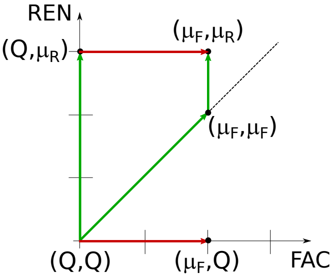
# Recurrence relations for scale dependence in QCD

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## Structure of QCD prediction

- Running coupling and $\beta$-function
- Structure of QCD prediction

## $\mu_R$ scale dependence

- Recurrence relation for $\mu_R$
- An example: running coupling

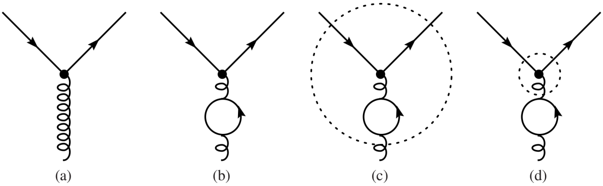
## $\mu_F$ scale dependence

- $\mu_F$ scale and DGLAP equation
- Moving in the $\mu_F - \mu_R$ plane
- An example: Deep Inelastic Scattering

## Conclusions


# Renormalization scale in QCD

Beyond tree level, we find divergent integrals in the UV region

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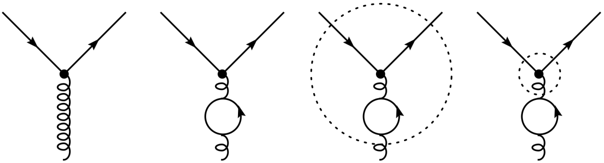

(a) (b) (c) (d)

→ a scale $\mu_R$ appears in the renormalization procedure

Evolution of the coupling $\alpha_s$ with $\mu_R$ → **QCD $\beta$-function**

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\ln\mu_R^2} = \beta(\alpha_s) = -(\beta_0\alpha_s^2 + \beta_1\alpha_s^3 + \dots), \quad \beta_0 = \frac{33-2n_f}{12\pi}$$

# Residual scale dependence

Expansion of an observable $\sigma$ in a perturbation series in the coupling $\alpha_s$ evaluated at $\mu_R$:

$$\sigma(Q) = \sum_{n=0}^{\infty} c_n \alpha_s^n$$

$\sigma$ **independent from** $\mu_R$, but true only for an *all-order* prediction! $\rightarrow$ fixed-order prediction retains a scale dependence

$$\sigma^{(k)}(Q, \mu_R) = \sum_{n=0}^{k} c_n(Q, \mu_R)\alpha^n(\mu_R)$$

The residual scale dependence is proved to be of $\mathcal{O}(\alpha_s^{k+1})$:

$$\sigma^{(k)}(Q, \mu_R) = \sigma^{(k)}(Q, Q) + \mathcal{O}(\alpha_s^{k+1})$$

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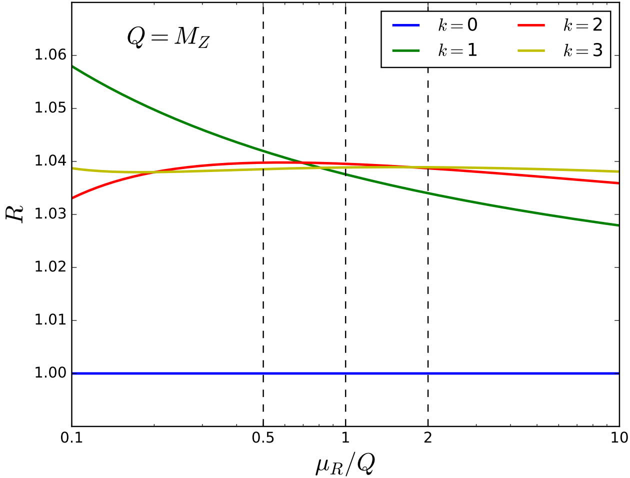
# Example of scale dependence

$$R = \sigma_{e^+e^-\to\text{had}}(Q)/\sigma_{e^+e^-\to\mu^+\mu^-}(Q) \propto 1 + c_1\,\alpha_s(Q) + c_2\,\alpha_s^2(Q) + \ldots$$



# Renormalization scale dependence

Observable $\sigma$ expanded in perturbation series:

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Once given $c_n(Q, Q)$, how can we find $c_n(Q, \mu_R)$? i.e. how can we shift the renormalization scale from $Q$ to a generic $\mu_R$?

