Statistics of the Radiating Relativistic Electrons

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Outline

- Preliminaries
- Statistical properties of recoils
- Straggling function
- Comparative analysis
- Verifications and illustrations
- Summary and outlook

Processes and Model

***'All models are wrong but some are useful!'* (George Box)**

quantum processes

- radiation in periodic fields
- ionization losses of relativistic electrons
- ionization losses of heavy particles
- ...

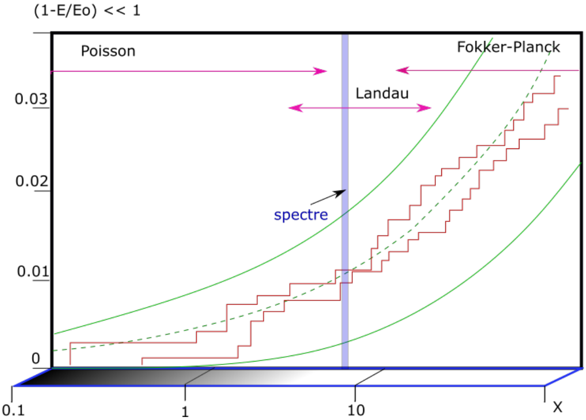
model

- Independent identically distributed (i.i.d.) recoils
- Compact support of the recoil spectrum $w(\omega)$, $0 < \omega_{min} \leq \omega \leq \omega_{max} < \infty$
- Normalised spectrum: $\int_{-\infty}^{\infty} w(\omega)\, d\omega = 1$
- Small magnitude of a recoil, $\omega_{max} \ll \gamma_{e/p}$

Key model suggestion: the recoil spectrum independent of the particle energy

Scheme of the Process. Straggling function

Mathematical formulation “Subordinate to compound Poisson Process”



straggling function

- minimum parameters
- $\rightarrow$ radiation spectrum
- $\rightarrow$ detector signal

Non-recoiled particles $= \exp(-x)$ (x is the average number of recoils per particle)

Rigorous Solutions for Arbitrary Recoil Spectrum

Characteristic functions, Bulyak, Shul'ga (2016, 2017)

evolution of spectrum

$$\hat{f} = \hat{f}_0 \exp[x(\check{w} - 1)]$$

straggling function

$$\hat{S}_x = \hat{w} \mathrm{e}^{x(\hat{w}-1)}$$

moments of spectrum

$$\overline{\gamma}(x) = \overline{\gamma_0} - x\,\overline{\omega}$$

$$\mathrm{Var}[\gamma](x) = \overline{(\gamma - \overline{\gamma})^2} = \mathrm{Var}[\gamma_0] + x\,\overline{\omega^2}$$

$$\ldots$$

SF moments

$$\overline{\epsilon} = (1 + x)\,\overline{\omega}$$

$$\mathrm{Var}[\epsilon] = (1 + x)\,\overline{\omega^2} - \overline{\omega}^2$$

$$\ldots$$

$\hat{g}$ is the Fourier transform, $\check{g}$ the inverse Fourier transform.

Density of energy losses distribution defined by

- recoil spectrum $w(\omega)$
- average number of recoils $x = \int_0^z P(z')\, dz' / \overline{\omega}$

Summary of Preliminary

- Mathematically the characteristic function is the solution for SF
- Practically, SF may be restored at limits
  - ‘Poisson’, $x \lesssim 1$, a few self states may be derived
  - ‘Fokker–Planck’, $x \to \infty$, a few first moments sufficient

How does SF evolve in between the limits ?

Analysis: BS $\Leftrightarrow$ α-stable

Goals: stability parameter, α, and scale c (half width at 1/e height)

CF of α-stable distribution

$$\hat{\phi}(s) = \exp\left\{ \mathrm{i}s\mu - |cs|^{\alpha}\left[1 - \mathrm{i}\beta\,\mathrm{sgn}(s)\Phi\right]\right\}$$

$$\Phi = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right), & \alpha \neq 1, \\ -\frac{2}{\pi}\log|s|, & \alpha = 1, \end{cases}$$

$\alpha \in (0,2]$	stability parameter
$\beta \in [-1,1]$	skewness parameter
$c \in (0,\infty)$	scale parameter
$\mu \in (-\infty,\infty)$	location parameter

CF of BS distribution

$$\hat{S}_x(s) = \hat{w}(s)\exp\left[x(\hat{w}(s) - 1)\right]$$

Physical reasons

$\alpha \in (1,2]$ $\quad = 2$ Gaussian $\quad x \to \infty$

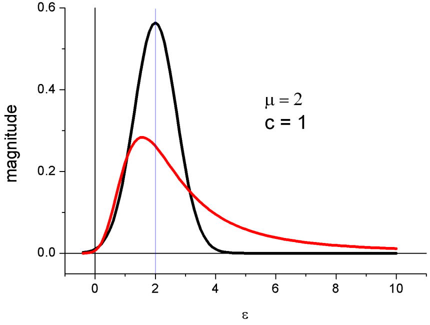
$c \sim \sqrt{x\overline{\omega^2}}$ FP $\quad x \to \infty$

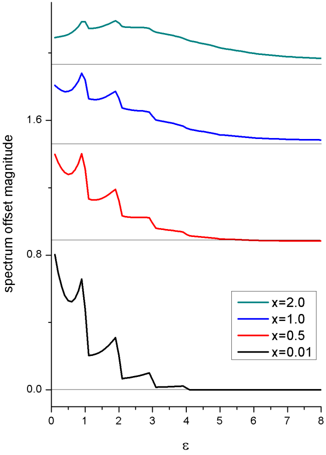
$\mu = x\overline{\omega}$ energy conservation

Analysis of Straggling Function

Central Limit Theorem: if $x \to +\infty$ distribution $\to$ Gaussian for i.i.d.

