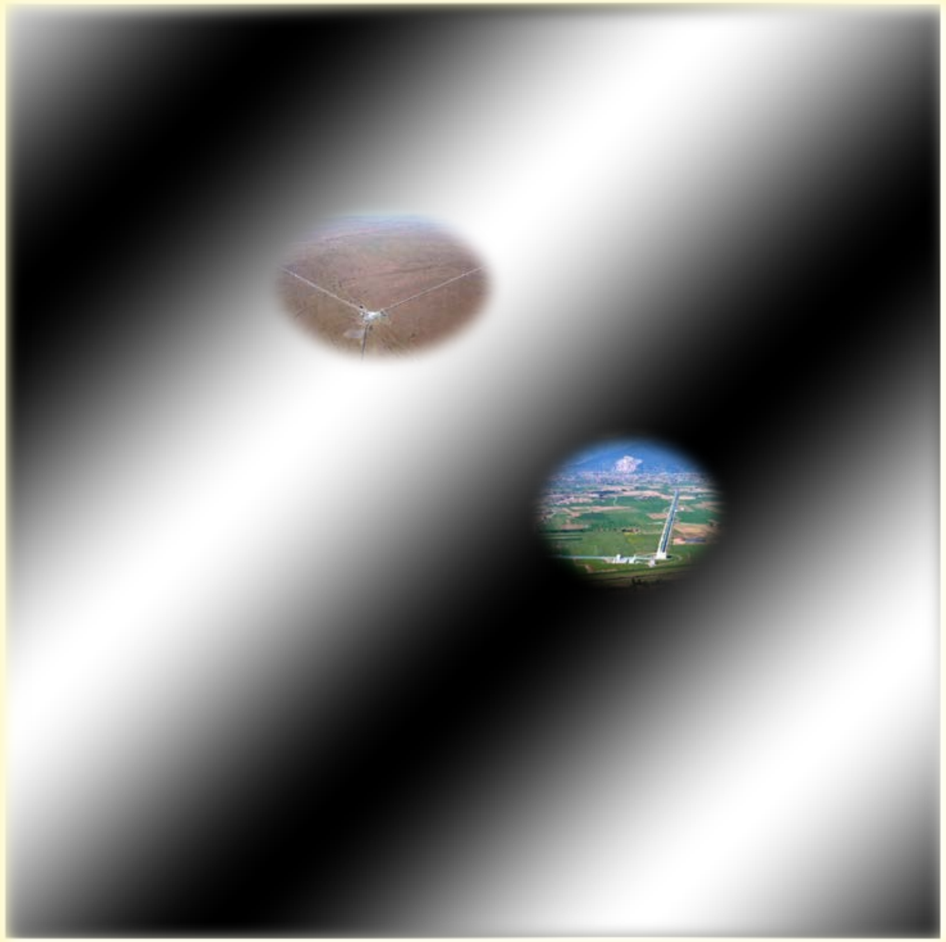
where the overlap reduction function γ_{AB} is defined by

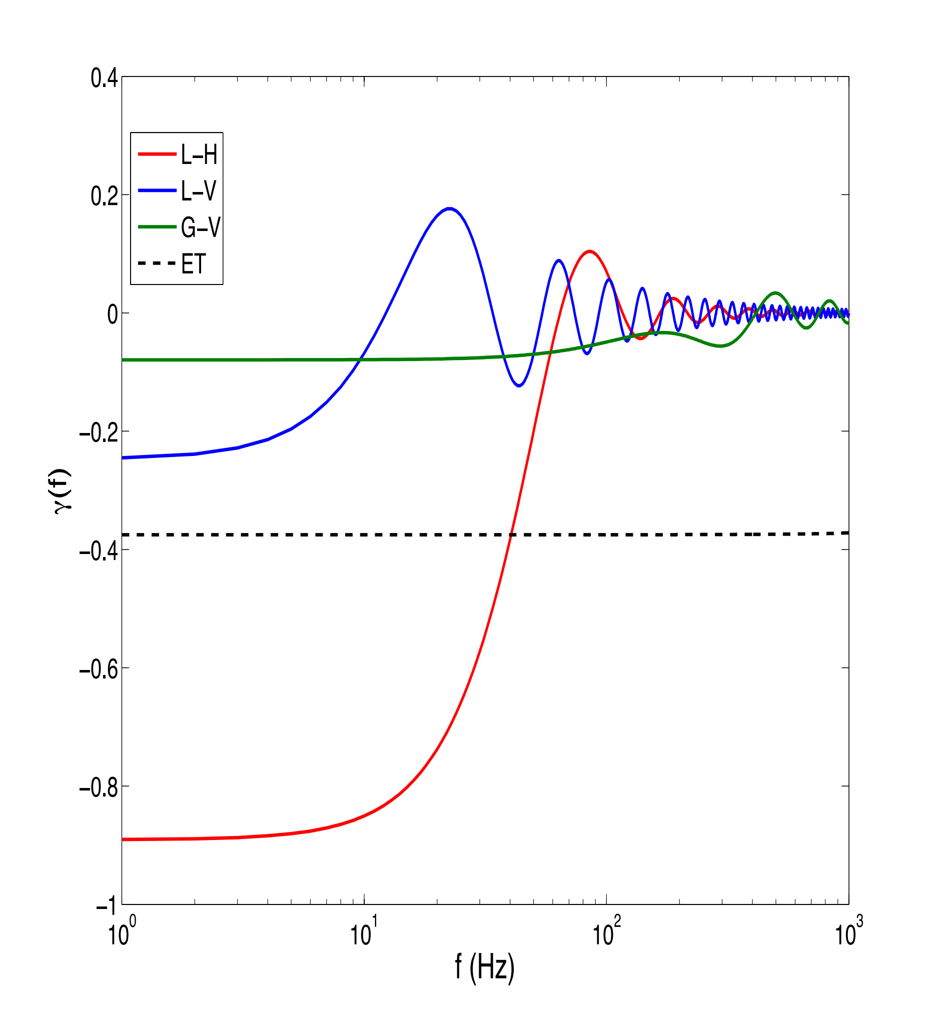
$$\gamma_{AB}(f) = \frac{1}{F}\sum_P \frac{d\hat{\Omega}}{4\pi} D_A^{ij}\varepsilon_{ij}^P\left(\hat{\Omega}\right) D_B^{kl}\varepsilon_{kl}^P\left(\hat{\Omega}\right) e^{2\pi i f\hat{\Omega}\cdot\Delta\vec{r}}$$