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- Imposing:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu_R^2}\sigma^{(k)}(Q) = \frac{\mathrm{d}}{\mathrm{d}\ln\mu_R^2}\sum_{n=0}^{k} c_n(Q,\mu_R)\alpha_s^n(\mu_R) = 0,$$

  and neglecting higher order terms which appear in the calculation;

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- Solving the $\beta$-function up to some order and expanding the result aroung the couping evaluated at $\mu_R^2$:

  $$\alpha_s(Q^2) = \alpha_s(\mu_R^2) + \beta_0 L \alpha_s^2(\mu_R^2) + \{\beta_1 L + \beta_0^2 L^2\}\alpha_s^3(\mu_R^2) + \dots$$

  where $L = \ln(\mu_R^2/Q^2)$, and then substituting this expression in $\sigma(Q)$.

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$$\sigma(Q) = \sum_{n=0}^{\infty} c_n(Q, Q)\alpha_s^n(Q) = \sum_{n=0}^{\infty} c_n(Q, \mu_R)\alpha_s^n(\mu_R)$$

Once given $c_n(Q, Q)$, how can we find $c_n(Q, \mu_R)$? i.e. how can we shift the renormalization scale from $Q$ to a generic $\mu_R$?

- Working with an *all-order* expression and imposing:

$$\frac{\mathrm{d}}{\mathrm{d}\ln \mu_R^2}\sigma(Q) = 0,$$

using identities well known in statistical mechanics:

$$\sum_{a=\lambda}^{\infty} \alpha^a \sum_{b=0}^{\infty} f_{a,b}\alpha^{b+1} = \sum_{a=\lambda+1}^{\infty} \alpha^a \sum_{b=\lambda}^{a-1} f_{b,a-1-b}$$

$\longrightarrow$ **recurrence relation!**

# Recurrence relation for $\mu_R$ scale dependence

[arXiv:1105.5152]

The $c_n(Q, \mu_R)$ coefficient is given by a polynomial of degree less than or equal to $n-1$ in the $\ln(\mu_R^2/Q^2)$ variable:

$$c_n(Q, \mu_R) = \sum_{l=0}^{n-1} c_{n,l} \left( \ln \frac{\mu_R^2}{Q^2} \right)^l \tag{1}$$

with $c_{n,l}$ given by the following recurrence relation:

$$c_{n,l} = \begin{cases} c_{n,0} = c_n(Q, Q) \\ c_{n,l} = \dfrac{1}{l} \displaystyle\sum_{m=l-1}^{n-1} m\beta_{n-1-m} c_{m,l-1} \end{cases} \tag{2}$$

# An example: running coupling

$$\alpha_s(\mu) = \sum_{n=0}^{\infty} c_n(\mu,\mu)\alpha_s(\mu) = \sum_{n=0}^{\infty} c_n(\mu,\mu_0)\alpha_s(\mu_0)$$

with $c_n(\mu,\mu) = \delta_{n1}$ as initial condition. First terms, with $L = \ln(\mu_0^2/\mu^2)$:

$$\begin{aligned}\alpha_s(\mu^2) = {} & \alpha_s(\mu_0^2) + \beta_0 L \alpha_s^2(\mu_0^2) + \{\beta_1 L + \beta_0^2 L^2\}\alpha_s^3(\mu_0^2) \\ & + \left\{\beta_2 L + \frac{5}{2}\beta_0\beta_1 L^2 + \beta_0^3 L^3\right\}\alpha_s^4(\mu_0^2) \\ & + \left\{\beta_3 L + \frac{3}{2}(\beta_1^2 + 2\beta_0\beta_2)L^2 + \frac{13}{3}\beta_0^2\beta_1 L^3 + \beta_0^4 L^4\right\}\alpha_s^4(\mu_0^2) \\ & + \mathcal{O}(\alpha_s^5)\end{aligned}$$

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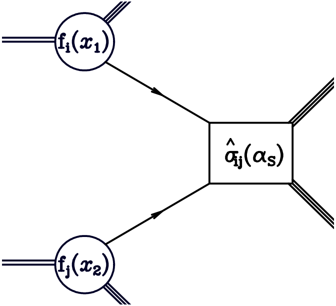
$$\begin{aligned}\alpha_s(\mu^2) = {} & \alpha_s(\mu_0^2) + \beta_0 L \alpha_s^2(\mu_0^2) + \{\beta_1 L + \beta_0^2 L^2\}\alpha_s^3(\mu_0^2) \\ & + \left\{\beta_2 L + \frac{5}{2}\beta_0\beta_1 L^2 + \beta_0^3 L^3\right\}\alpha_s^4(\mu_0^2) \\ & + \left\{\beta_3 L + \frac{3}{2}(\beta_1^2 + 2\beta_0\beta_2)L^2 + \frac{13}{3}\beta_0^2\beta_1 L^3 + \beta_0^4 L^4\right\}\alpha_s^4(\mu_0^2) \\ & + \mathcal{O}(\alpha_s^5)\end{aligned}$$

$$\sigma(Q) = \sum_{ij} \underbrace{\hat{\sigma}_{ij}(Q)}_{\text{Partonic cross section}} \otimes \overbrace{f_i \otimes f_j}^{\text{Parton Distribution Functions (PDFs)}}$$




