Quantum Features of Relativistic Particle Scattering and Radiation in Crystals

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Tuesday, 25 September - Hotel Continental Ischia
Plan

Introduction. *The need of quantum theory of incoherent scattering in crystals*

Wigner function description of both quantum and classical effects

– *classical motion description by eikonal method*

– *quantum scattering description in Born approximation*

Wigner function of incoherent scattering

– *small angle scattering description by averaged angle*

– *large angle scattering description by Moss cross-section modification*

Incoherent scattering contribution to radiation and pair production
Observed effects predicted by Prof. V.G. Baryshevsky and his School

Parametric radiation
Channeling radiation
Spin rotation in crystals
Radiative cooling
Synchrotron-type $e^+ e^-$ pair production
Multiple volume reflection in one crystal
Quasichanneling oscillations
The main thing still ubiquitously missed is quantum incoherent scattering theory.
THE PENETRATION OF ATOMIC PARTICLES THROUGH MATTER

BY

NIELS BOHR

KØBENHAVN
I KOMMISSION HOS EJNAR MUNKSGAARD
1948
Bohr condition $Z\alpha/\beta >> 1$

Or, according to (1.1.4) and (1.3.2),

$$\kappa = \frac{2 |e_1 e_2|}{\hbar v},$$  \hspace{1cm} (1.3.8)

we thus have

$$\kappa >> 1$$  \hspace{1cm} (1.3.9)

as the necessary and sufficient condition for the justification of the classical considerations
Landau and Lifshitz about quantum nature of particle scattering by atoms
§126. The quasi-classical case

It is of interest to investigate the manner in which the passage occurs from the quantum-mechanical theory of scattering to the limit of the classical theory.

If we can speak of classical scattering through an angle $\theta$ when the particle is incident at an impact parameter $\rho$, it is necessary that the quantum-mechanical indeterminacies of these two quantities should be relatively small: $\Delta \rho \ll \rho$, $\Delta \theta \ll \theta$. The indeterminacy in the scattering angle is of the order of magnitude $\Delta \theta \sim \Delta \rho / \rho$, where $\rho$ is the momentum of the particle and $\Delta \rho$ is the indeterminacy in its transverse component. Since $\Delta \rho \sim \hbar / \Delta \rho \gg \hbar / \rho$, we have $\Delta \theta \gg \hbar / \rho \rho$, and thus

$$\theta \gg \hbar / \rho mv.$$  \hfill (126.6)
The classical angle of deviation of the particle can be estimated as the ratio of the transverse momentum increment $\Delta p$ during the “collision time” $\tau \sim \rho/v$ and the original momentum $mv$. The force acting on the particle at a distance $\rho$ is $U'(\rho)$; hence $\Delta p \sim |U'(\rho)|\rho/v$, so that $\theta \sim |U'(\rho)|\rho/mv^2$.

Substitution in (126.6) gives the condition for quasi-classical scattering in the form

$$|U'(\rho)|\rho^2 \gg \hbar v.$$  \hfill (126.7)

For a Coulomb field, $U = \alpha/r$, the condition (126.7) is satisfied if $\alpha \gg \hbar v$. This is the opposite condition to that for which the Coulomb field can be regarded as a perturbation. We shall see, however, that the quantum theory of scattering in a Coulomb field leads to a result which, as it happens, is always in agreement with the classical result.

Since $\frac{Z\alpha}{\beta} \approx \frac{14}{137} \ll 1$ the single-atom scattering is “quantum” and can be treated as a perturbation.
INFLUENCE OF CRYSTAL LATTICE ON MOTION OF ENERGETIC CHARGED PARTICLES

BY

JENS LINDBAARD

København 1965
Appendix B
Quantal Corrections to Classical Description

Single Collision

The total uncertainty, $\delta \theta$, in scattering angle can be obtained in a way analogous to that used by Bohr\textsuperscript{12}).

With a wave packet of width $\delta r$ there are two contributions to $\delta \theta$, one from diffraction and one from classical uncertainty in position, i.e.

$$
(\delta \theta)^2 = \frac{\lambda^2}{4(\delta r)^2} + (\delta r)^2 \cdot (\vartheta'(p))^2.
$$

(B.1)

In order to obtain a well-defined orbit, we may demand $(\delta \theta)^2 < \vartheta^2$, or

$$
\lambda \frac{d}{dp} \left( \frac{1}{\vartheta(p)} \right) < 1,
$$

(B.3)

which formula in the case of Rutherford scattering, $\vartheta = b/p$, leads to the inequality of Bohr\textsuperscript{12})

$$
\kappa = \frac{2Z_1Z_2e^2}{\hbar v} > 1.
$$

(B.4)
QUANTAL TREATMENT OF DIRECTIONAL EFFECTS
FOR ENERGETIC CHARGED PARTICLES IN CRYSTAL LATTICES

PHILIP LERVIG, JENS LINDHARD and VIBEKE NIELSEN
Institute of Physics, University of Aarhus, Denmark

in continuation of previous work ¹,²)
1) J. Lindhard, Phys. Lett. 12 (1964) 126

single scattering of a charged particle by an atom is to be described
by quantal perturbation methods when the velocity is sufficiently high,
in particular at small angles of deflection ³).