Gauss $\alpha = 2$, Landau $\alpha = 1$, $\beta = 1$ distributions





SF in helical undulator, $K = 1$

Analysis of Straggling Function, cont.

BS has nonlinear logarithmic dependence of s. "Linearising" it at s_*, the root of

$$\Re[x\,(1-\hat{w}(s_*))] = 1 = |\pi c s_*|^{\alpha} \ ,$$

yields

$$\alpha(x) = \left.\frac{s D_s \Re[\hat{w}]}{1-\Re[\hat{w}]}\right|_{s=s_*} \ ,$$

where $D_s \cdot = \partial \cdot / \partial s$
Scaling parameter

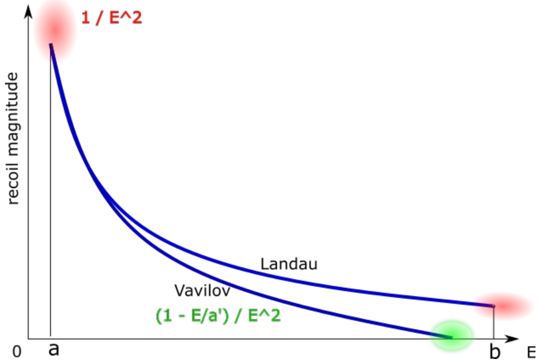
$$c(x) = [x m_{\alpha}[w]]^{1/\alpha} = \frac{1}{\pi s_*}$$

with α–moment of the recoil

$$m_{\alpha}[w] = \int \omega^{\alpha} w(\omega)\, \mathrm{d}\omega \ , \qquad \omega > 0$$

Validation of BS Distribution at Limits

Rigorous: if $x \to \infty$ then $\alpha \to 2$ Central Limit Theorem

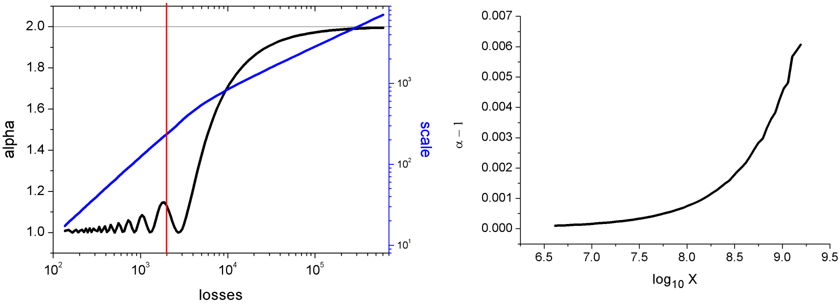


Landau distribution (L.D. Landau, 1944)

- Bounded Pareto distribution $\to$ Rutherford cross section when $a \to 0$, $b \to \infty$
- $\varpi \to 0$ such that $P = x\,\varpi$ finite
- $\lim_{a\to 0, b\to\infty} \alpha = 1$

No Contradiction Between CLT and Landau Distribution

Practical case: $a > 0,\ b < \infty \quad \rightarrow \quad \alpha > 1$

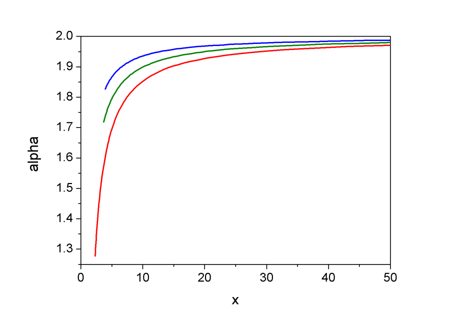


1 GeV electrons traversing tungsten target

Within model limits $\alpha - 1 \ll 1$

Helical Undulator Stability Parameter

$K = 0.01, 0.3, 1$



when K increases

- later the Gaussian distribution attracted
- thicker the tail
- faster the scale parameter increases

Some References

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Summary

General properties of straggling spectrum

$$\hat{S}_{\text{BS}} = \hat{w} \exp\left[x(\hat{w} - 1)\right]$$

$$\alpha(x) = \left. \frac{sD_s \Re[\hat{w}]}{1 - \Re[\hat{w}]} \right|_{s=s_*}$$

$$c(x) = \{x\, m_\alpha[w]\}^{1/\alpha}$$

when number of recoils increases

- (variable) stability parameter increases: $1 < \alpha(x) \leq 2$
- type of diffusion varies from ballistic to normal
- width of spectrum increases as $x^{1/\alpha}$, maximum rate at the beginning

Efficiency of a source of hard EM radiation – spectral brightness – inverse to the width of spectrum

Losses of particles-radiators occur due to tail(s) of the spectrum