- **Depends on distance (same features of strain correlations)**
- **Depends on orientation (via the overlap of detector tensors)**

Overlap reduction function (a.k.a. coherence)

$$\mathrm{SNR}_Y^2 := \frac{\mu_Y^2}{\sigma_Y^2} = 2T\int_0^\infty S_h^2(f)\frac{\gamma_{12}^2(f)}{S_{n,1}(f)S_{n,2}(f)}\,df$$





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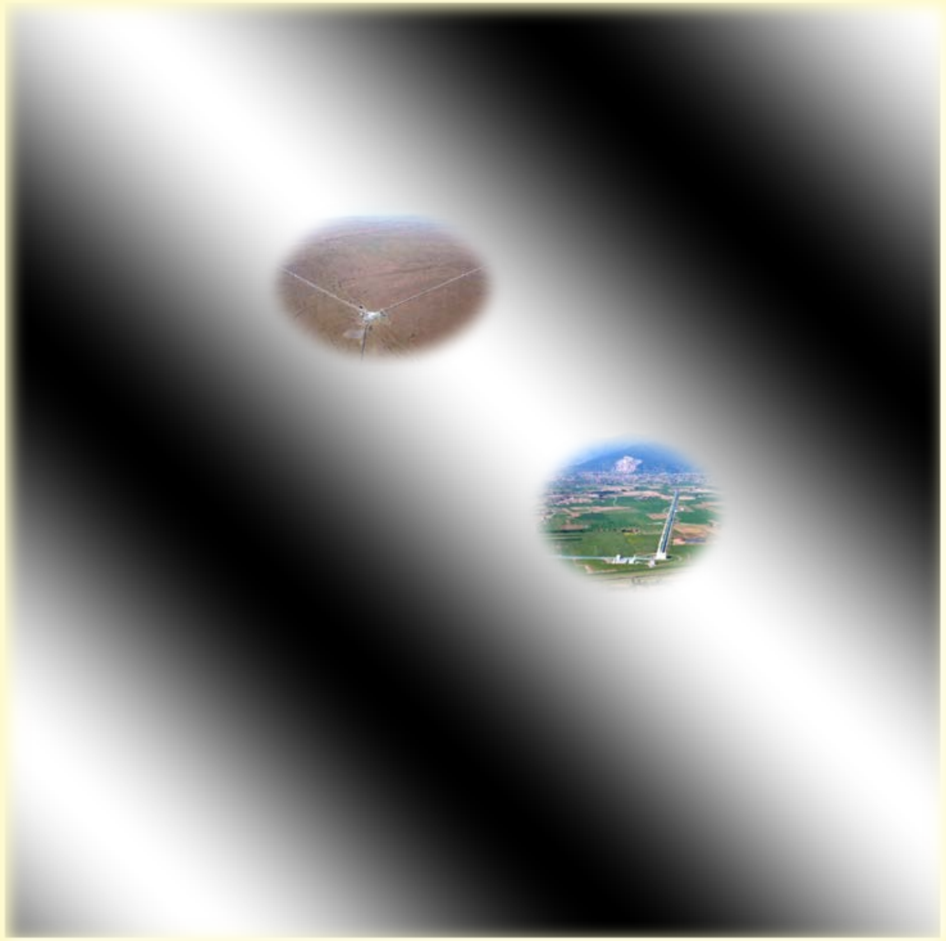
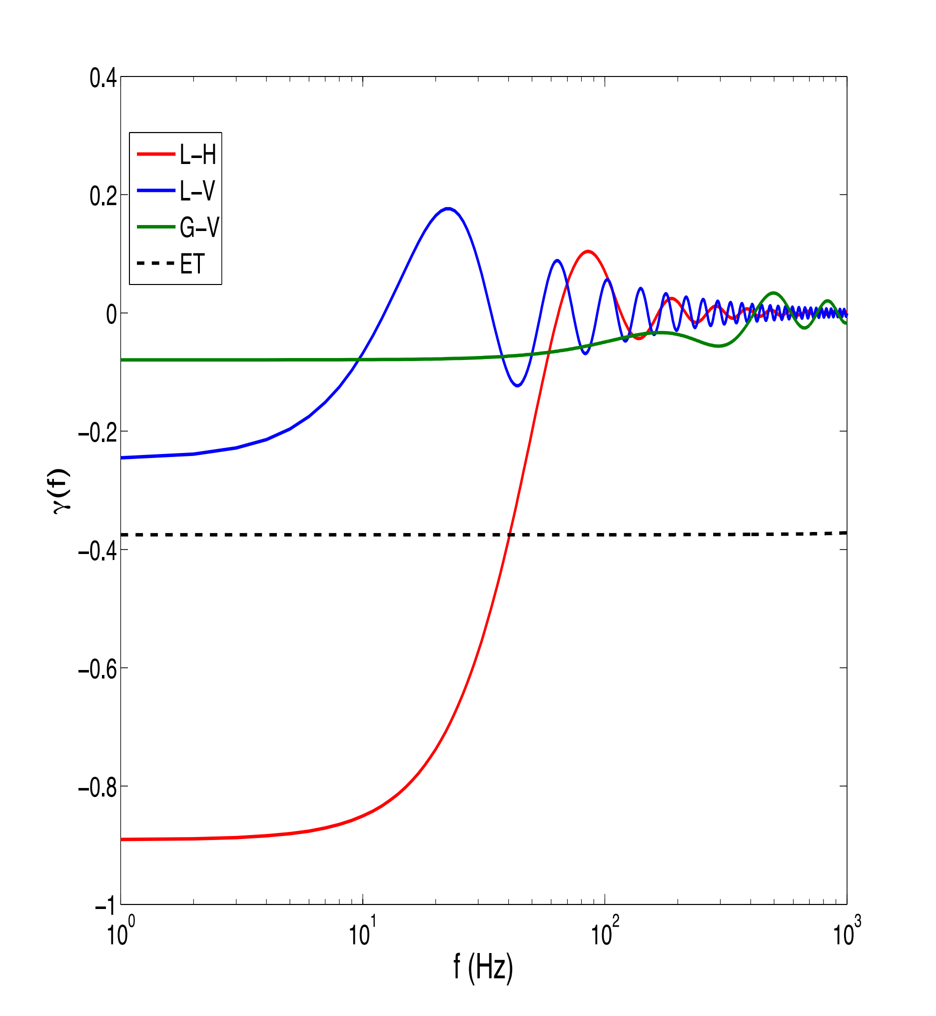
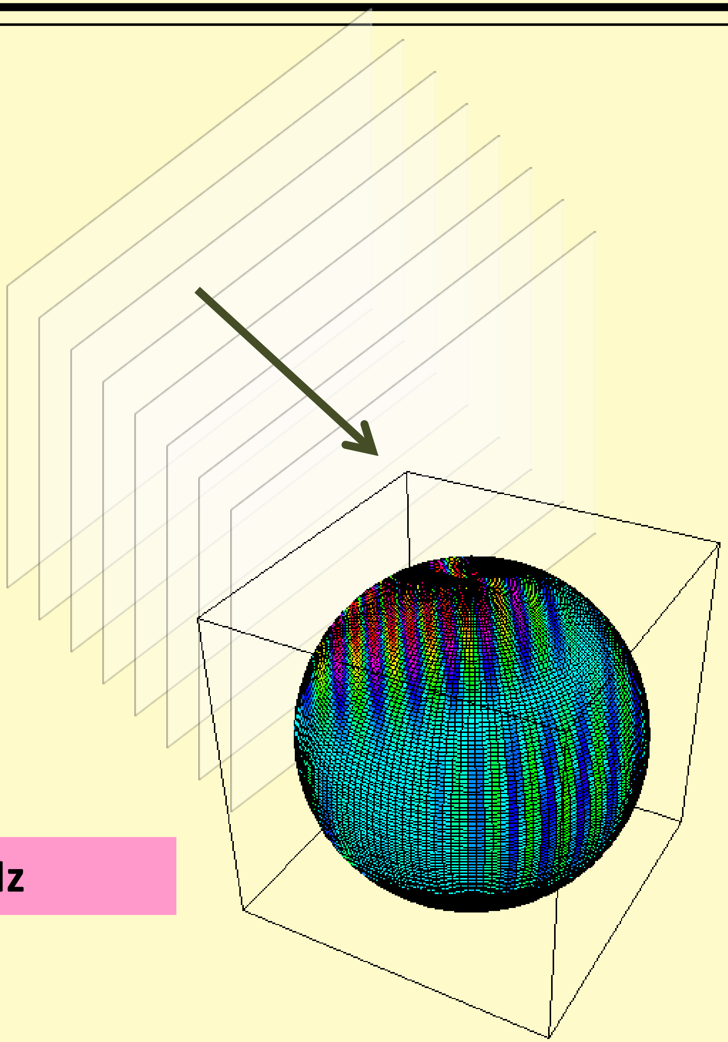




