VIIIth International Conference
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On the validity of the Baïer-Katkov formula

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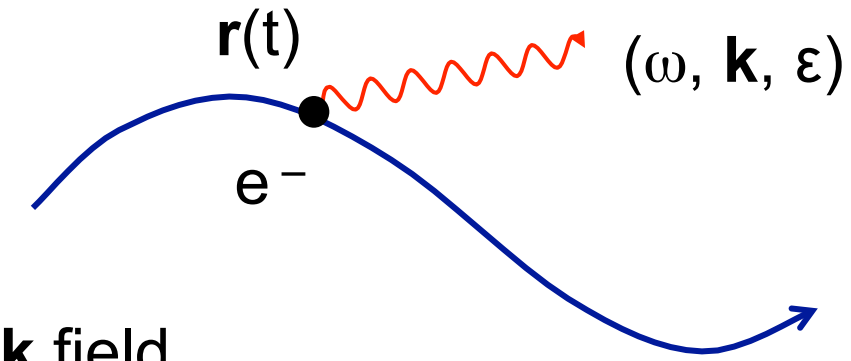
Classical radiation formula in vacuum (1/2)



applies to :

- Synchrotron radiation in **weak** field,
  e.g. in undulator
- **Soft** Compton effect (Thompson regime)
- **Soft** coherent Bremsstrahlung ($\omega << \gamma m$)
- Channeling Radiation (classical trajectory .and. $\omega << \gamma m$)

Natural units systems $\hbar = c = 1$; $\alpha = 1/137$

$m = 511$ KeV $p^0/m \equiv \gamma = (1-v^2)^{-1/2}$ $(v \equiv \beta)$

λ_C = *Compton wavelength* = $1/m$ = 386 fermi

Classical radiation formula (2/2) :

Spectral-angular distribution : $dN(\boldsymbol{\varepsilon}) / (d\omega d\Omega) = (\alpha\omega/4\pi^2) |\mathbf{a}\cdot\boldsymbol{\varepsilon}^*|^2$

$\boldsymbol{\varepsilon}$ = photon polarization

Right circular polarization (helicity $\Lambda = +1$) : $\boldsymbol{\varepsilon}_{(+)} = (1, i) /\sqrt{2}$

Left circular polarization (helicity $\Lambda = -1$) : $\boldsymbol{\varepsilon}_{(-)} = (1, -i) /\sqrt{2}$

$\mathbf{a} = \int \exp(i\phi)\, d\mathbf{r}_\perp$;

$\phi = k\cdot X$

4-vectors :
$X = (t,\mathbf{r})$
$k = (\omega,\mathbf{k})$
$p = (\gamma m, \gamma m\mathbf{v})$

$\mathbf{r}_\perp$ = perpendicular to $\mathbf{k}$; $\mathbf{v}_\perp = d\mathbf{r}_\perp /dt$

Ultrarelativistic approximation : $\phi = (\omega/2) \int^t (\gamma^{-2} + \mathbf{v}_\perp^2)\, dt'$

*The classical theory is **Spin-blind** and without **recoil effect***

Recoil and spin effects

… become important when $\omega/\gamma m$ is not small. It can happen in :

- fast varying field: $|F|^{-1}\,|dF/dt| \sim m$ in the instantaneous electron frame, where $F = F^{\mu\nu} = \{\mathbf{E},\mathbf{B}\}$. Examples:
  - hard Compton effect ($p.k \sim > m^2$)
  - bremstrahlung at impact parameter $b < \sim \gamma\,\lambda_C$

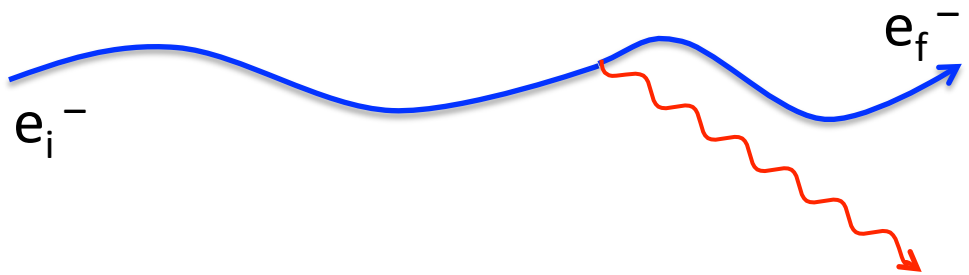
- *Strong Field*, i.e., $|F| \sim>$ $\mathbf{E_{crit}} = m^2/e = 1{,}32\ 10^{18}$ volt/m

in the electron frame. Examples:

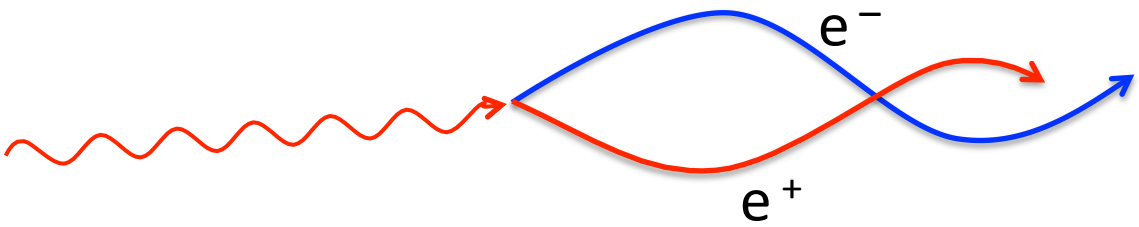
- near neutron stars,
- in channeling of electrons of energy ~> 100 GeV
- in very intense laser fields.

The "magic" Baier-Katkov formula (1/5)

... is a modification of the classical radiation formula which takes **recoil** and **spin** effects into account.



Using crossing symmetry, it can also apply to pair creation in strong field.



The "magic" Baier-Katkov formula (2/5)

1) *Recoil effect* : replace $\phi = k \cdot X$

by $\phi' = (\gamma/\gamma') \, k \cdot X$

2) *Spin dependence* : replace $\mathbf{a} \cdot \mathbf{e}^*$ by $\langle \lambda' | a_\Lambda | \lambda \rangle$

λ, λ' = electron helicities = $\pm 1/2$

Λ = photon helicity.

After summation over spins :

$$dN / (d\omega d\Omega) = (\alpha\omega/8\pi^2) \left\{ (1+\gamma^2/\gamma'^2) \left| \int \exp(i\phi') \, d\mathbf{r}_\perp \right|^2 + (1/\gamma' - 1/\gamma)^2 \left| \int \exp(i\phi') \, dt \right|^2 \right\}$$

The "magic" Baier-Katkov formula (3/5)

Helicity *non-flip* amplitudes

$\langle + | a_- | + \rangle = \int \exp(i\phi') \, (dx+idy) / \sqrt{2}$
$\langle - | a_+ | - \rangle = \int \exp(i\phi') \, (dx-idy) / \sqrt{2}$
} = as classical, but with ϕ'