We discuss whether it can be treated classically, like in the binary collision model, or if it should be treated quantum-mechanically. We give arguments for the latter opinion. We show that the quantum approach predicts a slower dechanneling than the classical one.
classical vs quantum scattering

\[ \frac{Z \alpha}{\beta} \ll 1 \]
quantum

\[ \frac{Z \alpha}{\beta} \gg 1 \]
classical
Quantum vs classical cross sections

Relativistic analog to

\[
\theta(\rho) = \frac{2Z\alpha}{\varepsilon a\rho} \int_{\rho}^{\infty} \frac{\exp(-r/a)}{\sqrt{r^2 - \rho^2}} r \, dr = \frac{2Z\alpha}{\varepsilon a} K_1(\rho/a)
\]

\[
\frac{d\sigma_{\text{Class}}}{d\Omega} = \frac{\rho(\theta)}{\theta} \left| \frac{d\theta(\rho)}{d\rho} \right|^{-1} = \frac{4Z^2\alpha^2}{\varepsilon^2\theta^4} \left[ \frac{b}{K_1'(b)K_1^3(b)} \right], \quad b = \rho/a
\]

\[
\frac{d\sigma_{\text{Quant}}}{d\Omega} = \frac{4Z^2\alpha^2}{\varepsilon^2\theta^4} \frac{1}{\left(1 + \frac{\hbar^2}{\theta^2} p^2 a^2\right)^2} = \frac{4Z^2\alpha^2}{\varepsilon^2\theta^4} \left(1 + \frac{\hbar^2}{\theta^2} p^2 a^2\right)^{-2}
\]
Indeed, classical approach overestimates scattering.
Typical scattering angles $\theta_{\text{min}}$ and $\theta_u$

Momentum transfer $q_\perp = \hbar Z^{1/3} / a_0$

and scattering angle $\theta_{\text{min}} = \hbar Z^{1/3} / a_0 p$

are determined by the atom screening radius $a_0 / Z^{1/3}$

Momentum transfer $q_\perp = \hbar / u_1$

and scattering angle $\theta_u = \hbar / u_1 p \mp (2 \div 3) \theta_{\text{min}}$

are determined by atomic vibration amplitude $u_1$

which determines additional effective screening

(scattering suppression by correlations)
Modified Dechanneling Theory and Diffusion Coefficients

M. Kitagawa and Y. H. Ohtsuki

Department of Physics, Waseda University, Nishi-Ohkubo 4, Shinjuku, Tokyo, Japan
(Received 20 February 1973)

A new type of dechanneling theory is presented by constructing a Fokker-Planck equation. The damping term, which is not renormalized to the diffusion coefficient, and a new term in addition to the usual diffusion equation appear in the Fokker-Planck equation. Making use of the general expression of the diffusion coefficient given by Ohtsuki, some simple analytic expressions for diffusion coefficients due to the many-phonon excitations, the one-electron excitation, and the plasmon excitation are presented.

\[
\langle \Delta p_x^2 \rangle = \frac{8 \pi Z^2 Z^2 e^4 N}{v_x^2} L_n \tag{3.8}
\]

Diffusion coefficients for the planar case are also derived from Eq. (3.1) with same assumptions taking into account \( \vec{h}_z = (0, h_y) \),

\[
\langle \Delta p_y^2 / \Delta z \rangle = \langle \Delta p_x^2 / \Delta z \rangle_{\text{random}} P(y), \tag{3.11}
\]

where \( P(y) \) is the distribution function of the atom at \( y \) from the channeling plane,

\[
P(y) = \frac{d_y}{(2\pi)^{1/2} \rho_y} \exp \left( -\frac{y^2}{2\rho_y^2} \right). \tag{3.12}
\]
Presently used: 

**Kitagawa-Ohtsuki approximation**

\[
\left\langle \frac{\Delta p^2_1}{\Delta z} \right\rangle \approx P(r_1) \times \left\langle \frac{\Delta p^2_1}{\Delta z} \right\rangle_{\text{random}}
\]

\[
P(r_1) = \frac{r_1^2}{\rho_1^2} \exp \left( -\frac{r_1^2}{\rho_1^2} \right), \quad P(y) = \frac{d_y}{(2\pi)^{1/2}\rho_y} \exp \left( -\frac{y^2}{2\rho_y^2} \right).
\]

**we:** + suppression by coherent scattering

\[
\left\langle d\vartheta^2_s(z)/dz \right\rangle = n \int_0^{\vartheta_2} \int_0^{2\pi} \vartheta^2 \frac{d\sigma}{d\Omega} \left[ 1 - \exp(-p^2\vartheta^2u^2_1) \right] d\varphi \vartheta d\vartheta
\]
Kitagawa-Ohtsuki approximation

\[ d\Sigma_{KO} \equiv n(\rho) \cdot d\sigma_{Mott}(\vec{q}_\perp) = \]
\[ \exp\left(-\frac{\rho^2}{2u_1^2}\right) \times \frac{4\alpha^2Z^2d\vec{q}_\perp}{2\pi u_1^2 \left(q_\perp + \frac{\hbar^2}{a_F^2}\right)^2} \]