Targeted search: reconstruct a map of the gravitational wave uminosity in the sky

$$C(t,f) = \oint_{S^2} \mathcal{L}(\theta,\phi,f)\mathcal{F}(\theta,\phi,f,t)\,\sin\theta d\theta d\phi$$

- Correct the direction-dependent modulation
- Cross-Correlate
- At least 3 detectors needed to close the inverse problem.
- Angular resolution limited by λ/D



Virgo+WA 200 Hz

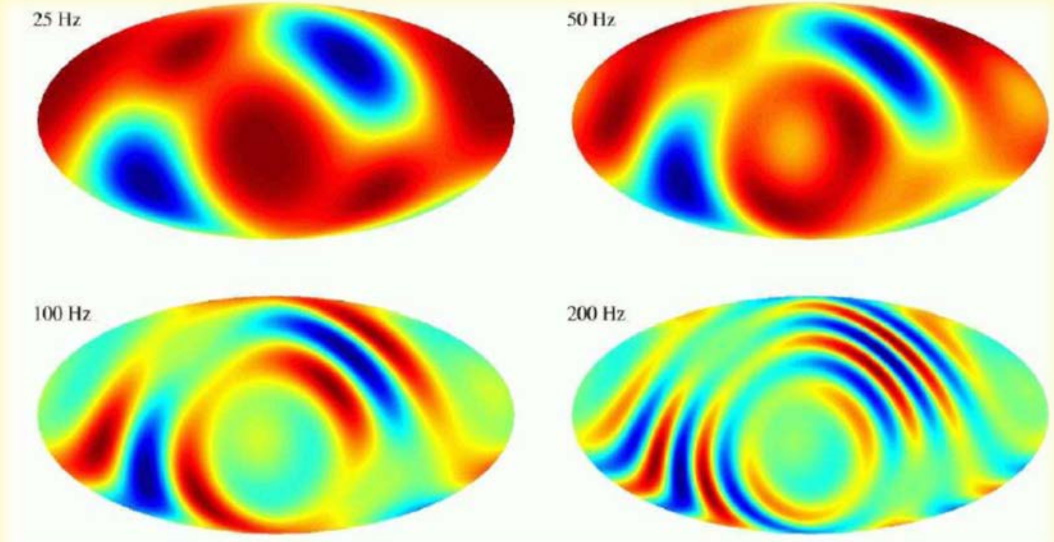
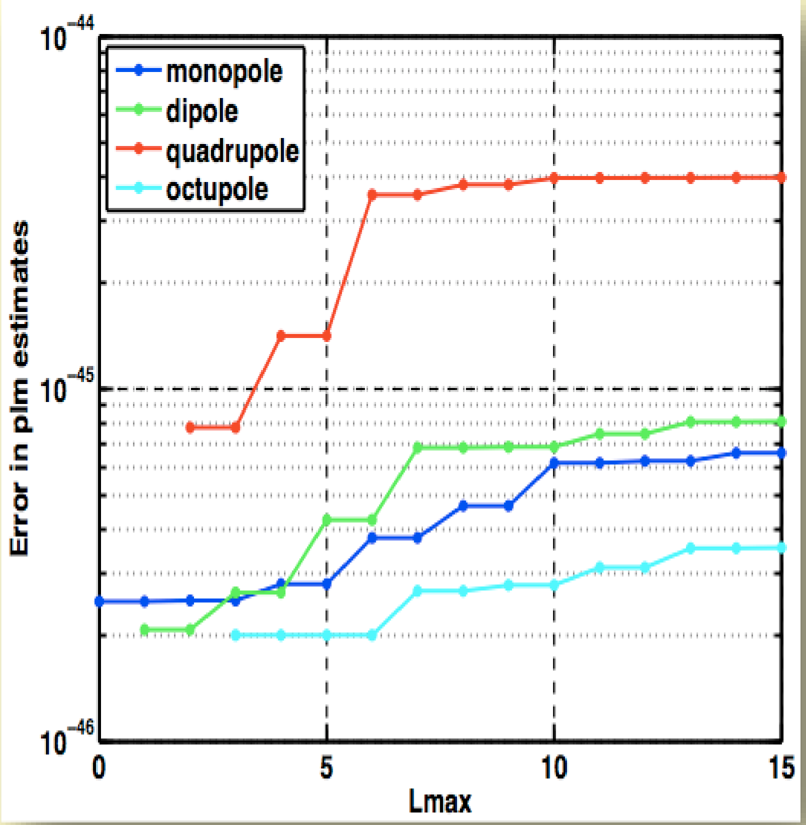