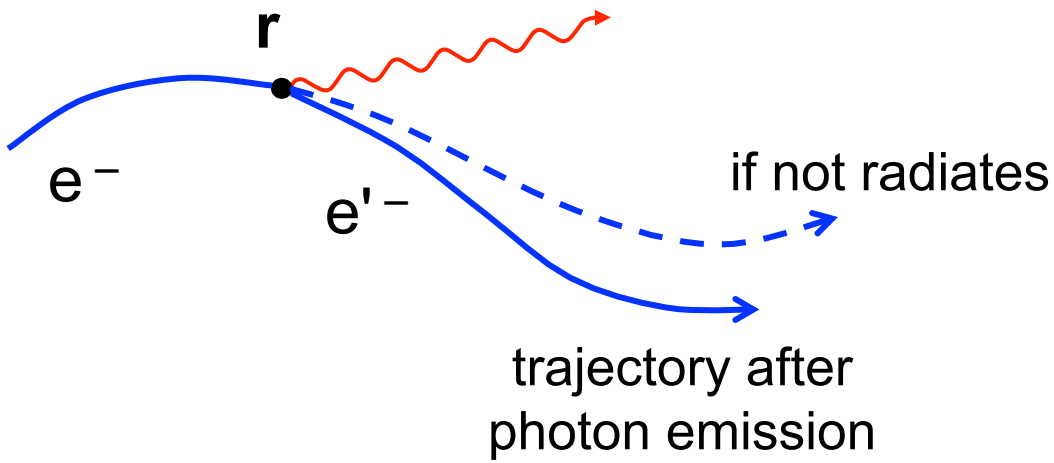
$\langle - | a_- | - \rangle = (\gamma/\gamma') \langle + | a_- | + \rangle$
$\langle + | a_+ | + \rangle = (\gamma/\gamma') \langle - | a_+ | - \rangle$

Helicity-*flip* amplitudes

$\langle - | a_+ | + \rangle = - \langle + | a_- | - \rangle = 2^{-1/2} (1/\gamma' - 1/\gamma) \int \exp(i\phi') \, dt$
$\langle - | a_- | + \rangle = - \langle + | a_+ | - \rangle = 0$ (too much imbalance of helicity !)

The "magic" B-K formula (4/5)

« magic » because it does not depend on the final electron trajectory !



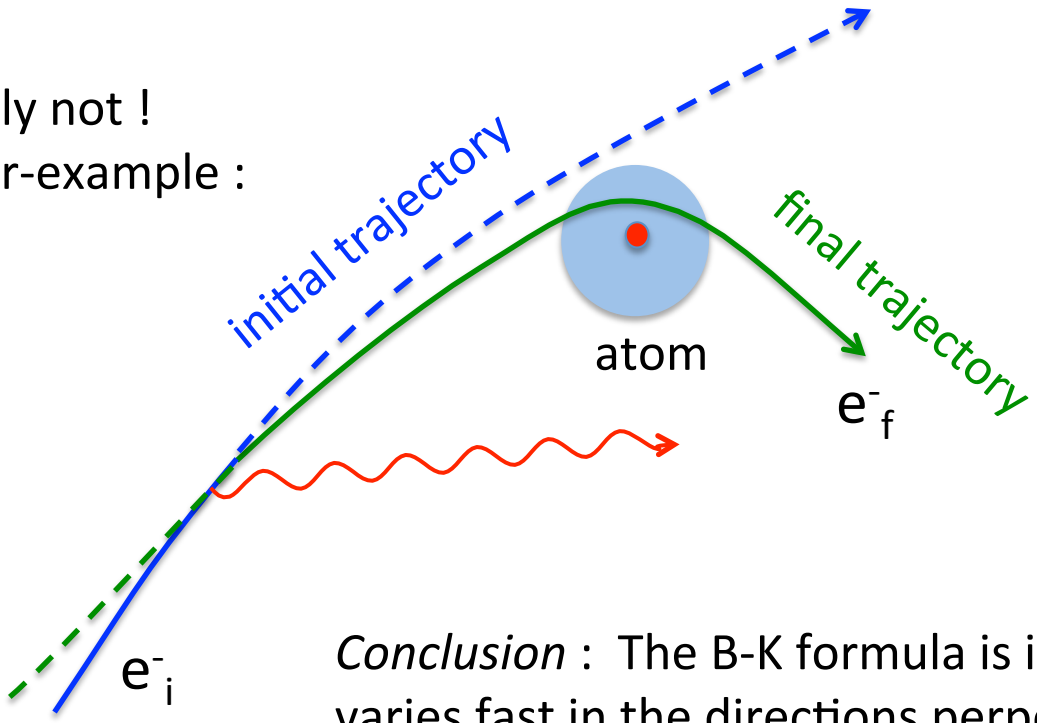
. . . whereas the quantum formula involves
both the initial and final wave functions :

$$\langle f|a|i\rangle = \int d^3\mathbf{r}\ \psi_f^*(\mathbf{r})\ \boldsymbol{\varepsilon}^*.\boldsymbol{\alpha}\ \psi_i(\mathbf{r})\ e^{i\mathbf{k}.\mathbf{r}}$$

The "magic" B-K formula (5/5)

Should we conclude that the full information about the final trajectory is contained in the initial one ?

Certainly not !
Counter-example :

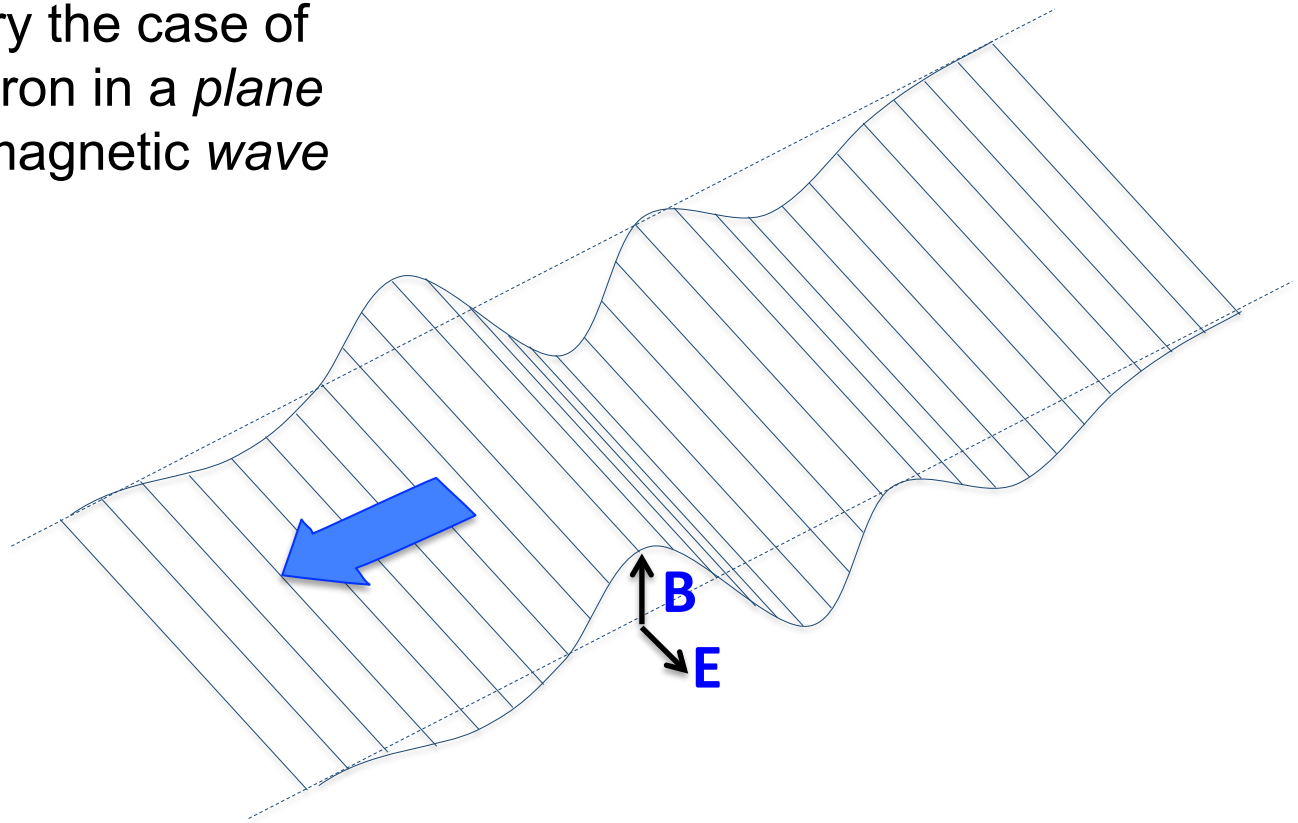


Conclusion : The B-K formula is inaccurate if the field varies fast in the directions perpendicular to the trajectory.