- is, in fact, refined in the present talk
- is used everywhere for normalization
Single-atom scattering modification in crystals (M. L. Ter-Mikaelian)

\[ \varphi = \sum_i \frac{Ze}{|r - r_i|} \exp \left( \frac{-|r - r_i|}{R} \right). \]

\[ d\sigma = d\sigma_{BH} \left| \sum_i \exp \left( \frac{iqr_i}{\hbar} \right) \right|^2, \quad q = p_1 - p_2 - \hbar k \]

averaging over thermal vibrations:

\[ \left| \sum_i \exp \left( \frac{iqr_i}{\hbar} \right) \right|^2 = N \left[ 1 - \exp \left( \frac{-q^2u^2}{\hbar^2} \right) \right] \]

\[ + \exp \left( \frac{-q^2u^2}{\hbar^2} \right) \left| \sum_i \exp \left( \frac{iqr_{10}}{\hbar} \right) \right|^2. \]

Using this equality, we obtain for bremsstrahlung cross section in a crystal the expression [see formulas (3) and (12) in Ref. 4]

\[ d\sigma = d\sigma_{BH} \left\{ N \left[ 1 - \exp \left( \frac{-q^2u^2}{\hbar^2} \right) \right] \right. \]

\[ + \exp \left( \frac{-q^2u^2}{\hbar^2} \right) \left| \sum_i \exp \left( \frac{iqr_{10}}{\hbar} \right) \right|^2 \] (7.4)
M. L. Ter-Mikaelian approach is equivalent to the plane wave approximation and does not take into consideration both particle and nuclei transverse distribution nonuniformity.
Essential points for KO refining:

- Strong particle *transverse localization* (trajectories)
- Strong *nonuniformity* of transverse distribution of the scatterers
Our approach is founded on Wigner function (WF) as well as both eikonal and Born approximations and quantum-mechanically describes incoherent particle scattering in non-uniform media of atomic strings and planes.
This new approach:

- refines Kitagawa-Ohtsuki approximation
- Uses ideas/works of W.R. Hamilton, N. Bohr, E. Wigner, J. Lindhard, M. L. Ter-Mikeilian

Has something common with works of A.I. Akhiezer & N.F. Shul’ga and X. Artru
Primary formulae

\[ P(z, z_0, \tilde{\rho}, \tilde{q}_\perp) = \]
\[ \pi^{-2} \int \Psi^*(z, z_0, \tilde{\rho} + \bar{\chi}) \Psi(z, z_0, \tilde{\rho} - \bar{\chi}) \exp(2i\tilde{q}_\perp \bar{\chi}) d^2\chi \]

\[ \Psi \approx \Psi_{eik}(\vec{r}) \left[ 1 - i \int \delta U_{at}(\tilde{\rho}' - \tilde{\rho}_{at}, z') \frac{dz'}{\nu} \right] \]
Eikonal (≈ “classical”)
wave function and
Wigner function
Dirac equation solution by optic eikonal method

\[
\begin{bmatrix}
\Delta + p^2 - 2\varepsilon U(\vec{r})
\end{bmatrix} \psi(\vec{r}) = \begin{bmatrix}
\Delta + p^2 n^2
\end{bmatrix} \psi(\vec{r}) = 0
\]

\[
\psi(\vec{r}) = f(\vec{r}) \exp( i p \Im(\vec{r}) )
\]

\[
n^2 = 1 - 2\varepsilon U / p^2
\]

\[
(\vec{\nabla} \Im)^2 = n^2
\]
Light rays vs particle trajectories

\[
(\vec{\nabla} \mathfrak{Z})^2 = n^2 \quad n^2 = 1 - 2\varepsilon U / p^2
\]

\[
\vec{s} = \frac{d\vec{r}}{ds} = \begin{vmatrix} \vec{\nabla} \mathfrak{Z} \end{vmatrix}
\]

\[
n \frac{d\vec{r}}{ds} = \vec{\nabla} \mathfrak{Z} \quad \frac{d}{ds} \left( n \frac{d\vec{r}}{ds} \right) = \vec{\nabla} n
\]

\[
\varepsilon \frac{d^2\vec{r}}{dt^2} = -\vec{\nabla} U + \left( \vec{\nu} \vec{\nabla} U \right) \vec{\nu} \quad \rightarrow \quad \varepsilon \frac{d^2\vec{r}}{dt^2} = -\vec{\nabla} U + \left( \vec{s} \vec{\nabla} U \right) \vec{s},
\]

\[
\mathfrak{I}(\vec{r}) = \mathfrak{I}(\vec{r}(s, s_0)) = \int_{s_0}^{s} n(\vec{r}(s')) ds' \approx -\int_{s_0}^{s} U(\vec{r}(s')) ds' / \nu
\]
Pre-exponential factor in \( \psi(\vec{r}) = f(\vec{r}) \exp(ip\Im(\vec{r})) \)

\[
\frac{\partial f}{\partial s} = \frac{1}{n} (\vec{\nabla} f) \vec{\nabla} \Im = (\vec{\nabla} f) \frac{d\vec{r}}{ds} = -\frac{1}{2n} f \Delta \Im + \frac{i\Delta}{2n} f
\]