Figure 1. The antenna pattern for the cross-correlated LIGO detectors in the Earth-fixed frame, $\mathcal{F}(\bar{\theta},\bar{\phi},f)$, with $g_I = 0$ and $f = 25, 50, 100$ and 200 Hz.

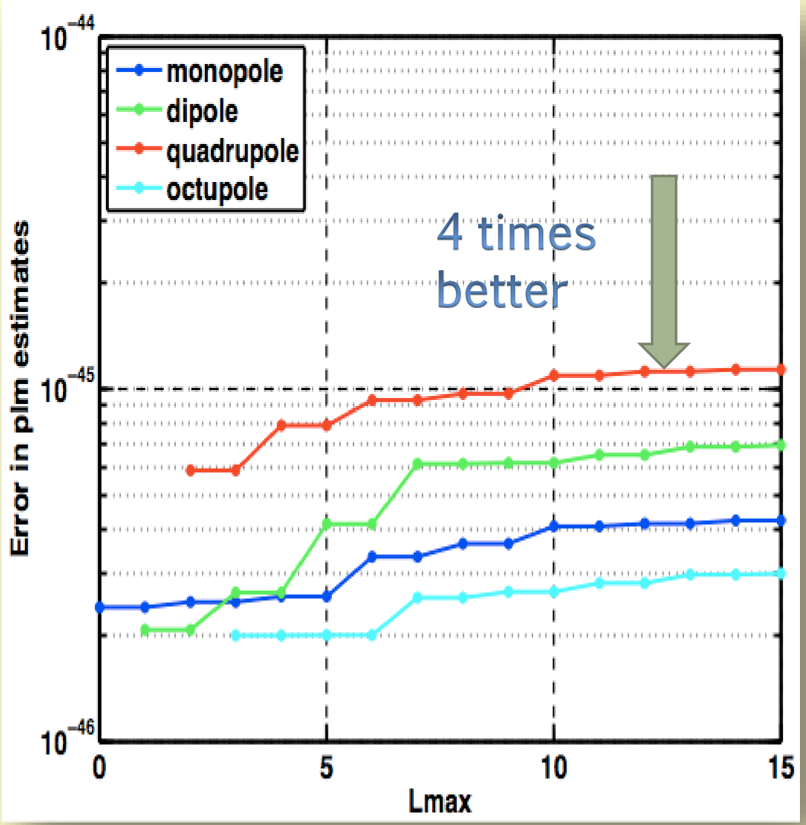
Cornish N.J., 2001, "Mapping the Gravitational Wave Background", CQG, 18, 4277; astro-ph/0105374

Detectors' number and reconstruction error

HL



HLV



Non Gaussian Stochastic Background

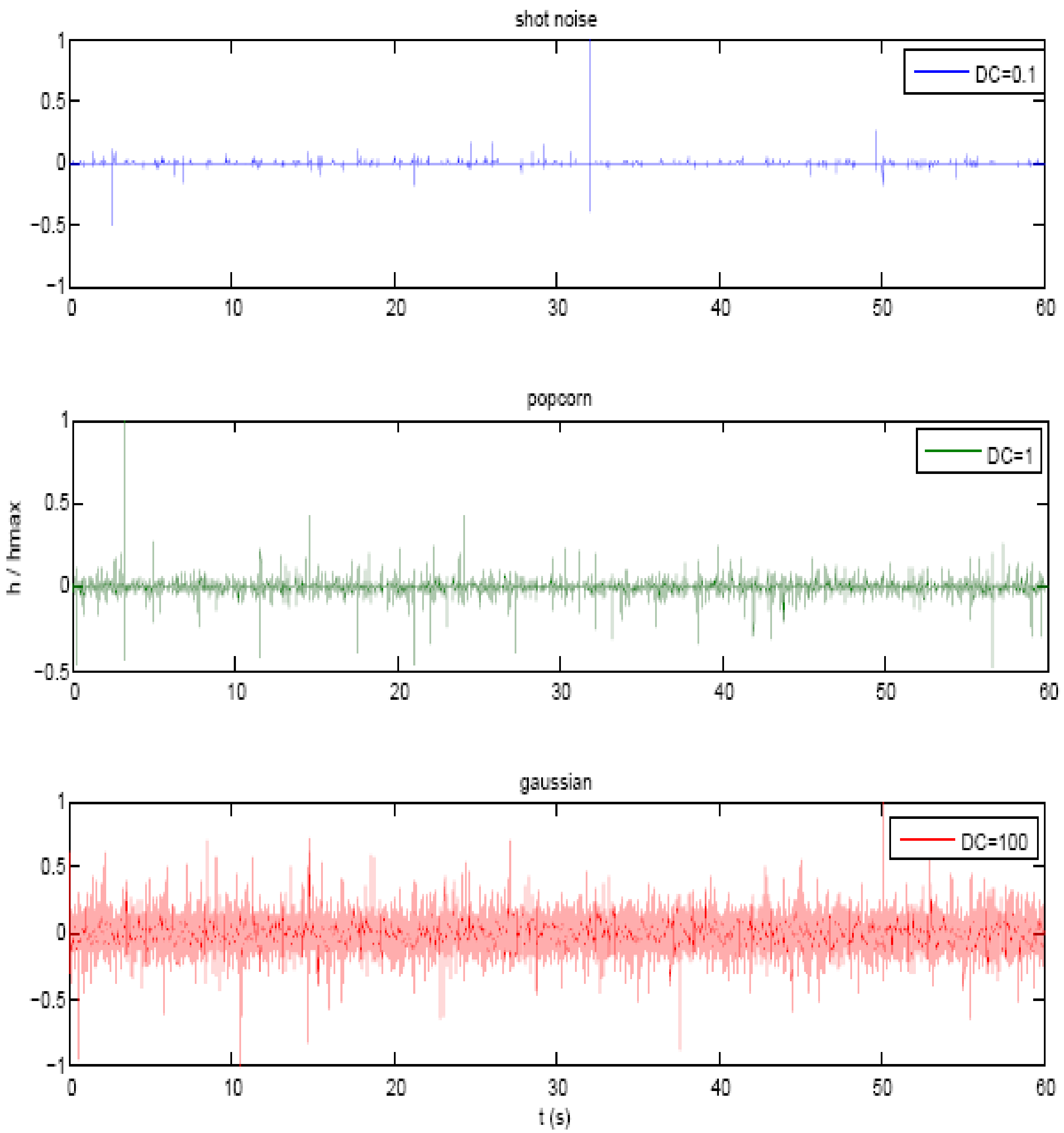
Duty cycle D(z): key parameter to characterize the detection regime. Given the average event duration τ