- Does the B-K formula becomes exact when the field is invariant along a plane ?

Plane wave external field

Let us try the case of an electron in a *plane* electromagnetic *wave packet* :



Then the Dirac wave function of the electron has a simple form, found by Volkov

Plane wave paket - Volkov wave function

Representation where α_z and Σ_z are diagonal: $\psi = (\chi,\xi) = (\chi_+ , \chi_- , \xi_+ , \xi_-)$

χ = component of α_z = +1
ξ = component of α_z = -1

+ or – is the sign of the *helicity* $\Sigma_z/2$

Light-like coordinates: $X = (\eta , \mathbf{r}_T , X^-)$ with $\eta = X^+ = t+z$ and $X^- = t-z$

Potential 4-vector of the wave packet: $(0, \mathbf{A}_T(\eta), 0)$
Mechanical $\mathbf{p}_T = \gamma m \mathbf{v}_T$
Canonical $\mathbf{P}_T = \mathbf{p}_T - e\mathbf{A}_T$ $\quad P^+ = p^+ ; P^- = p^-$

Mass-shell condition:
$p^+p^- = m^2 + \mathbf{p}_T^2$

Eigenstates of P^+ and $\mathbf{P}_T$:

$$\Psi(X) = \exp(-i\, p^+X^-/2 + i\, \mathbf{P}_T.\mathbf{X}_T)\, \psi(\eta)$$
$$\mathbf{p}_T(\eta) = \mathbf{P}_T + e\mathbf{A}_T(\eta)$$
$$p^-(\eta) = [m^2 + \mathbf{p}_T^2(\eta)]/p^+$$

Solution of the Dirac equation:

$$\chi(\eta) = \chi(-\infty) \exp\{ (-i/2) \int_{-\infty}^{\eta} d\eta'\ p^-(\eta') \}$$
$$\xi(\eta) = (1/p^+)\, [m + \boldsymbol{\sigma}.\mathbf{p}_T(\eta)\, \sigma_z]\, \chi(\eta)$$

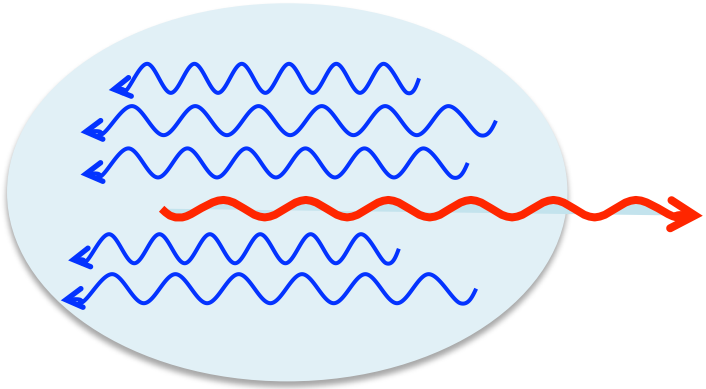
Volkov transition amplitude – 1) the phase factor

Photon momentum $k = (k^+ , \mathbf{k}_T , k^-)$.

Frame where $\mathbf{k}_T = 0$, $k^- = 0$

Mom. conservation $P' + k = P$

$p'^+ = p^+ - k^+$; $\mathbf{P'}_T = \mathbf{P}_T$; $\mathbf{p'}_T(\eta) = \mathbf{p}_T(\eta)$



$$\langle f| a |i\rangle = \int d\eta\, \psi_f^*(\eta)\, \boldsymbol{\varepsilon}^*.\boldsymbol{\alpha}\, \psi_i(\eta)\, \exp(ik^-\eta/2)$$

$$= \int d\eta\, \exp[\, i\phi'(\eta)\,]\ \ \langle \lambda_f|Q(\boldsymbol{\varepsilon},\eta)|\lambda_i\rangle\ ,$$

The phase factor

$$\phi'(\eta) = (p^+/p'^+) \int_{-\infty}^{\eta} d\eta'/2\ \ [m^2+\mathbf{p}_T^2(\eta)] / (p^+)^2$$

equals the Baïer-Katkov phase in the ultra-relativistic limit,

$\omega\, dt \Rightarrow k^+ d\eta'/2$; $\gamma \Rightarrow p^+/(2m)$; $\mathbf{v}_T(t) \Rightarrow 2\, \mathbf{p}_T(\eta)/p^+$; $\gamma/\gamma' \Rightarrow p^+/p'$

Transition amplitude – 2) the spin factor $\langle \lambda_f | Q(\boldsymbol{\varepsilon},\eta) | \lambda_i \rangle$

helicity $\lambda = \sigma_z /2$ *applied to the* $\alpha_z = +1$ *component* χ

$\langle + | Q(\mathbf{-},\eta) | + \rangle = \sqrt{2}\ [p_x(\eta) + i\, p_y(\eta)] / p^+$

$\langle - | Q(\mathbf{-},\eta) | - \rangle = (p^+/p'^+)\ \langle + | Q(\mathbf{-},\eta) | + \rangle$

$\langle - | Q(\mathbf{+},\eta) | + \rangle = \sqrt{2}\ m\ (1/p'^+ - 1/p^+)$

$\langle - | Q(\mathbf{-},\eta) | + \rangle = 0$, etc.

Recall Baïer-Katkov :

$\langle + | a_- | + \rangle = \int \exp(i\phi')\ (dx+idy)\ /\sqrt{2}$

$\langle - | a_- | - \rangle = (\gamma/\gamma')\ \langle + | a_- | + \rangle$

$\langle - | a_+ | + \rangle = 2^{-1/2}\ (1/\gamma' - 1/\gamma)\ \int \exp(i\phi')\ dt$

$\langle - | a_- | + \rangle = 0$

Application of the B-K formula

- ‘Exact’ case : Compton back scattering on laser field

- Approximate cases : when the electron is ultra-relativistic so that in its frame the field of the radiator is nearly equivalent to a plane electromagnetic wave
- (Weizsäcker-Williams approximation). Examples:

> Undulator radiation
> Synchrotron radiation
> Coherent bremsstrahlung on planes
> Bremsstrahlung on a single atom

In these cases, the Baïer-Katkov formula can also be obtained by the WKBJ method.

What about channeling radiation ?

B-K formula in channeling radiation

In channeling, the potential strongly depends on the transverse coordinates. The B-K formula is *a priori* inapplicable.

The spectral lines predicted by the B-K formula do not coincide with the ones of radiative transition between transverse states (X.A. in Channeling 2014).