\[
f(\vec{\rho}, s) = \frac{p}{2\pi is} \int \exp\left(-\frac{p}{2is} (\vec{\rho} - \vec{\rho}(s))^2\right) f(\vec{\rho}_0) d\vec{\rho}_0 \quad \text{diffraction}
\]

\[
f = \exp\left(-\frac{1}{2} \int_0^s \nabla^2 \Im \, ds\right) \quad \text{focusing / defocusing}
\]
Eikonal WF asymptotically (only!) reduces to a delta-function corresponding to the classical limit

\[1D: \int \exp \left( i \int [U(x + \chi, z) - U(x - \chi, z)] \frac{dz}{v} \right) \exp(2i q x) d\chi = \]

\[2\pi \left( \int U'' dz / v \right)^{-1/3} Ai \left( 2 \left( q + \int U' dz / v \right) \left( \int U'' dz / v \right)^{-1/3} \right) \approx \pi \delta \left( q + \int U' dz / v \right) \]

\[P_{eik}(\vec{\rho}, \vec{q}_\perp, z, z_0) \approx \frac{d}{d \vec{q}_\perp} n \left( \vec{q}_\perp + \int_{z_0}^{z} \vec{\nabla} U(\vec{\rho}, z') dz' \right) \neq n \delta \left( \vec{q}_\perp + \int_{z_0}^{z} \vec{\nabla} U(\vec{\rho}, z') dz' \right) ! \]

where \( \vec{q}_\perp = -\int_{z_0}^{z} \vec{\nabla} U(\vec{\rho}, z') dz' \) corresponds to classical motion

however \( \vec{q}_\perp \) uncertainty is not zero!
Resulting WF undergoes factorization in the classical:

\[ P(z > z', z_0, \bar{\rho}, \bar{q}_\perp) = \]

\[ P_{eik}(z, z'_0, \bar{\rho}, \bar{q}_\perp) \times P_{at}(\bar{\rho}(z'), \bar{q}_\perp) \times P_{eik}(z'_{-0}, z_0, \bar{\rho}, \bar{q}_\perp) \]

making it possible to introduce local WF of incoherent scattering:

\[ P(z > z', z_0, \bar{\rho}, \bar{q}_\perp) = \]

\[ P_{eik}(z, z'_0, \bar{\rho}, \bar{q}_\perp) \times P_{at}(\bar{\rho}(z'), \bar{q}_\perp) \times P_{eik}(z'_{-0}, z_0, \bar{\rho}, \bar{q}_\perp) \]
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Wigner function description of both quantum and classical effects
- *classical motion description by eikonal method*
- *quantum scattering description in Born approximation*

**Wigner function of incoherent scattering**
- *small angle scattering description by averaged angle*
- *large angle scattering description by Moss cross-section modification*

Incoherent scattering contribution to radiation and pair production
Wigner function of incoherent scattering

\[ P_{at}(\tilde{\rho} - \tilde{\rho}_{at}(z'), \tilde{q}_\perp) \square P_{Born}(\tilde{\rho} - \tilde{\rho}_{at}(z'), \tilde{q}_\perp) = \int \left[ 1 + i \int_{z'-0}^{z'+0} \delta U_{at}(\tilde{\rho} + \tilde{\chi} - \tilde{\rho}_{at}(z'), z'')\frac{dz''}{\nu} \right] \]

\[ \times \left[ 1 - i \int_{z'-0}^{z'+0} \delta U_{at}(\tilde{\rho} - \tilde{\chi} - \tilde{\rho}_{at}(z'), z'')\frac{dz''}{\nu} \right] \exp(2i\tilde{q}_\perp\tilde{\chi})\frac{d^2\chi}{\pi^2} \]

\[ \frac{d\Sigma(\tilde{\rho})}{d\tilde{q}_\perp} \equiv n_{nucl}(\tilde{\rho}) \frac{d\sigma(\tilde{\rho})}{d\tilde{q}_\perp} = \frac{1}{n_p} \frac{d}{dz} \frac{dn_p(\tilde{\rho})}{d\tilde{q}_\perp} = \int \left[ P_{at}(z', \tilde{\rho} - \tilde{\rho}_{at}, \tilde{q}_\perp) - 1 \right] \exp\left( -\frac{\rho_{at}^2}{2u_1^2} \right) \frac{d^2\rho_{at}}{2\pi u_1^2 d} \]

\[ = \left( \frac{2Z\alpha}{\pi v} \right)^2 \frac{1}{d} \int_{-\infty}^{\infty} \cos(2\tilde{k}\tilde{\rho}) \frac{\exp(-2\kappa^2u_1^2) - \exp[-(q^2 + \kappa^2)u_1^2]}{[(\tilde{q} + \tilde{k})^2 + \kappa_s^2][(\tilde{q} - \tilde{k})^2 + \kappa_s^2]} d\tilde{k} \]