$$D(z) = \int_0^z [(1+z')\tau] \left[\frac{dR^O}{dz'}(z')\right] dz'$$

Observed event duration

Inverse observed time interval between events

- ***D*** $\ll$ **1: resolved sources**
  - Burst data analysis, optimal filtering
- ***D*** $\sim$ **1: popcorn noise**
  - Maximum Likelihood statistic *(Drasco et al. 2003)*, Probability Event Horizon *(Coward et al. 2005)*
- ***D*** $\gg$ **1:** ***Gaussian stochastic background***
  - Cross correlation statistic (isotropic/anisotropic)



PARAMETER ESTIMATION

Some examples

Bayesian approach

- If we know the statistical properties of the noise (and of the signal, in needed)

$$P\left(s_1, \cdots, s_N | \vec{\alpha}\right)$$

Data (known)

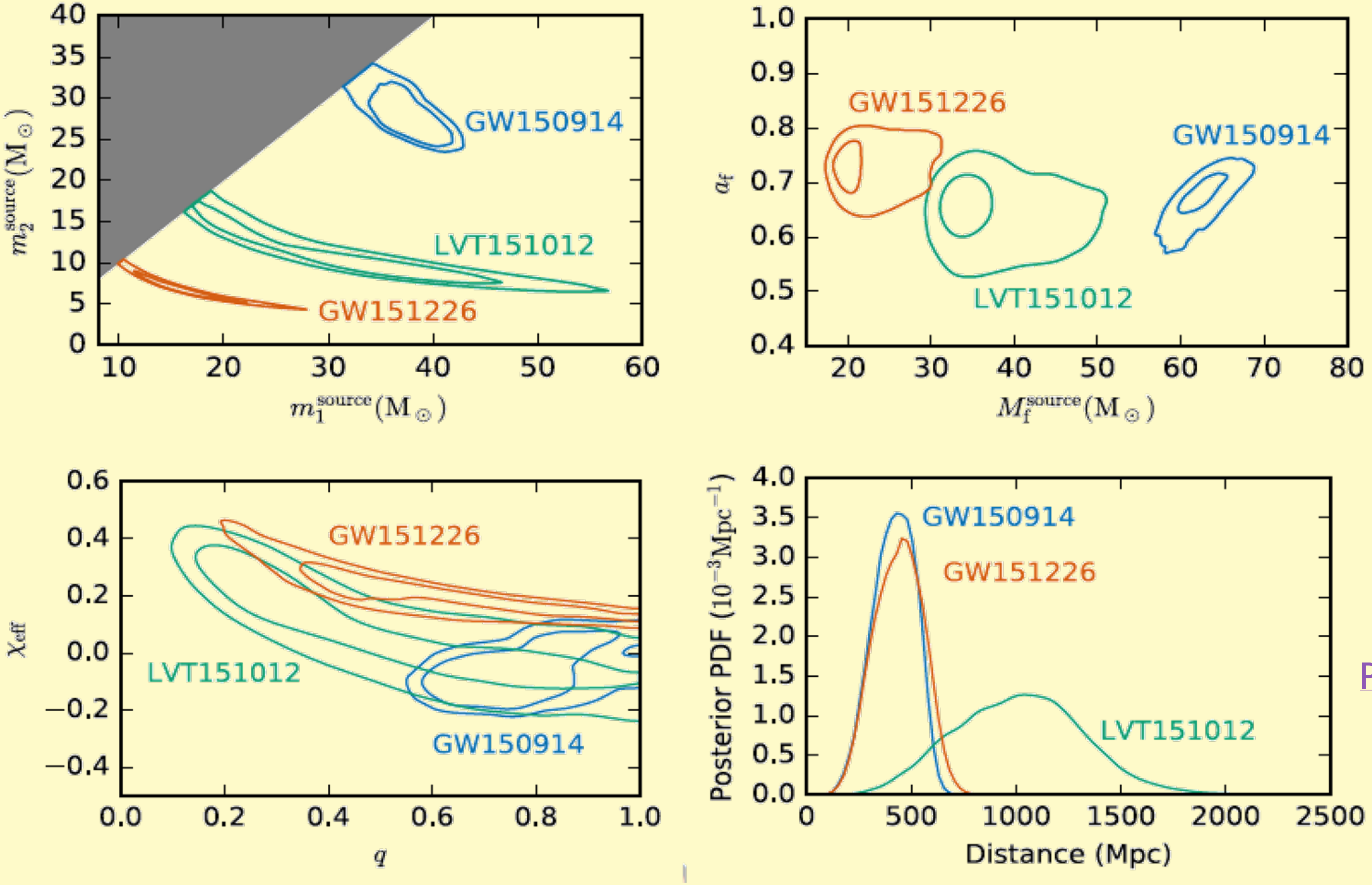
parameters (unknown)

