Dispersion relations for different types of radiation from periodic structures: similarities and differences

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Outline

1. Types of radiation in periodic structures.
2. Role of frequency dependent permittivity.
3. Small-angle X-ray TR from periodic stacks, end/butt TR.
4. About one asymptotic expression widely-used in periodic structures.
Function of dielectric permittivity

\[ \varepsilon(\omega) = 1 + \int_0^\infty dt \ f(t) \exp(i\omega t) \]

X-ray: \[ \varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} + i\chi(\omega) \]

\[ \varepsilon(\omega) = 1 + \chi'(\omega) + i\chi''(\omega) \]

Absorption – other sign – different consideration for X-ray CR needed?

L.D. Landau, et al., Electrodynamics of Continuous Media (Second Edition), V. 8 in Course of Theoretical Physics, 1984, Pergamon Press Ltd.
Smith-Purcell radiation

Smith-Purcell dispersion relation

\[ \lambda = \frac{d}{m} \left( \frac{1}{\beta} - \cos \theta \right), \quad m = 1, 2, \ldots. \]

Electron in vacuum \( \epsilon(\omega) \) in dispersion relation

\[ \omega d (\gamma^{-2} + \theta^2) = 4\pi \gamma m, \quad m = 1, 2, \ldots. \]

\( \theta \ll 1 \)
\( \gamma \gg 1 \)

One frequency \( \omega_1 \) at one angle \( \theta_1 \)

S.J. Smith, E.M. Purcell, Physical Review (1953)
Undulator radiation

Dispersion relation (ultrarelativistic)

\[
\lambda = \frac{\lambda_u}{2\gamma^2} \left(1 + \frac{K^2}{2} + \gamma^2 \theta^2\right)
\]

Electron in vacuum\[\Rightarrow\]NO \(\varepsilon(\omega)\) in dispersion relation

One frequency \(\omega_1\) at one angle \(\theta_1\)

Resonant TR dispersion relation

\[ \omega d \left( \gamma^{-2} + \theta^2 \right) + a \omega \frac{\omega_p^2}{\omega^2} = 4\pi cm, \quad m = 1, 2, \ldots \]

Two frequencies \( \omega_{1,2} \) at one angle \( \theta_1 \)


Parametric X-ray radiation

\[
\frac{\omega}{c} \beta \left( \frac{1}{\beta} - \cos \theta \sqrt{\varepsilon(\omega)} \right) = \beta g
\]

Electron in matter \( \varepsilon(\omega) \) in dispersion relation

\[
\omega^2 \left( \gamma^{-2} + \theta^2 \right) - 2 \omega \frac{vg}{\beta} + \omega_p^2 = 0
\]

\( \theta << 1 \)
\( \gamma >> 1 \)

Two frequencies \( \omega_{1,2} \) at one angle \( \theta_1 \)

Spectral distribution

\[
\frac{d^2W(n, \omega)}{d\Omega \, d\omega} = \frac{d^2W_1(n, \omega)}{d\Omega \, d\omega} \frac{\sin^2(Nd\varphi/2)}{\sin^2(d\varphi/2)}
\]

\[
\frac{\sin^2(d\varphi N/2)}{\sin^2(d\varphi/2)} \xrightarrow{N \gg 1} 2\pi N \sum_m \delta(d\varphi - 2\pi m)
\]

\[
\frac{dW(\omega)}{d\omega} = 2\pi N \int d\theta \frac{d^2W_1(n, \omega)}{d\Omega \, d\omega} \sum_m \delta(d\varphi - 2\pi m)
\]

**NO frequency dispersion**

\[
\theta_1 = \sqrt{\frac{4\pi cm}{d\omega} - \gamma^{-2}}
\]