Single nucleus limit \( q_\perp \gg \hbar/u_1 \)

\[ \frac{d\Sigma(\tilde{\rho})}{d\tilde{q}_\perp} \rightarrow n_{nucl}(\tilde{\rho}) \frac{d\sigma_{Mott}(q_\perp)}{d\tilde{q}_\perp} = \left( \frac{2Z\alpha}{v q^2} \right)^2 \frac{1}{2\pi u_1^2 d_{inat}} \exp\left( -\frac{\rho^2}{2u_1^2} \right) \]
Essential expression of the presentation describes Coulomb scattering by nonuniformly distributed screened nuclei.
\[
\frac{d\Sigma(\vec{\rho})}{d\vec{q}_\perp} \equiv \frac{d}{dz}\frac{dn(\vec{\rho})}{n d\vec{q}_\perp} = \left(\frac{2Z\alpha}{\pi\nu}\right)^2 \frac{1}{d} \times \\
\int \cos(2\kappa\vec{\rho}) \frac{\exp(-2\kappa^2 u_1^2) - \exp[-(q^2 + \kappa^2)u_1^2]}{[(\vec{q} + \kappa)^2 + \kappa_s^2][((\vec{q} - \kappa)^2 + \kappa_s^2]} d\kappa
\]

versus

\[
d\Sigma_{KO} \equiv n(\rho) \cdot d\sigma(\vec{q}_\perp) = \\
\frac{\exp\left(-\rho^2 / 2u_1^2\right)}{2\pi u_1^2} \times \frac{4\alpha^2 Z^2 d\vec{q}_\perp}{q_\perp^2 + \frac{\hbar^2}{a_F^2}} \left(\frac{q_\perp^2 + \frac{\hbar^2}{a_F^2}}{q_\perp^2 + \frac{\hbar^2}{a_F^2}}\right)^2
\]
New Wigner function greatly differs from KO at $q_\perp \approx u_1$.
WF can be negative 1
WF can be negative 2

WF completely loses probabilistic interpretation at $q_\perp \sim u_1$, $\rho > 2.5 u_1$ ($< 5\%$ of particles)
All the effects of small angle incoherent scattering boils down to the average square of the scattering angle.

\[
\frac{d\theta_{\rho,\varphi}^2(\vec{\rho})}{dz} = \int_{q<q_{\text{max}}} \frac{d\Sigma(\vec{\rho})}{d\vec{q}_\perp} \frac{q_{\rho,\varphi}^2}{p^2} d\vec{q}
\]
.. which WF allows one to readily evaluate

\[
\frac{d\theta_{\rho,\varphi}^2(\vec{\rho})}{dz} = \int_{q<q_{\text{max}}} d\Sigma(\vec{\rho}) \frac{q_{\rho,\varphi}^2}{p^2} d\vec{q}
\]
Average scattering angle \textbf{square} becomes positively determined at small momentum transfer.
Average scattering angle square considerable exceeds KO limit at large q and ρ
WF high \( q_\perp \) Mott limit:

\[
d\Sigma \xrightarrow{q_\perp \gg \frac{\hbar}{u_1}} d\Sigma_{KO}
\]

\[
\equiv n_{nucl}(\rho) \cdot d\sigma(\bar{q}_\perp) = \frac{\exp(-\rho^2/2u_1^2)}{2\pi u_1^2} \times \frac{4\alpha^2 Z^2 d \bar{q}_\perp}{\left(q_\perp^2 + \hbar^2 / a_F^2\right)^2}
\]
Mott cross section modification by nonuniformity of string atom distribution

\[ d\sigma_{\text{mod}} = d\sigma_{\text{Mott}} \left\{ 1 - \frac{\rho^2}{2q^2u_1^4} \exp\left( -\frac{\rho^2}{2u_1^2} \right) \left[ \frac{2(\bar{q}\rho)^2}{\rho^2 p^2} - 1 \right] + \ldots \right\} \]
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Incoherent scattering contribution to radiation and pair production
Development of the consistent theory of relativistic particle incoherent scattering in crystals will mean the same for radiation and pair production processes.
Key points

Radiation (pair production), accompanying single scattering:
- has to be simulated separately
- is suppressed by coherent scattering;
coherent scattering also suppresses Landau-Pomeranchuk effect
Key simulation points:

Trajectory simulations in most realistic potentials

Simulation of **incoherent scattering** on both nuclei and electrons

Separate simulation of **single** and **multiple** scattering

Direct integration of **Baier-Katkov formula**

**Infinite** trajectories, **density** effect…
Radiation process simulations from the “First Principles”

The general expression for radiation intensity

\[
\frac{d^2 I}{d\omega d^2\theta} = \frac{\alpha \omega^2 d\omega}{8\pi^2 \varepsilon'^2} \times \int \int dt_1 dt_2 \left[ (\varepsilon^2 + \varepsilon'^2)(\mathbf{v}_\perp(t_1) - \theta)(\mathbf{v}_\perp(t_2) - \theta) + \omega^2 / \gamma^2 \right] \\
\exp \left\{ i \frac{\omega \varepsilon}{2\varepsilon'} \left[ \int_{-\infty}^{t_1} (\gamma^{-2} + (\mathbf{v}_\perp(t') - \theta)^2) dt' + \int_{-\infty}^{t_2} (\gamma^{-2} + (\mathbf{v}_\perp(t'') - \theta)^2) dt'' \right] \right\}
\]

contains two integrals

\[
A = \int \exp \left\{ i \frac{\omega \varepsilon}{2\varepsilon'} \int_{-\infty}^{t} [\gamma^{-2} + (\mathbf{v}_\perp(t') - \theta)^2] dt' \right\} dt,
\]