- Using Bayes' theorem we get

$$P\left(\vec{\alpha}|s_1, \cdots, s_N\right) = \frac{P\left(s_1, \cdots, s_N | \vec{\alpha}\right) P\left(\vec{\alpha}\right)}{P\left(s_1, \cdots, s_N\right)}$$

- This is the «mother of all information»
- Take away message: waveforms can contain very detailed information about the parameters of the sources

Parameters of the BBH systems

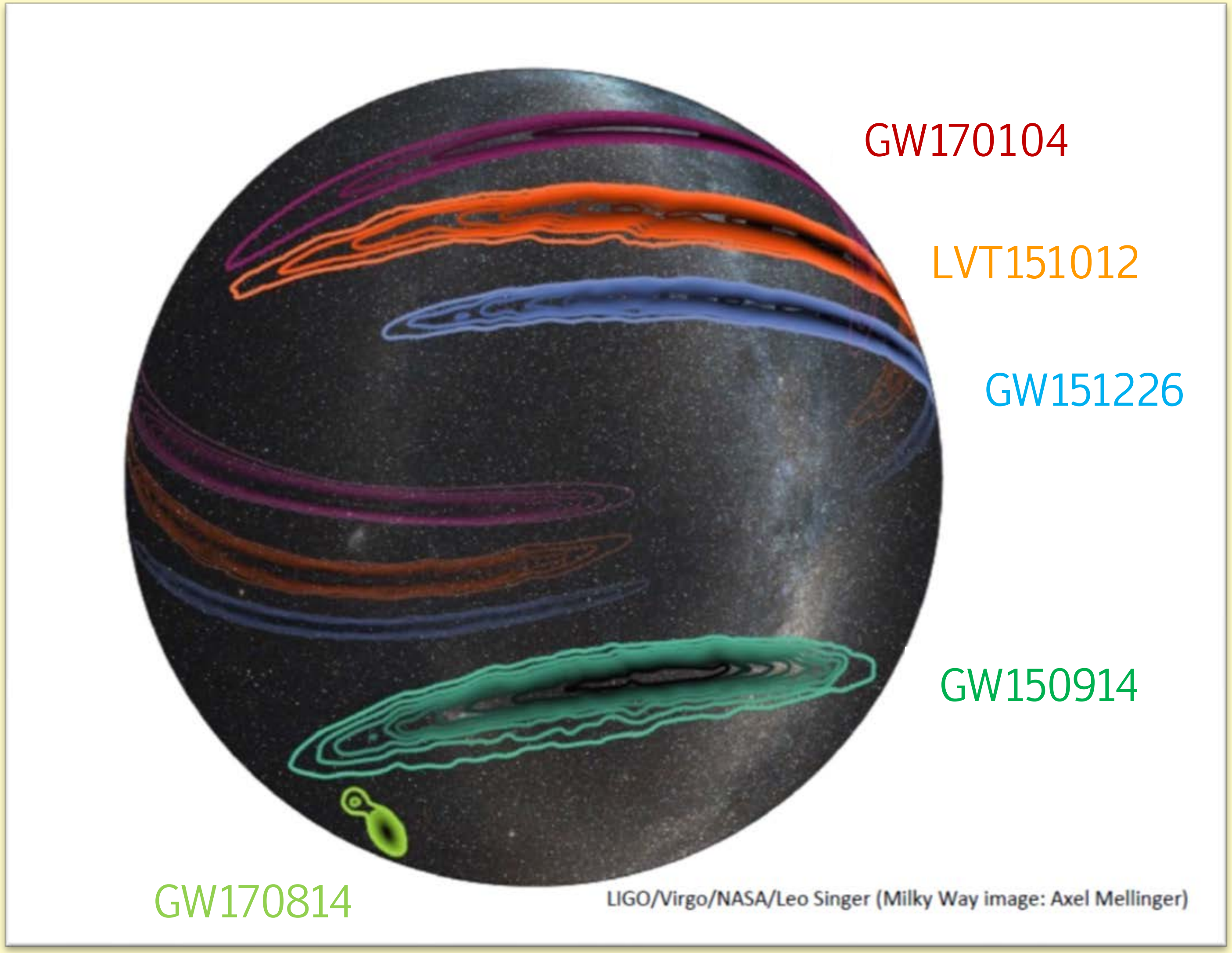


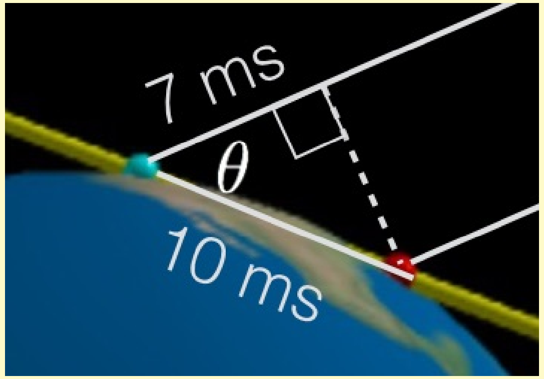
50% and 90% credible regions

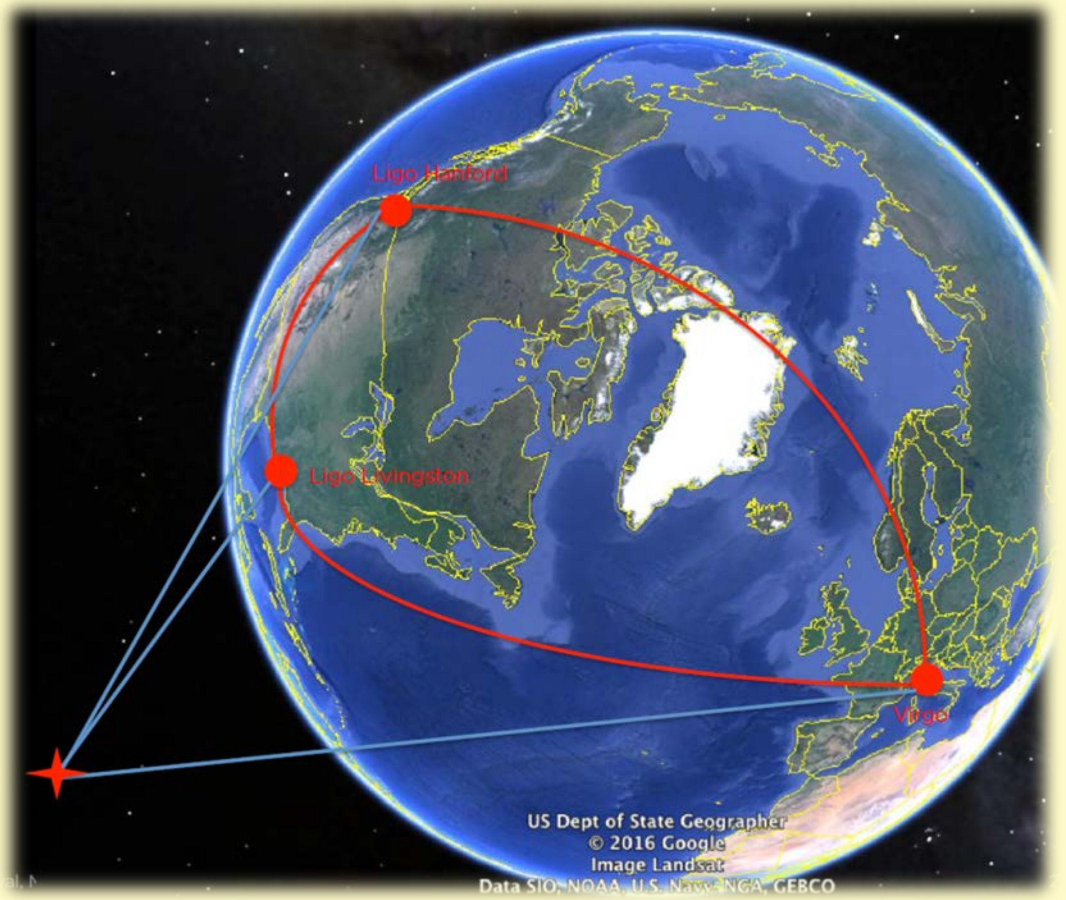
Phys. Rev. Lett. 116, 061102 (2016)