"Multiplicative" convolution:

$$(a \otimes b)(x) = \int_x^1 \frac{\mathrm{d}y}{y} a(y) b\left(\frac{x}{y}\right)$$

$$a \otimes b = b \otimes a$$

$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$

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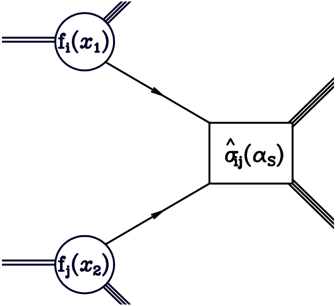
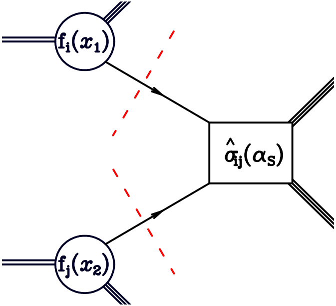



Divergence due to collinear emission of gluon

$$\sigma(Q) = \sum_{ij} \underbrace{\hat{\sigma}_{ij}(Q, \mu_F)}_{\text{Partonic cross section}} \otimes \overbrace{f_i(\mu_F) \otimes f_j(\mu_F)}^{\text{Parton Distribution Functions (PDFs)}}$$

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*absorbed* into the PDF $f$ at a scale $\mu_F$

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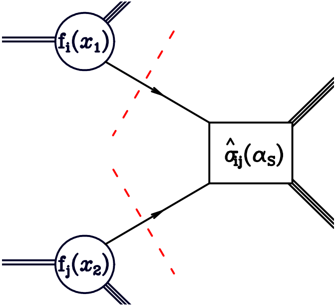


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Evolution of PDF given by **DGLAP equation**

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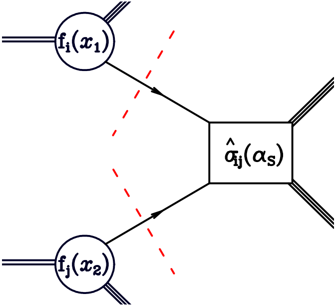


Divergence due to collinear emission of gluon

↓

*absorbed* into the PDF $f$ at a scale $\mu_F$

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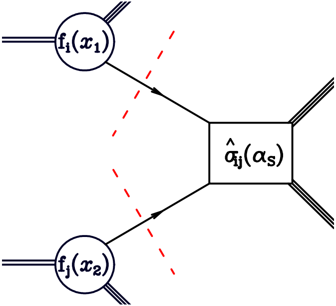
Evolution of PDF given by **DGLAP equation**

$f = q, \bar{q}, g$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu_F^2}\begin{pmatrix} q_i \\ g \end{pmatrix} = \sum_{q_k,\bar{q}_k} \begin{pmatrix} P_{ik}(\alpha_s(\mu_F)) & P_{ig}(\alpha_s(\mu_F)) \\ P_{gk}(\alpha_s(\mu_F)) & P_{gg}(\alpha_s(\mu_F)) \end{pmatrix} \otimes \begin{pmatrix} q_k \\ g \end{pmatrix}$$

$$\sigma(Q) = \sum_{ij} \left( \sum_{n=0}^{\infty} \hat{\sigma}_{ij}^{(n)}(Q, \mu_F, \mu_R) \alpha_s^{(n+\lambda)}(\mu_R) \right) \otimes f_i(\mu_F) \otimes f_j(\mu_F)$$