\[
B = \int (\mathbf{v}_\perp(t) - \theta) \exp \left\{ i \frac{\omega \varepsilon}{2\varepsilon'} \int_{-\infty}^{t} [\gamma^{-2} + (\mathbf{v}_\perp(t') - \theta)^2] dt' \right\} dt
\]

and slowly decreases with radiation angle \( \theta \), complicating its numerical integration.
Radiation at sharp change of particle trajectory

\[
\frac{d\mathcal{E}}{d\omega d\phi} = \frac{e^2}{4\pi^2 c} [\mathbf{k}, I]^2,
\]

\[
I = \frac{ic}{\omega} \int_{-\infty}^{\infty} dt \, e^{i(\omega/c)(ct-n \cdot r(t))} \frac{d}{dt} \frac{v(t)}{c - n \cdot v(t)}.
\]

\[
I \approx \frac{ic}{\omega} \left( \frac{v'}{c - n \cdot v'} - \frac{v}{c - n \cdot v} \right),
\]

\[
\frac{d\mathcal{E}}{d\omega} = \frac{2e^2}{\pi c} \left( \frac{2\xi^2 + 1}{\xi \sqrt{\xi^2 + 1}} \ln \left( \xi + \sqrt{\xi^2 + 1} \right) - 1 \right).
\]
Single scattering effects are treated separately

\[ A = \int_{-\infty}^{\infty} \exp \{ i \varphi(t) \} dt = \frac{i}{\varphi(+0)} - \frac{i}{\varphi(-0)} + \]

\[ i \sum_{i=1}^{N} \left\{ \left[ \frac{1}{\varphi(t_i+0)} - \frac{1}{\varphi(t_i-0)} \right] \exp i \varphi(t_i) - \frac{2\check{\varphi}(t_i)}{\check{\varphi}^3(t_i)} \sin \left[ \frac{\varphi(t_i-0) - \varphi(t_{i-1}+0)}{2} \right] \exp i \varphi(t_i) \right\}, \]

\[ \tilde{B} = \int_{-\infty}^{\infty} \left[ \tilde{\varphi}_\perp(t) - \tilde{\theta} \right] \exp \{ i \varphi(t) \} dt = \left[ \frac{i}{\varphi(+0)} - \frac{i}{\varphi(-0)} \right] \left( \tilde{\varphi}_\perp(0) - \tilde{\theta} \right) + \]

\[ i \sum_{i=1}^{N} \left\{ \left[ \frac{\tilde{\varphi}_\perp(t_i)+\check{\varphi}_\perp(t_i)-\tilde{\theta}}{\varphi(t_i+0)} - \frac{\tilde{\varphi}_\perp(t_i)-\tilde{\theta}}{\varphi(t_i-0)} \right] \exp i \varphi(t_i) - \right\}, \]

\[ \frac{2}{\check{\varphi}_\perp^2(t_i)} \left[ \ddot{\varphi}_\perp(t_i) - \left( \tilde{\varphi}_\perp(t_i) - \tilde{\theta} \right) \frac{\check{\varphi}(t_i)}{\check{\varphi}(t_i)} \right] \sin \left[ \frac{\varphi(t_i-0) - \varphi(t_{i-1}+0)}{2} \right] \exp i \varphi(t_i) \}

where \( \omega' = \varepsilon/(\varepsilon - \omega), \check{\varphi}(t) = \omega' \left( \tilde{\varphi}_\perp(t_i) - \tilde{\theta} \right) \right) \hat{\varphi}_\perp(t) \) and \( \check{t}_i = (t_i + t_{i-1})/2. \)
Both surface and refraction effects can be easily included

derivatives of the phase $\varphi(t)$ on the left and on the right of the entrance crystal surface

$$\varphi(-0) = \frac{\omega'}{2} \left[ \gamma^{-2} + \left( \vec{v}_\perp(0) - \vec{\theta} \right)^2 \right],$$

$$\varphi(+0) = \frac{\omega'}{2} \left[ \gamma^{-2} + \frac{\omega_p^2}{\omega^2} + \left( \vec{v}_\perp(0) - \vec{\theta} \right)^2 \right];$$
on the left and on the right of each inter-step border

$$\varphi(t_i - 0) = \frac{\omega'}{2} \left[ \gamma^{-2} + \frac{\omega_p^2}{\omega^2} + \left( \vec{v}_\perp(t_i) - \vec{\theta} \right)^2 \right],$$

$$\varphi(t_i + 0) = \frac{\omega'}{2} \left[ \gamma^{-2} + \frac{\omega_p^2}{\omega^2} + \left( \vec{v}_\perp(t_i) + \vec{\vartheta}_i - \vec{\theta} \right)^2 \right];$$

and on the right from the exit surface

$$\varphi(t_N + 0) = \frac{\omega'}{2} \left[ \gamma^{-2} + \left( \vec{v}_\perp(T) + \vec{\vartheta}_N - \vec{\theta} \right)^2 \right].$$
Local approximation, axial case with incoherent scattering for $\mathcal{Q}_s(l_{coh}) < m/\varepsilon$