**Frequency dispersion**

\[
\theta_1 = \sqrt{\frac{4\pi cm}{\omega d} - \gamma^{-2} - \frac{a}{d} \frac{\omega^2}{\omega_p}}
\]

\[
\frac{dW(\omega)}{d\omega} = A \sum_m \frac{d^2W_1(n, \omega)}{d\Omega \, d\omega} \bigg|_{\theta = \theta_1}
\]

no qualitative difference
Angular distribution

\[ \frac{dW(\theta)}{d\theta} = 2\pi N \int d\omega \frac{d^2W_1(n,\omega)}{d\Omega d\omega} \sum_m \delta(d\varphi - 2\pi m) \]

dispersion relation

**NO frequency dispersion**

\[ \omega_1 = \frac{4\pi cm}{d(\gamma^{-2} + \theta^2)} \]

**Frequency dispersion**

\[ \omega_1 = \frac{2\pi cm + \sqrt{(2\pi cm)^2 - ad\omega_p^2(\theta^2 + \gamma^{-2})}}{d(\theta^2 + \gamma^{-2})} \]

\[ \omega_2 = \frac{2\pi cm - \sqrt{(2\pi cm)^2 - ad\omega_p^2(\theta^2 + \gamma^{-2})}}{d(\theta^2 + \gamma^{-2})} \]

\[ \frac{dW(\theta)}{d\theta} = A \sum_m \left( \frac{d^2W_1(n,\omega)}{d\Omega d\omega} \bigg|_{\omega=\omega_1} + \frac{d^2W_1(n,\omega)}{d\Omega d\omega} \bigg|_{\omega=\omega_2} \right) \]

qualitative difference!

*angular dispersion would lead to similar effect in spectral distribution*
**Experiment**

G.A. Naumenko et al.,
JETP Letters (2017)

**Theory**


X-ray transition radiation

\[ d = a + 2 \text{ mm} \]
\[ a = 62 \mu\text{m} \]
\[ \gamma = 4 \cdot 10^4 (20 \text{ GeV}) \]

150 polypropylene foils

Courtesy A. Savchenko
\[ \omega_1 = \frac{2\pi cm + \sqrt{(2\pi cm)^2 - ad\omega_p^2(\theta^2 + \gamma^{-2})}}{d(\theta^2 + \gamma^{-2})} \]

\[ \omega_2 = \frac{2\pi cm - \sqrt{(2\pi cm)^2 - ad\omega_p^2(\theta^2 + \gamma^{-2})}}{d(\theta^2 + \gamma^{-2})} \]

\[ \omega_1 = \omega_2 \quad \text{at} \quad \theta_0 = \frac{\sqrt{4\pi^2 c^2 m^2}}{a\omega_p^2(a + b)} - \gamma^{-2} \]

\[ \theta_0 = \sqrt{\frac{4\pi^2 c^2 m^2}{a\omega_p^2 (a + b)}} - \gamma^{-2} \]

\( m = 3 \)
\( \theta_0 = 0.48 \text{ mrad} \)
\( \theta = 0.47 \text{ mrad} \)

\( d = a + 2 \text{ mm} \)
\( a = 62 \text{ \mu m} \)
\( \gamma = 4 \cdot 10^4 (20 \text{ GeV}) \)

\( m = 2 \)
\( \theta_0 = 0.32 \text{ mrad} \)
\( \theta = 0.31 \text{ mrad} \)

150 polypropylene foils
Role of surface

Often we consider the radiation process in an infinite media/structure. But what of the surface? how does it change the radiation characteristics?

\[
\lambda = \frac{d}{m} \left( \frac{1}{\beta} - \sqrt{\varepsilon} \cos \theta' \right)
\]

\[
\sqrt{\varepsilon(\omega)} \cos \theta' = \cos \theta
\]

\[
\lambda = \frac{d}{m} \left( \frac{1}{\beta} - \cos \theta \right)
\]

\[
\lambda = \frac{d}{m} \frac{1}{2} \left( \gamma^{-2} + \theta^2 \right)
\]

\[
\sqrt{\varepsilon(\omega)} \cos \theta' = \sqrt{\varepsilon(\omega)} - \sin^2 \theta
\]

\[
\lambda = \frac{d}{m} \left( \frac{1}{\beta} - \sqrt{\varepsilon(\omega)} - \sin^2 \theta \right)
\]

\[
\lambda = \frac{d}{m} \frac{1}{2} \left( \gamma^{-2} + \theta^2 + \frac{\omega_p^2}{\omega^2} \right)
\]