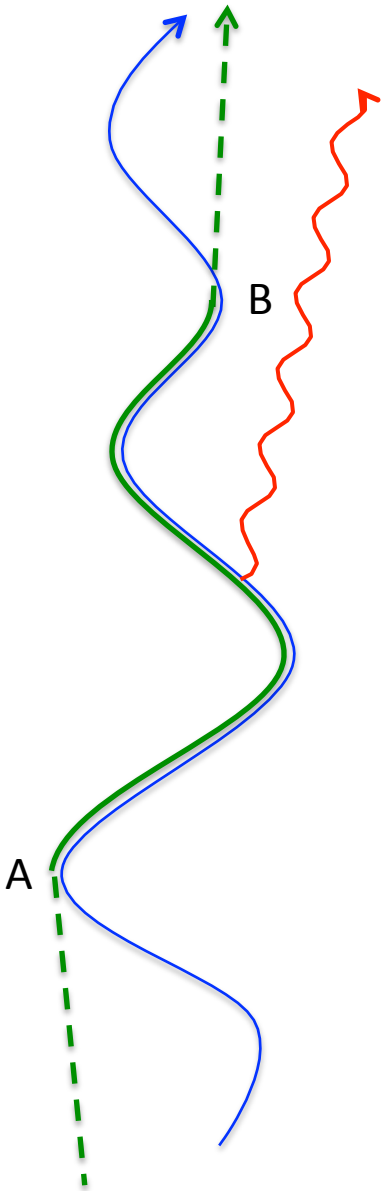
Radiative capture :

above-barrier state => channeled state

is certainly a bad case for the B-K formula: the final motion is too different from the initial one.

Nevertheless, B-K formula is usefull in channeling radiation, if one applies it by small parts of the trajectory, like **AB** here (completed by the 2 semi-infinite lines).

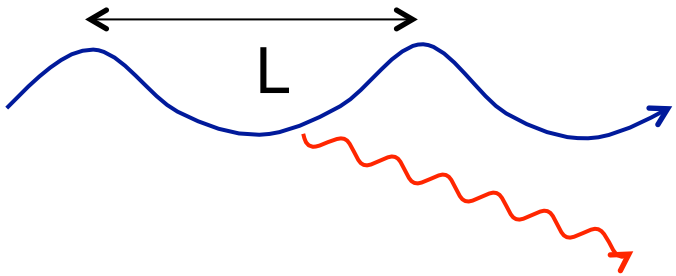
It makes the spectrum continuous, but provides an average energy loss. This is the method used in the Fortan program FOT.



---------- Thank you ! ----------

Fine test of the B-K formula in channeling radiation

- Assume periodic trajectories and compare the spectral lines predicted by B-K with "exact" spectral lines.



Conserv. of energy and P_L :

$$(\omega/\gamma\gamma' + \omega\theta^2)/2 = E_T - E'_T \qquad (1)$$

B-K formula for the ν^{th} harmonic

$$(\omega/\gamma\gamma' + \omega'\theta^2)/2 = 2\nu\pi/L - \omega'\langle \mathbf{v}_T^2 \rangle /2 \qquad (2)$$

$\nu = n - n'$ = decrease of transverse quantum number
n is given by the Bohr quantization rule $L \cdot E \cdot \langle \mathbf{v}_T^2 \rangle = 2n\pi$
$\Rightarrow$ The R-H-S of (1) and (2) are equal for $\omega << E$.

$\Rightarrow$ B-K formula does not give the exact detailed spectrum in the hard region of CR.

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Quantum versus classical approach of dechanneling and other incoherent processes at high energy in aligned crystals

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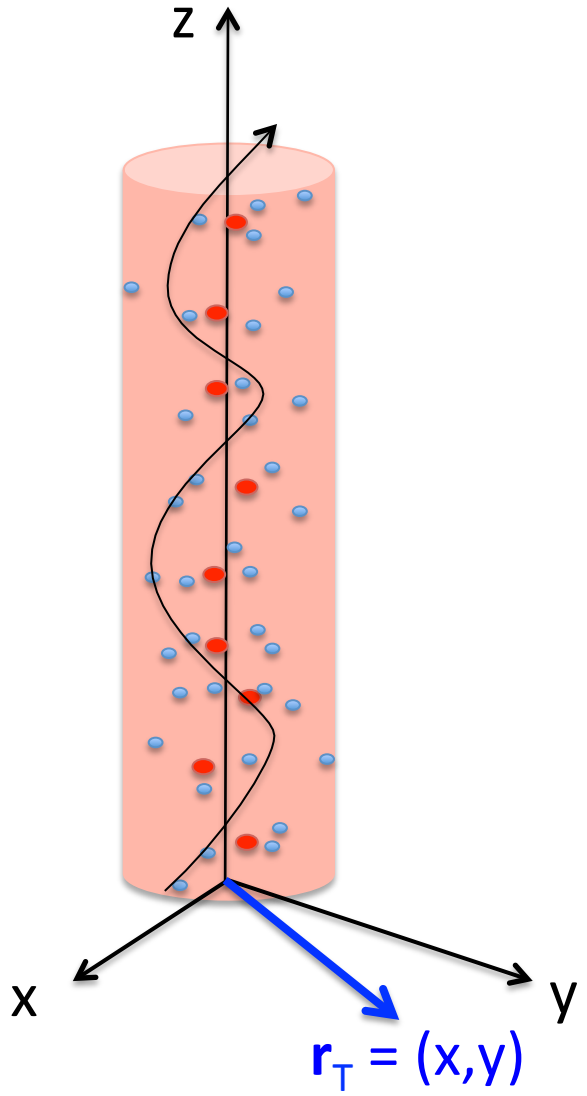
(Axial channeling)

Continuous Lindhard potential:

$V_{Lin}(\mathbf{r}_T) = \langle\langle V(t, \mathbf{r}_T, z)\rangle_t\rangle_z$

created by the charge density

$\rho_{Lin}(\mathbf{r}_T) = \langle\langle \rho(t,\mathbf{r}_T,z)\rangle_t\rangle_z$



V_{Lin} conserves :

> the projectile energy ε,

> the longitudinal momentum p_z

> the transverse energy $\varepsilon_T = \mathbf{p}_T^2/(2\varepsilon) - e\, V_{Lin}(\mathbf{r}_T)$

Residual potential:

$\delta V(t,\mathbf{r}_T,z) = V(t,\mathbf{r}_T,z) - V_{Lin}(\mathbf{r}_T)$

- Breaks ε , p_z and ε_T .
- Responsible for *dechanneling* and *incoherent bremsstrahlung*.

Residual charge density

$\delta\rho(t,\mathbf{r}_T,z) = \rho(t,\mathbf{r}_T,z) - \rho_{Lin}(\mathbf{r}_T)$

Frozen approximation:
Neglect the t-dependence of δρ and δV :