Phys. Rev. X 6, 041015

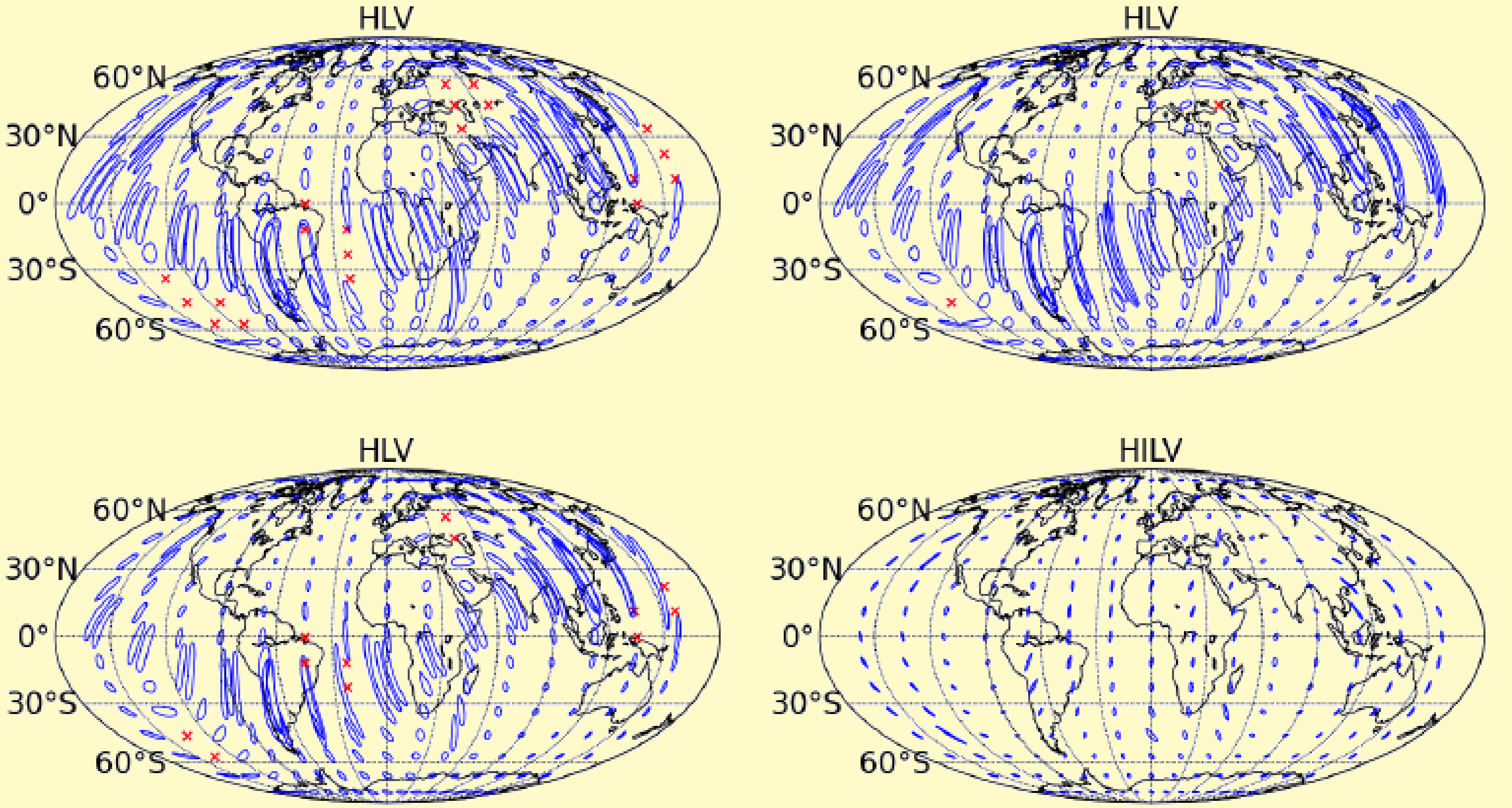
Localization

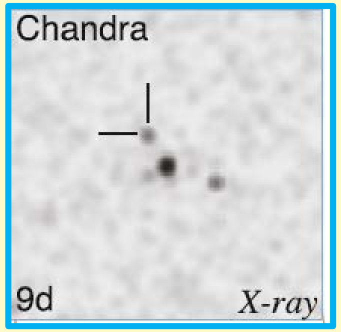


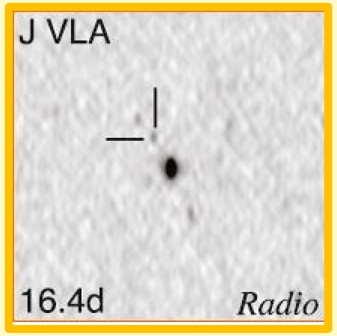




Localization



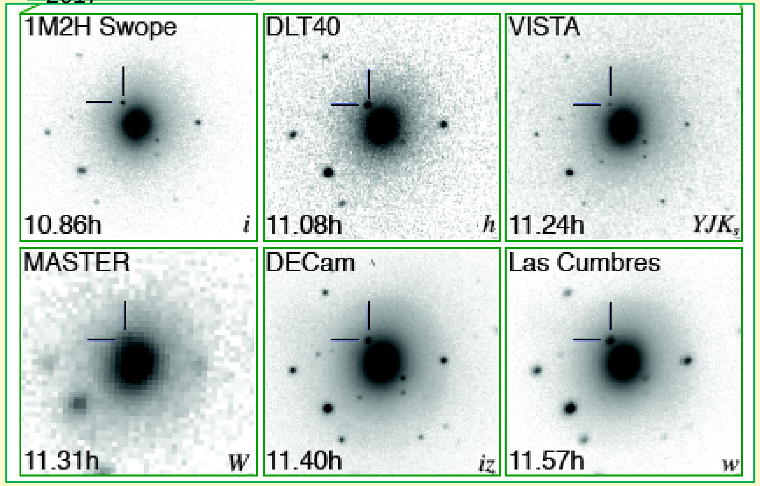