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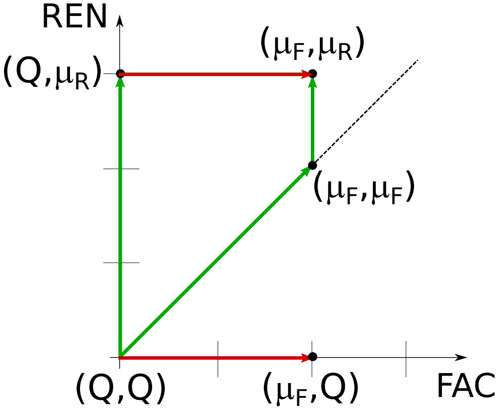
$$\hat{\sigma}_{ij}^{(n)}(Q, Q, Q) \xrightarrow{?} \hat{\sigma}_{ij}^{(n)}(Q, \mu_F, \mu_R)$$

$$\hat{\sigma}_{ij}^{(n)}(Q,Q,Q) \xrightarrow{?} \hat{\sigma}_{ij}^{(n)}(Q,\mu_F,\mu_R)$$

$$\boxed{\hat{\sigma}_{ij}^{(n)}(Q,Q,Q) \to \hat{\sigma}_{ij}^{(n)}(Q,\mu_F,\mu_F) \to \hat{\sigma}_{ij}^{(n)}(Q,\mu_F,\mu_R)}$$



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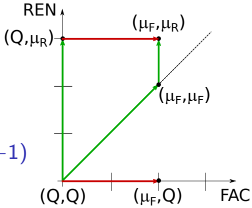
$$\hat{\sigma}_{ij}^{(n)}(Q,\mu_F,\mu_F) = \sum_{l=0}^{n} \hat{\sigma}_{ij}^{(n,l)} \left(\ln \frac{\mu_F^2}{Q^2}\right)^l$$

$$\hat{\sigma}_{ij}^{(n,l)} = \frac{1}{l} \sum_{m=l-1}^{n-1} \beta_{n-1-m}(m+\lambda)\hat{\sigma}_{ij}^{(m,l-1)} - \left\{ \sum_k \hat{\sigma}_{jk}^{(m,l-1)} \otimes P_{ki}^{(n-1-m)} + i \leftrightarrow j \right\}$$

$$\hat{\sigma}_{ij}^{(n,0)} \equiv \hat{\sigma}_{ij}^{(n)}(Q,Q,Q)$$



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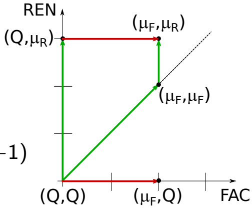

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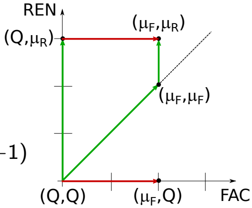


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$$\hat{\sigma}_{ij}^{(n,0)} \equiv \hat{\sigma}_{ij}^{(n)}(Q,Q,Q)$$

→ only symbolic, they must be properly convoluted with PDFs

# An example: short-distance coefficients in DIS

Only one convolution (here $q$ is quark singlet PDF):

$$F(Q) \supset \sum_{n=0}^{\infty} C_q^{(n)}(Q, \mu_F, \mu_R)\alpha_s^n(\mu_R) \otimes q(\mu_F)$$

First terms, with $L_M = \ln(\mu_F^2/Q^2)$ and $L_R = \ln(\mu_R^2/\mu_F^2)$:

$$\begin{aligned}
C_q^{(0)}(Q,\mu_R,\mu_F) &= c_q^{(0,0)} \\
C_q^{(1)}(Q,\mu_R,\mu_F) &= c_q^{(1,0)} + L_M(c_q^{(0,0)} \otimes P_{qq}^{(0)}) \\
C_q^{(2)}(Q,\mu_R,\mu_F) &= c_q^{(2,0)} + L_M\left[c_q^{(0,0)} \otimes P_{qq}^{(1)} + c_q^{(1,0)} \otimes \left(P_{qq}^{(0)} - \beta_0\mathbb{I}\right) + c_g^{(1,0)} \otimes P_{gq}^{(0)}\right] \\
&\quad + L_M^2\left[\frac{1}{2}\left(c_q^{(0,0)} \otimes P_{qq}^{(0)}\right) \otimes \left(P_{qq}^{(0)} - \beta_0\mathbb{I}\right) + \frac{1}{2}\left(c_q^{(0,0)} \otimes P_{qg}^{(0)}\right) \otimes P_{gq}^{(0)}\right] \\
&\quad + L_R\beta_0 c_q^{(1,0)} + L_R L_M \beta_0 c_q^{(0,0)} \otimes P_{qq}^{(0)}
\end{aligned}$$

where $c_i^{(n,0)} \equiv C_i^{(n)}(Q,Q,Q)$ (note that $c_g^{(0)} = 0$).

## Conclusions

- Recurrence relations:
  - general way to obtain scale dependence;
  - straightforward implementation into any HEP generator;
  - checked against literature (e.g. top pair production);
  - code in preparation with a series of examples and *ready-to-use* scale dep. for some important observables.
- Applications?
  - theoretical uncertainty estimates;
  - useful for studies about scale choice.