$$\delta\rho(\mathbf{r}_T,z) = \delta\rho(t,\mathbf{r}_T,z)_{\,t=0}$$

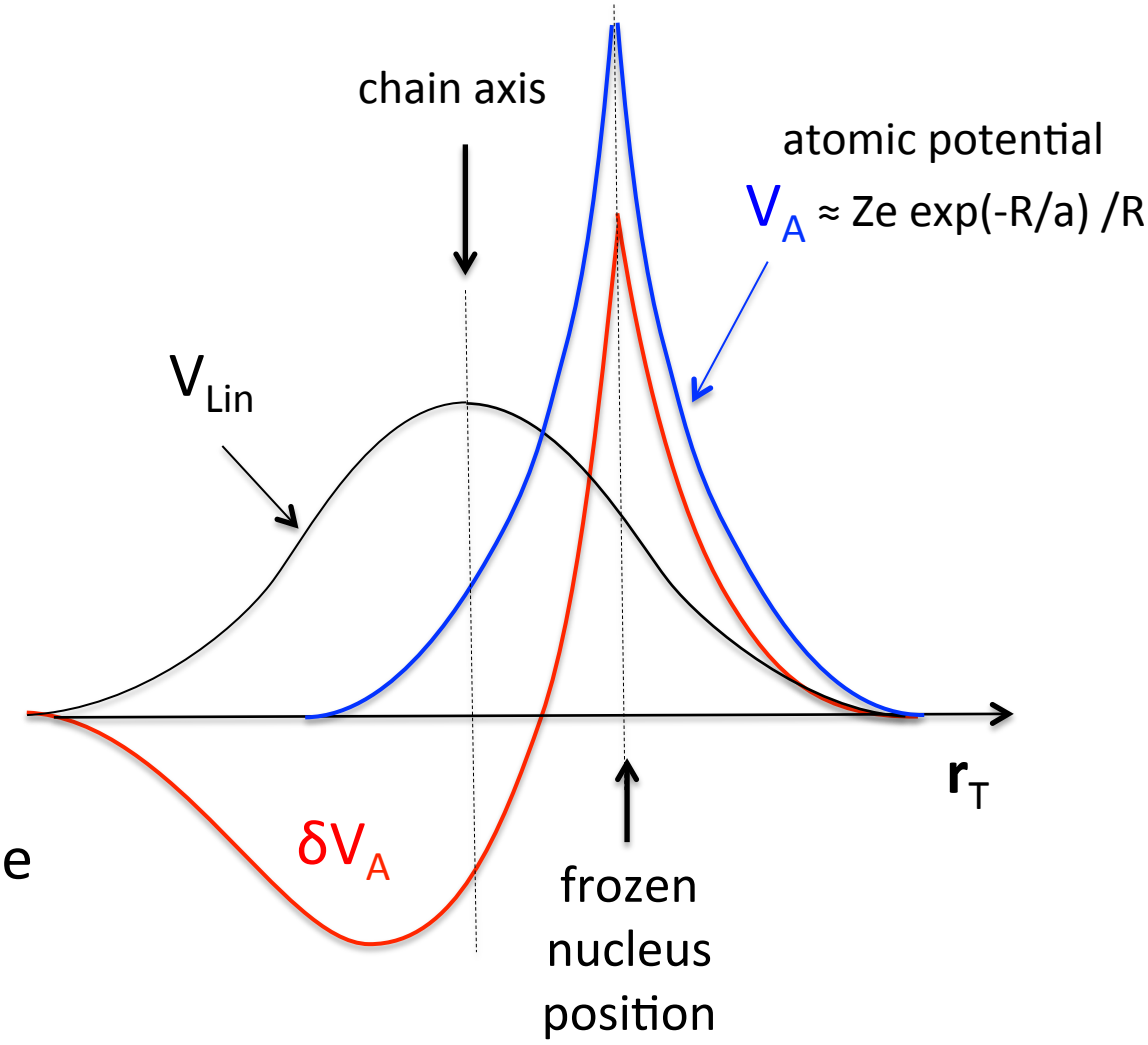
Justification : the projectile moves much faster than the crystal constituents

Three contributions to δρ and δV (Beloshitsky, …)

1) periodic modulation in z (negligible effects)
2) vibrations of the atomic chain (phonons)
3) electron motion inside the atoms

$$\delta V = \delta V_{period} + \delta V_A + \delta V_e$$

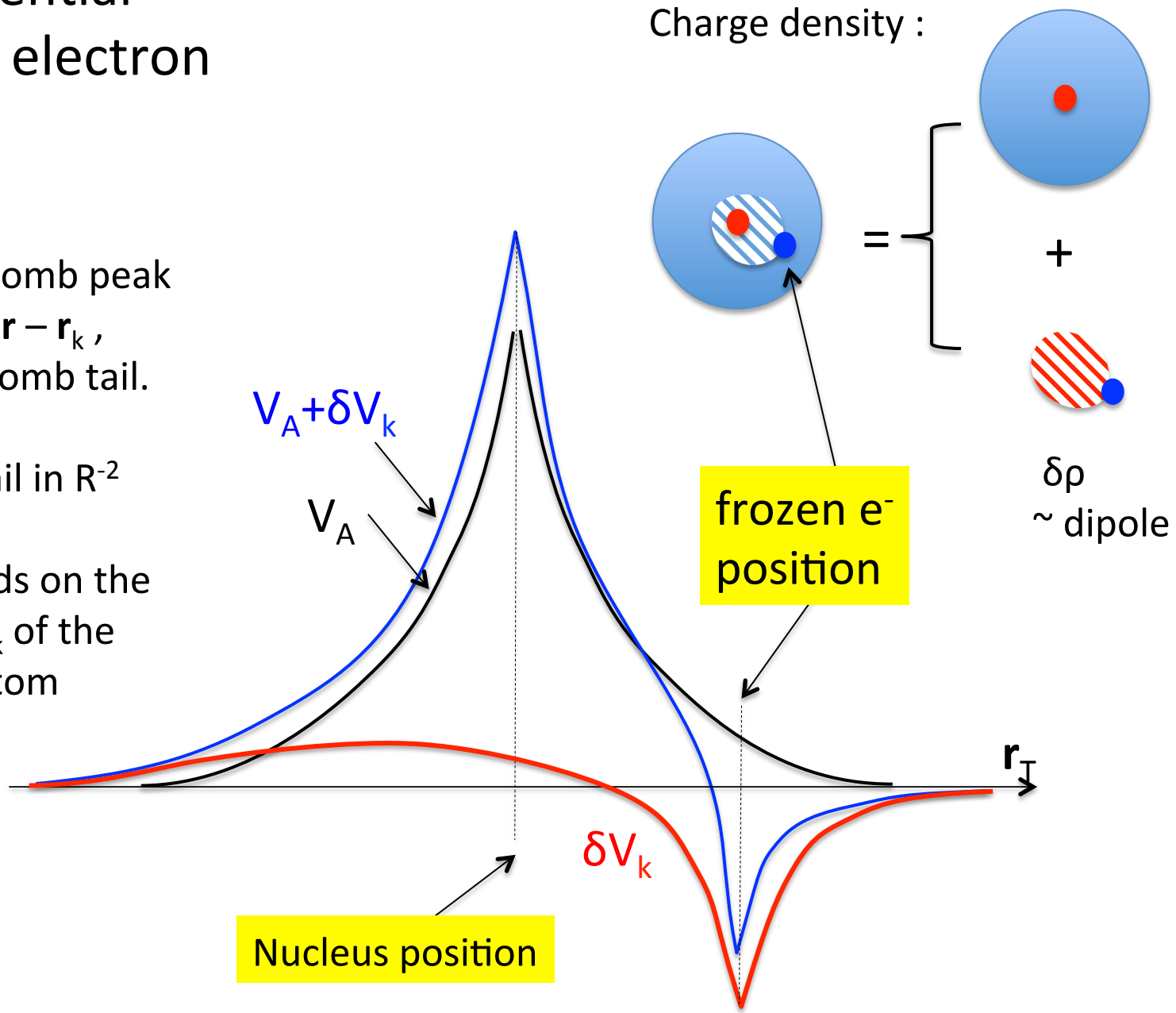
Residual potential δV_A from the vibrations of the atomic chain



- δV_A is of short range
- Its shape depends on the transverse position $\mathbf{r}_{N\,T}$ of the frozen nucleus
- It is generally dissymmetric
- zero mean value : $\int d^3\mathbf{r}\ \delta V_A = 0$

Residual potential from the k^{th} electron of an atom

- δV_k has the Coulomb peak in -e/R with $\mathbf{R} = \mathbf{r} - \mathbf{r}_k$, but not the Coulomb tail.
- It has a dipole tail in R^{-2}
- Its shape depends on the frozen position $\mathbf{r}_k$ of the electron in the atom
- It is generally dissymmetric