D.V. Karlovets, JETP (2011)
End/butt radiation

\[
\frac{d^2 W(n, \omega)}{d\Omega \, d\omega} = \frac{d^2 W_1(n, \omega)}{d\Omega \, d\omega} \frac{\sin^2 (N d \varphi / 2)}{\sin^2 (d \varphi / 2)} = 2\pi N \sum_m I_1(\theta, \omega) \delta(d \varphi(\theta, \omega) - 2\pi m)
\]

\[
\frac{\sin^2 (d \varphi N / 2)}{\sin^2 (d \varphi / 2)} \xrightarrow{N \gg 1} 2\pi N \sum_m \delta(d \varphi - 2\pi m)
\]

\[
\frac{dW(\omega)}{d\omega} = 2\pi N \sum_m \int d\theta I_1(\theta, \omega) \delta(d \varphi(\theta, \omega) - 2\pi m)
\]

\[
a \frac{\omega}{4c} \left(\gamma^{-2} + \theta^2 + 1 - \varepsilon(\omega)\right) \ll 1
\]

\[
b \frac{\omega}{4c} \left(\gamma^{-2} + \theta^2\right) \ll 1
\]

The periodic target = equivalent plate

\[
\frac{dW(\omega)}{d\omega} = 2\pi N \sum_{m} \int_{\theta_c}^{\theta_c} d\theta I_1(\theta, \omega) \delta(d \varphi(\theta, \omega) - 2\pi m) + \int_{0}^{\theta_c} d\theta I_1(\theta, \omega) 2\pi N \sum_m \delta(d \varphi(\theta, \omega) - 2\pi m)
\]

\[\propto \sum_{m} I_1(\theta_m, \omega) \text{ – resonant radiation}
\]

\[\propto I_1(\theta = \gamma^{-1}; \omega) \text{ – end / butt radiation}
\]

Garibian, Yan Shi, 1983

analytically numerically, saddle-point method
Side maxima

\[
\frac{\sin^2(d\phi N/2)}{\sin^2(d\phi/2)} \xrightarrow{N \gg 1} 2\pi N \sum_m \delta(d\phi - 2\pi m)
\]

\[
\frac{\sin^2(d\phi N/2)}{\sin^2(d\phi/2)} = 2\pi N \sum_m \delta(d\phi - 2\pi m) + \sum_{m} \sum_{p=1}^{N-2} \exp\left(-\frac{1}{\pi} \left[ N(d\phi - 2\pi m) - \pi(2p+1) \right]^2 \right) \frac{1}{\sin^2\left(\pi m + \pi \frac{2p+1}{2N} \right)}
\]
Side maxima

\[
\left( \frac{\sin(Mg)}{\sin(g)} \right)^2 \approx \sum_{r=-\infty}^{+\infty} \left[ M^2 \exp\left( -\frac{(g - \pi r)^2}{\pi M^{-2}} \right) + \sum_{p=1}^{M-2} \frac{1}{\sin^2(g_p)} \exp\left( -\frac{(g - g_p)^2}{\pi M^{-2}/4} \right) \right]
\]

\[g_p = \pi \left( r + \frac{2p + 1}{2M} \right)\]

Red curve - exact

Black curve – approximate one

for different number of periods M
Conclusions

• Frequency dependent of dielectric function can play an important role in radiation in periodic media/structures: it leads to new roots of dispersion relation, i.e. new types of radiation waves.

• For example, the effect is responsible for the sharp spikes in angular X-ray transition radiation.

• The role of the surface is considerable: radiation characteristics differ for radiation at small angles and large ones.

• In the case of Smith-Purcell effect in forward direction, we expect that its dispersion relation will contain the properties of the material of the target – like it happens in transition radiation.

• The “thin” structure of the widely used asymptotic expression is defined, the simple and easy-to-use expression is suggested. It must play a key part for the phenomena similar to “end transition radiation”, in all types of radiation in periodic media, and can be responsible for the fine structure of the radiation peaks, or even their existence when the main ones are suppressed.
Thank you for your attention!