Coulter et al. 2017

Yang et al. 2017

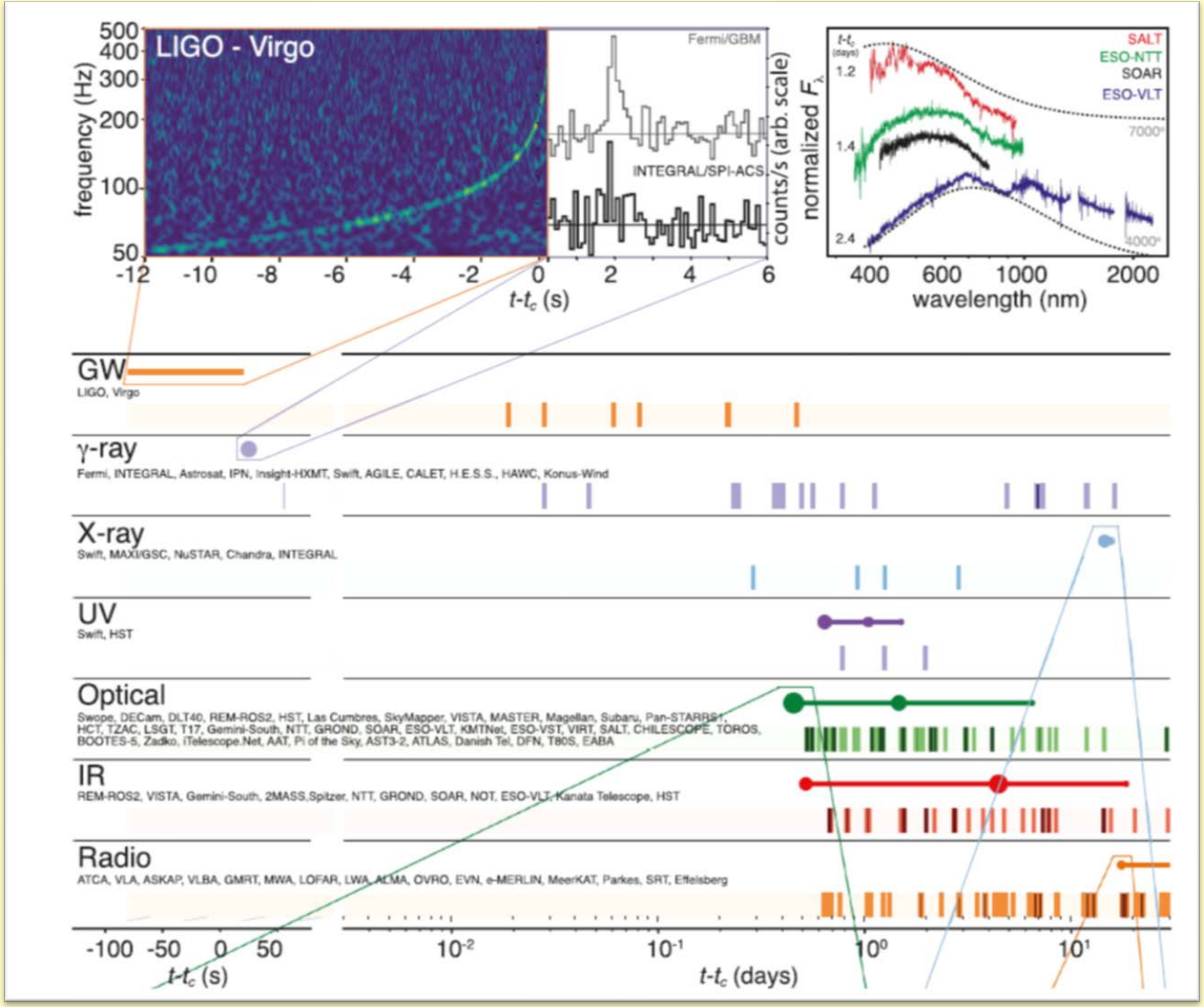
Tanvir et al. 2017



Lipunov et al. 2017

Allam et al. 2017

Accavi et al. 2017



[Astrophys. J. Lett. 848, L12 (2017)]