Monte Carlo simulation - 1

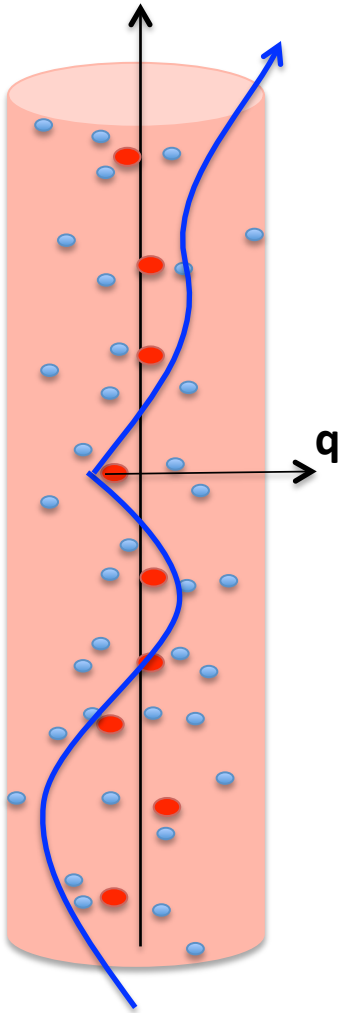
Semi-quantum method :

Represent the effect of δV by *kinks* of the trajectory,
where $\mathbf{p}_T$ receives a sudden jump $\mathbf{q}$ $(= \mathbf{q}_T)$

- between two kinks: classical motion in V_{Lin} .
- kinks are generated at the rate

$$dN(\mathbf{q}) / (dz\, d^2\mathbf{q}) = \rho_N(\mathbf{r}_T) \,.\, d\sigma_{N\,(Qu)} / d^2\mathbf{q} + \rho_e(\mathbf{r}_T) \,.\, d\sigma_{e\,(Qu)} / d^2\mathbf{q}$$

where $\sigma_{(Qu)}$ is the *quantum-mechanical cross section* on δV_k
$\rho_{e \text{ or } N}(\mathbf{r}_T)$ is the density of electrons or nuclei,
averaged on z
(the Lindhard charge density is $\rho_{Linhard} = Ze\, \rho_N - e\, \rho_e$)



- To simplify we ignore the shape dependence of the δV_k 's on the $\mathbf{r}_k$'s
- we ignore the correlations between the $\mathbf{r}_k$'s
- The random quantity is the vector $\mathbf{q}$

Monte Carlo simulation - 2

Classical method : ''binary collisions''

Use the classical equation of motion in the full frozen potential

$V(\mathbf{r}_T,z) = V_{Lin} + \delta V = \sum_k e_k / |\mathbf{r} - \mathbf{r}_k|$

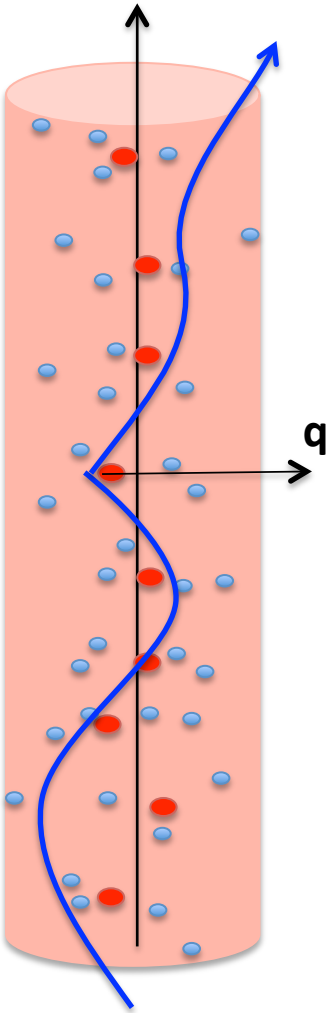
Each constituent *k* sitting near the trajectory changes $\mathbf{p}_T$ by

$$\Delta\mathbf{p}_T = (e/v) \int dz \ \mathrm{grad}_{\mathbf{b}} V_k(\mathbf{b},z)$$

(straight-line approximation). $\mathbf{b}$ = *impact parameter* = $(\mathbf{r}\text{-}\mathbf{r}_k)_T$

$$\Delta\mathbf{p}_T = \Delta\mathbf{p}_{T\,(\text{continuous})} + \mathbf{q}$$

$\Delta\mathbf{p}_{T\,(\text{continuous})} \rightarrow V_{Lin}$; $\mathbf{q} \rightarrow \delta V$



- To simplify we ignore the correlations between the $\mathbf{r}_k$'s
- The random quantity is the vector $\mathbf{r}_{k,T}$

Classical vs. quantum cross sections on δV_k

We re-formulate the classical method like the semi-quantum one, *neglecting the variations of the density* $\rho_k(\mathbf{r}_{k,T})$ *of consituent* k, *near the trajectory* (at least within the range b_{max} of δV_k .
This rough approximation gives

$$dN(\mathbf{q})/(dz\, d^2\mathbf{q}) \approx \rho_N(\mathbf{r}_T) \,.\, d\sigma_{N\,(Cl)}/d^2\mathbf{q} + \rho_e(\mathbf{r}_T) \,.\, D\sigma_{e\,(Cl)}/d^2\mathbf{q}$$

where $\sigma_{(Cl)}$ is the *classical cross section* on δV_k

Then it suffices to compare $d\sigma_{(Cl)}/d^2\mathbf{q}$ and $d\sigma_{(Qu)}/d^2\mathbf{q}$.

Recall of the main properties of δV_k :
- Coulomb singularity at the origin, $-e\,\delta V_k \sim \alpha_k/r$ ($\alpha_k = 1/137$ or $-Z/137$)
- short range, or decreasing at least like r^{-2}
- dissymmetry in $\mathbf{b}_T$.

General comparison between classical and quantum cross sections

(straight-line approximation) (Born approximation)

Average vector $\mathbf{q}$: $\int d^2\mathbf{q}\, \mathbf{q}\ d\sigma/d^2\mathbf{q} = 0$ (both in classical and quantum models)

Weighted cross sections: $M^{(n)} = \int d^2\mathbf{q}\ |\mathbf{q}|^n\ d\sigma/d^2\mathbf{q}$; $M^{(0)}$ = total cross section;

$M^{(n)}_{(Cl)}$ is better re-written as $\int d^2\mathbf{b}\ |\mathbf{q}_{Cl}(b)|^n$

Examples :

- For a spherically symmetric and monotonic potential V(r) :

$$M^{(1)}_{(Cl)} = 2\pi^2 \int r\, dr\, V(r)$$
$$M^{(1)}_{(Qu)} = 4\pi^2 \int r^2\, dr\, V^2(r)$$

- For any potential: $M^{(2)}_{(Cl)} = M^{(2)}_{(Qu)}$

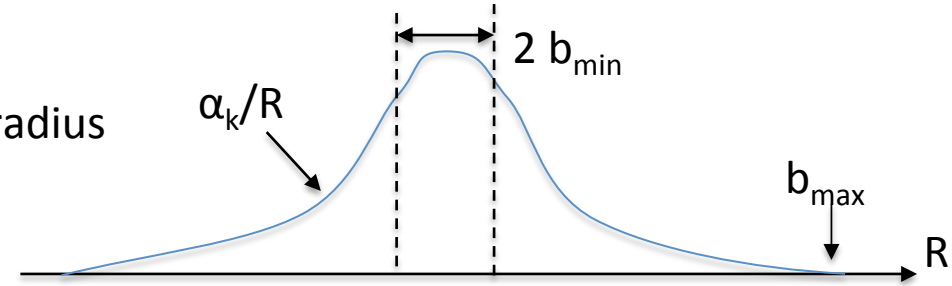
$M^{(2)}$ from δV_k governs the **rate of increase** of the **transverse energy**. Should we conclude that the classical and quantum models predict *the same dechanneling length* ?