J. Physique 48 (1987) 1009-1016

Classification
Physics Abstracts
12.20 — 41.70 — 61.80

On the theory of electron-positron pair production in crystals

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(Reçu le 30 octobre 1986, accepté le 16 janvier 1986)

Abstract. — The theory describing both coherent and incoherent radiational processes when energetic $\gamma$-quanta, electrons or positrons move through crystals in directions nearly parallel to the crystal axes or planes is presented. It is shown that, within the logarithmic approximation, the local probabilities of incoherent radiation processes are proportional to the cross-sections of the same processes pertaining to a separate nucleus placed in a uniform field. When the coherent pair production rate increases, the total probability of incoherent pair production starts to decrease proportionally to the $\gamma$-quantum energy to the $-2/3$ power.
Single vs multiple scattering

**Multiple**

\[ v = v_\| (1 - v_\perp^2(\tau)/2) + v_\perp(\tau), \]
\[ v_\perp(\tau) = \theta + w_\tau + \delta v (r + n v \tau), \tag{13} \]
\[ |\delta v| < \delta_s (l_{\text{coh}}) = (\sigma_s l_{\text{coh}})^{1/2}, \tag{4} \]
\[ \left\langle \left( \int_0^\tau \delta v_\perp \, d\tau' \right)^2 \right\rangle = \sigma_s \tau^2 |\tau|/3, \tag{19} \]
\[ \sigma_s = \sigma_s(\rho) = 8 \pi n(\rho) \left( \frac{Z e^2}{\varepsilon_\pm} \right)^2 \ln (\theta_{\text{max}}/\theta_{\text{min}}). \tag{5} \]

**Single**

\[ \delta v(t) = \begin{cases} \theta, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{26} \]
\[ |\theta| > \delta_s (l_{\text{coh}}) \]

\[ d\sigma(\theta) = 8 \pi \left( \frac{Z \alpha}{\varepsilon_+} \right)^2 \frac{d\phi}{\phi^3}. \tag{28} \]
\[ \sigma_z = 8 \pi n(\rho) \left( \frac{Z e^2}{\varepsilon_\pm} \right)^2 \ln (\theta_z/\theta_{\text{max}}). \tag{30} \]

\[ \frac{dW_{\delta,s}}{d\varepsilon_+} = \frac{dW_\delta}{d\varepsilon_+} + \sigma_s \frac{\alpha e^2}{30 \pi m^2 \omega} \frac{d\varepsilon_+}{d\varepsilon_+} F(x, \varepsilon_+ / \omega), \tag{22} \]
\[ \frac{dW_{Z,s}}{d\varepsilon_+} = \sigma_z \frac{\alpha e^2}{30 \pi m^2 \omega} \frac{d\varepsilon_+}{d\varepsilon_+} F(x, \varepsilon_+ / \omega), \tag{29} \]

\[ F(x, \varepsilon_+ / \omega) = (x^4 Y - 3 x^2 Y' - x^3) + (x^4 Y + 3 x Y - 5 x^2 Y' - x^3) e_\pm^2 / \omega^2, \]

\[ Y(x) = \int_0^\infty \sin (ux + u^3/3) \, du, \quad x = \left( m^2 \omega / e\varepsilon_+ \varepsilon_- \right)^{2/3}, \quad \kappa = e\varepsilon_\omega / m^3, \quad Y'' - x Y = -1. \]
Pair production probability in crystals

\[ \frac{dW_{\text{loc}}}{d\varepsilon_+} = \frac{dW_{6,\varepsilon}}{d\varepsilon_+} + \frac{dW_{Z,\varepsilon}}{d\varepsilon_+} = \frac{dW_6}{d\varepsilon_+} + \sigma \frac{\alpha \varepsilon_+^2}{30 \pi m^2 \omega} \frac{d \varepsilon_+}{\sigma} F(x, \varepsilon_+/\omega), \]

\[ \sigma = \sigma(\rho) = \sigma_s + \sigma_Z = 8 \pi n(\rho) \left( \frac{Z\alpha}{\varepsilon_+} \right)^2 \ln \left( \frac{\theta_+}{\theta_{\min}} \right). \]

low energy limit:

\[ \frac{dW_{\text{sm}}}{d\varepsilon_+} = \frac{\ln (183 Z^{-13}) - B}{\ln (183 Z^{-13})} \cdot \frac{dW_{BH}}{d\varepsilon_+}, \quad 2B = (1 + y) \exp(y) \int_y^\infty \exp(-t) \frac{dt}{t} - 1, \quad y = u^2/a_F^2 \]

\[ \frac{dW_{\text{loc}}(\rho)}{d\varepsilon_+} = \frac{dW_6(n(\rho))}{d\varepsilon_+} + \frac{2 \alpha^3 Z^2 n(\rho)}{15 m^2 \omega} \left[ 2 \ln (183 Z^{-13}) - \theta (1 - x) \ln x + 1 - (1 + y) \times \right. \]
\[ \times \exp(y) \int_y^\infty \exp(-t) \frac{dt}{t} \left. \right] [(x^4 Y - 3 x^2 Y^2 - x^3) + (x^4 Y + 3 x Y - 5 x^2 Y^2 - x^3) (\varepsilon_+^2 + \varepsilon_-^2)/\omega^2], \]

\[ \frac{dW}{d\varepsilon_+} = n_0 d \int \frac{dW_{\text{loc}}(\rho), n(\rho)}{d\varepsilon_+} d^2\rho. \]
Local approximation, axial case with incoherent scattering:

Coherent and incoherent PP in Ge<110>100K

Experiment on PP in Ge<110>100K
Landau-Pomeranchuk effect suppression by strong crystalline field \( \mathcal{C}_s(l_{coh}) > m/\varepsilon \)

\[
\frac{dW}{d\omega} = \frac{ie^2}{2\pi} \int_0^\infty \left[ \frac{m^2}{\varepsilon^2} + \frac{(\varepsilon^2 + \varepsilon'^2)}{4\varepsilon\varepsilon'} \theta^2(\tau) \right] \exp(-i\omega\tau)
\]

\[
\times \prod_{i=x,y} \exp\left\{ \frac{ib}{2} \left[ \frac{1}{\tau} \left( \int_0^\tau dt \theta_1(t) \right)^2 - \int_0^\tau dt \theta_1(t) \right] \right\} \frac{d\tau}{\tau},
\]

\[
a = m^2\omega/2\varepsilon\varepsilon', \quad b = \omega\varepsilon/\varepsilon'.
\]

\[
d^x\mathcal{P}_N = \frac{d\theta_1 \ldots d\theta_N}{(2\pi\Delta(\sigma_x\sigma_y)^{1/2})^N} \exp\left\{ -\left[ \frac{(\theta_1 - \omega\Delta)^2}{2\sigma_x\Delta} + \frac{\theta_1^2}{2\sigma_y\Delta} \right] \right. \\
- \left. \cdots - \left[ \frac{(\theta_N - \omega\Delta - \theta_{N-1} \tau)^2}{2\sigma_x\Delta} + \frac{(\theta_N - \theta_{N-1})^2}{2\sigma_y\Delta} \right] \right\}.
\]

\[
\langle \frac{dW}{d\omega} \rangle = -\frac{e^2}{\pi} \text{Im} \int_0^\infty \left\{ \frac{m^2}{\varepsilon^2} + \frac{\varepsilon^2 + \varepsilon'^2}{2\varepsilon\varepsilon'} \left[ \frac{2\varepsilon^2}{r_x^2} \text{th} \left( \frac{r_x\varepsilon}{2} \right) + \sum_{i=x,y} \frac{\sigma_i}{r_i} \text{th} \left( \frac{r_i\varepsilon}{2} \right) \right] \right\} \exp\left\{ -iaz - \frac{w^2}{2\sigma_x} + \frac{w^2}{r_x\sigma_x} \text{th} \left( \frac{r_x\varepsilon}{2} \right) \right\} dz.
\]

\[
\langle dW \rangle = \int d^\infty \Phi \int d\omega \{ \Phi(z) \} = \lim_{N \to \infty} \int \ldots \int d^x\mathcal{P}_N \frac{dW\{ \Phi(z) \}}{d\omega},
\]

\[
= \lim_{N \to \infty} \int \ldots \int d^x\mathcal{P}_N \frac{dW\{ \Phi(z) \}}{d\omega}.
\]
Landau-Pomeranchuk effect suppression by strong crystalline field

generalizes the expression obtained by Migdal for description of the Landau-Pomeranchuk effect for the probability of emission of a γ ray by \( e^\pm \) moving in an amorphous beam, and the Klepikov-Nikishov-Ritus expression for the probability of emission of a γ ray by \( e^\pm \) moving in a uniform electromagnetic field.

\[
\frac{dW}{d\omega} = -\frac{e^2}{\pi} \text{Im} \int_0^\infty \left\{ \frac{m^2}{\varepsilon^2} + \frac{(\varepsilon^2 + \varepsilon'^2)}{2\varepsilon\varepsilon'} \left[ \frac{2w^2}{r_z^2} \text{th}^2 \left( \frac{r_z^2}{2} \right) \right] \right\} \\
+ \sum_{i=x,y} \frac{\sigma_i}{r_i} \text{th} \left( \frac{r_i z}{2} \right) \right\} \\
\times \left( \frac{r_x r_y}{\text{sh} r_x z \text{sh} r_y z} \right)^{\eta_x} \exp \left\{ -iaz - \frac{w^2 z}{2\sigma_x} + \frac{w^2}{r_x \sigma_x} \text{th} \left( \frac{r_z^2}{2} \right) \right\} dz.
\]

1118 Sov. Phys. JETP 63 (6), June 1986
The role of incoherent scattering in radiation processes at small angles of incidence of particles on crystallographic axes or planes

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FIG. 2. Pair-production probability obtained after transition to the cross channel and integration over the energy of the $e^+ (e^-)$ in the expressions (18), (22), and (33). The approach of the probability $W(\rho_1)$ to the probability $W_E(\rho_1)$ (and not to the probability $W_s(n(\rho_1)) \approx W_s(n(0))$ (see Eq. (24))) illustrates the absence of the Landua-Pomeranchuk effect.
A consistent theory of relativistic particle incoherent scattering in crystals is suggested
Thank you for your attention!