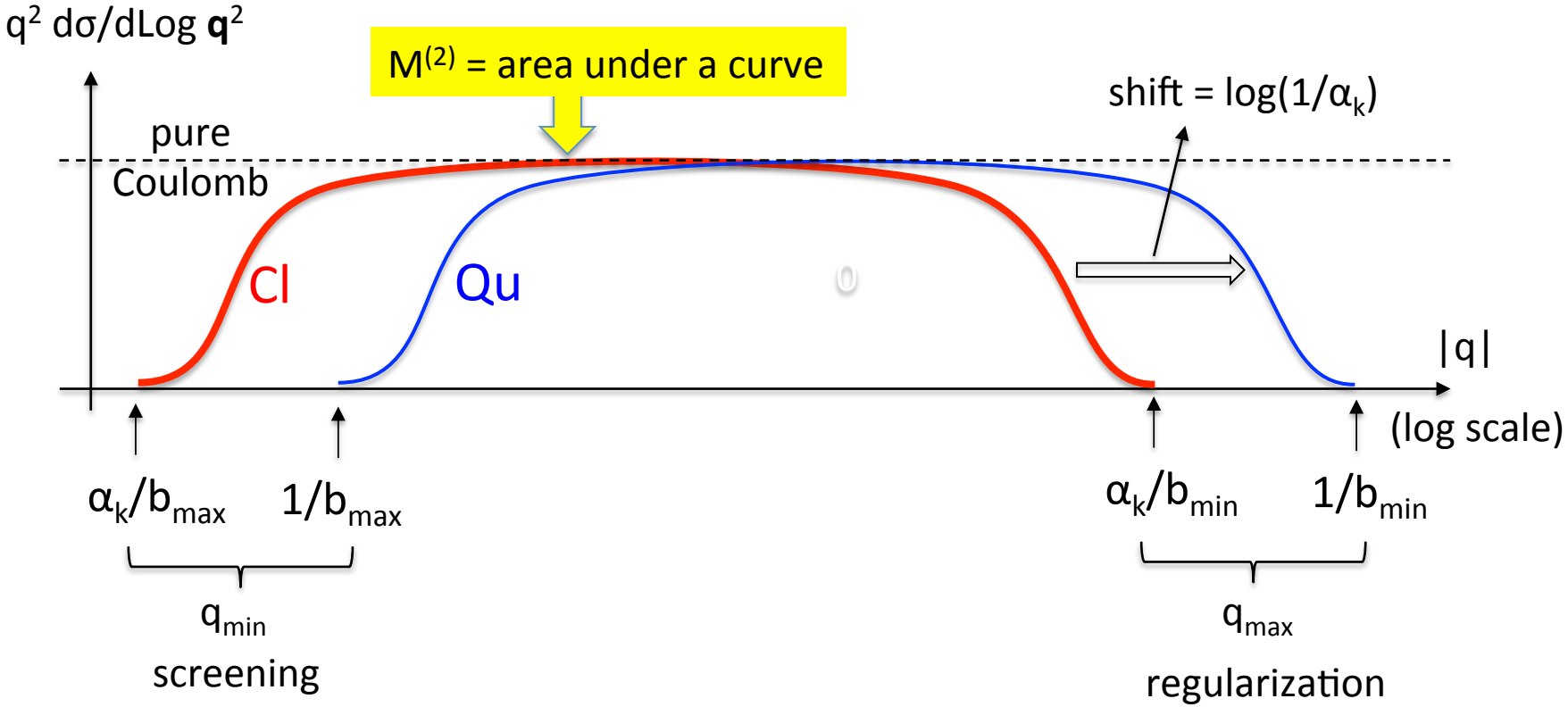
Simple model for δV_A :

$\alpha_k = Z/137$

b_{min} ~ nuclear radius

$b_{max} \sim a_{TF}$

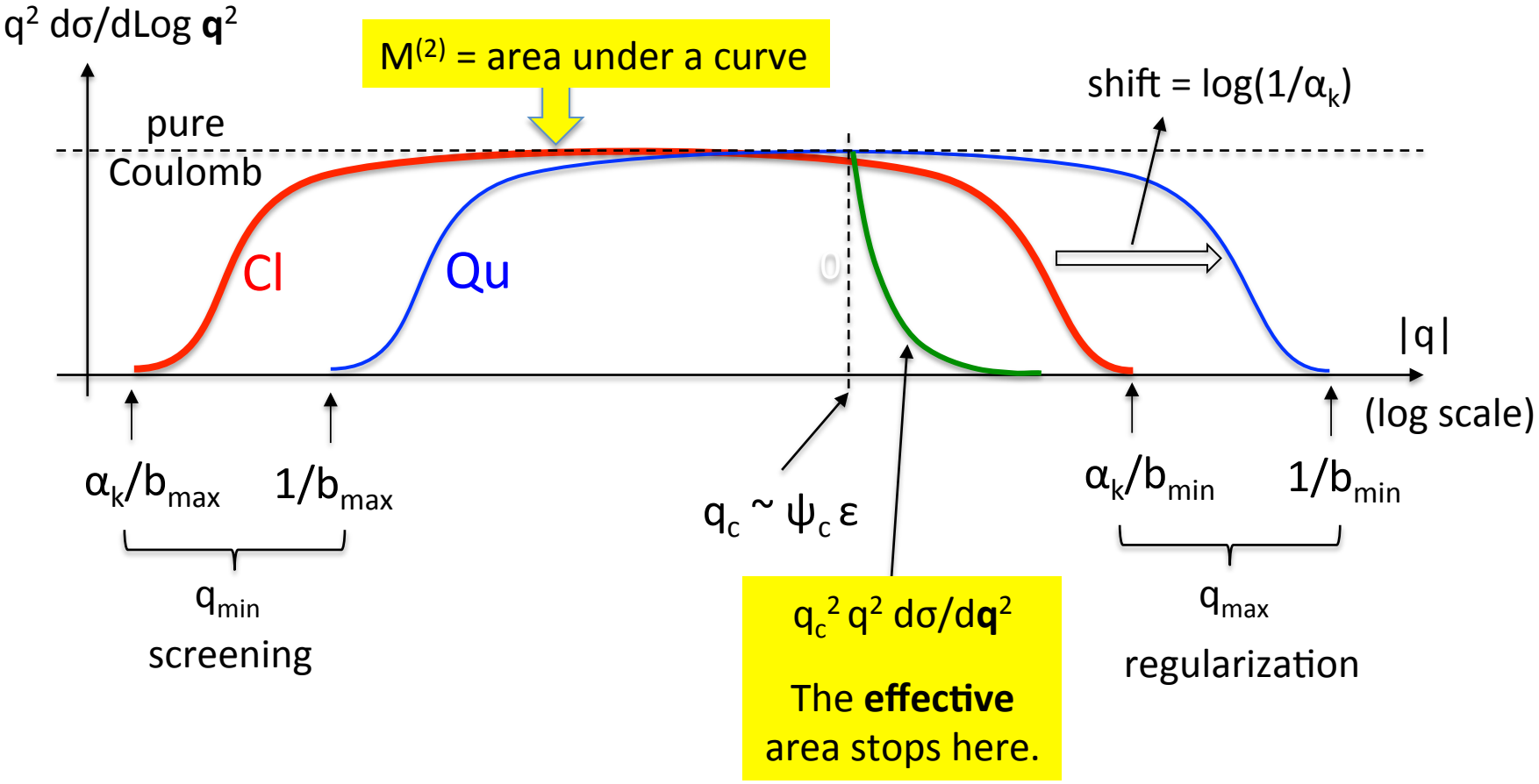




For $q > q_c \sim \psi_c \varepsilon$ dechanneling occurs at once, no matter the excess $q - q_c$.

Therefore the dechanneling rate is sensitive to $M^{(2)}_{eff} = \int d^2\mathbf{q} \cdot d\sigma/d^2\mathbf{q} \cdot \mathbf{min(q^2, q_c^2)}$

Conclusion: $M^{(2)}_{eff}$ (Qu) > $M^{(2)}_{eff}$ (Cl) (assuming $\alpha_k < 1$)



Rough estimation :

$$M^{(2)}_{eff}(Cl) \;/\; M^{(2)}_{eff}(Qu) \;=\; \log(q_c\, b_{max}/\alpha_k) \,/\, \log(q_c\, b_{max})$$

Typical order of magnitude

$b_{max} \sim 1$ Å $= 1/(2\text{keV})$; $\alpha_k \sim 1/10$ (Silicium) ;

$\varepsilon \sim 1$ GeV ; $\psi_c \sim 1$ mrad => $q_c \sim 1$ MeV

$$\Rightarrow\; M^{(2)}_{eff}(Cl) \;/\; M^{(2)}_{eff}(Qu) \approx 4/3$$

For electrons, $\alpha_k = 1/137$ => $M^{(2)}_{eff}(Cl) \;/\; M^{(2)}_{eff}(Qu) \approx 5/3$

Our final opinion :

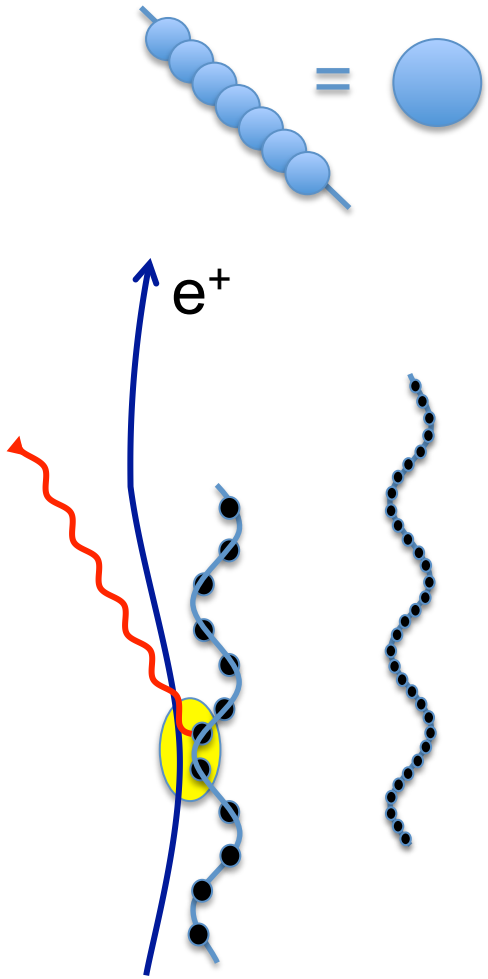
The quantum method is *dirty* (full of approximations). The classical method is *cleaner* but far from reality. Neglecting quantum effects *underestimates* the dechanneling length.

Effects of correlations between the constituents

Roughly speaking, the Born approximation becomes bad, and the Classical scattering theory becomes good, when α_k becomes > ~1.

When a group of N successive atoms is nearly aligned along the particle trajectory, it acts like *one **super-nucleus** of charge NZ*. If *NZ* /137 > ~1, coherent scattering on this group should be treated classically. This is the case of atoms displaced by a long-wavelength phonon.

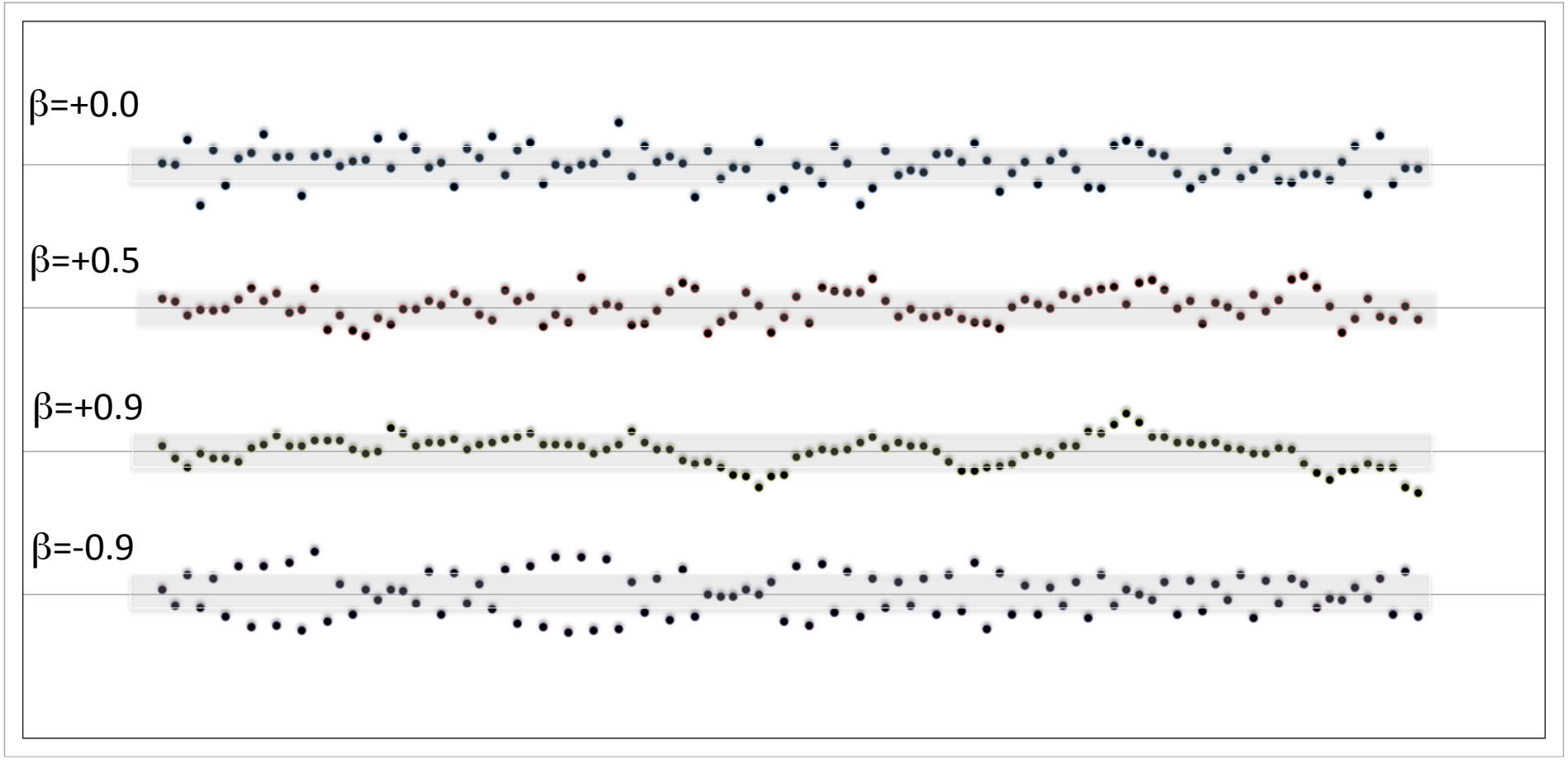
We obtain a 'semi-incoherent' process (X.A. Channeling 2016)



Thus, δV_A should be separated in two contributions:

- That produced by long-wavelength phonons, which can be treated classically,
- That produced by short-wavelength phonons, which must be treated quantum-mechanically.

Atomic chain vibration (simulated)



Conclusions

- The residual potential δV, responsible for incoherent processes, is usually of shorter range and weak ($\alpha_k << 1$). It should better be treated with the quantum Born approximation than treated classically.
- However, the inverse is true for the part of δV_A which is due to long-wavelength phonons.

Thank you !