Gravitational Waves: a theoretical primer

Valeria Ferrari

SAPIENZA
UNIVERSITÀ DI ROMA

INFN

Gravitational waves: foundations and beyond
Physics Department, Sapienza University of Rome
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Some basic facts about general relativity: the role of the metric tensor is twofold:

1) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

it allows to compute the distance between spacetime points, i.e. it describes the geometrical properties of the spacetime

2) $g_{\mu\nu}$

is the generalisation of the Newtonian potential (it follows from the equivalence principle), therefore it describes the gravitational field

Gravitational waves as a perturbation of a flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| << |\eta_{\mu\nu}|$$

We need to solve Einstein's equations for the perturbation $h_{\mu\nu}$

Einstein's equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$G_{\mu\nu} = \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right)$$ Einstein tensor

$T_{\mu\nu}$ energy-momentum tensor
source of the perturbation

Alternative way of writing Einstein's equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

This is the form of Einstein's eqs we are going to use

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\gamma\alpha} \left[g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} \right] \qquad {}_{,\alpha} \equiv \frac{\partial}{\partial x^{\alpha}}$$

Christoffel symbols

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} - \underline{\Gamma^{\alpha}{}_{\kappa\nu}\Gamma^{\kappa}{}_{\beta\mu} + \Gamma^{\alpha}{}_{\kappa\mu}\Gamma^{\kappa}{}_{\beta\nu}}$$

Riemann tensor

non linear part

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}$$

Ricci tensor and scalar curvature

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| << |\eta_{\mu\nu}|$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| << |\eta_{\mu\nu}|$$

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2}\eta^{\gamma\alpha}\left[h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}\right] + O(h^2)$$

$$R_{\mu\nu} = \frac{\partial}{\partial x^\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^{\alpha}{}_{\mu\alpha} + \Gamma^{\alpha}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\alpha}$$

Ricci tensor

non linear part

Einstein's equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

become

$$\left\{\Box_F h_{\mu\nu} - \left[\frac{\partial^2}{\partial x^\lambda \partial x^\mu}h^\lambda_\nu + \frac{\partial^2}{\partial x^\lambda \partial x^\nu}h^\lambda_\mu - \frac{\partial^2}{\partial x^\mu \partial x^\nu}h^\lambda_\lambda\right]\right\} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right)$$

$$\Box_F = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} = -\frac{\partial^2}{c^2\partial t^2} + \nabla^2$$

Maxwell's equations written for the vector potential

$$\Box_F A_\alpha - \frac{\partial^2 A^\beta}{\partial x^\alpha \partial x^\beta} = -\frac{4\pi}{c} J_\alpha$$

but we know that $A'_\alpha = A_\alpha + \frac{\partial \Phi}{\partial x^\alpha}$ is still a solution

and if we choose the scalar function ϕ such that $\frac{\partial}{\partial x^\beta} A'^\beta = 0$ **Lorenz gauge**

$$\Box A_\alpha = -\frac{4\pi}{c} J_\alpha$$

do we have an analogous of the Lorenz gauge for gravitational waves?

$$\left\{\Box_F h_{\mu\nu} - \left[\frac{\partial^2}{\partial x^\lambda \partial x^\mu} h^\lambda_\nu + \frac{\partial^2}{\partial x^\lambda \partial x^\nu} h^\lambda_\mu - \frac{\partial^2}{\partial x^\mu \partial x^\nu} h^\lambda_\lambda\right]\right\} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T\right)$$

our gauge freedom: diffeomorfism invariance

$$x^{\mu\prime} = x^\mu + \epsilon^\mu(x) \qquad \epsilon_{\mu,\nu} = O(h)$$

harmonic gauge condition

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x^\mu} h^\mu{}_\nu = \frac{1}{2}\frac{\partial}{\partial x^\nu} h^\mu{}_\mu$$

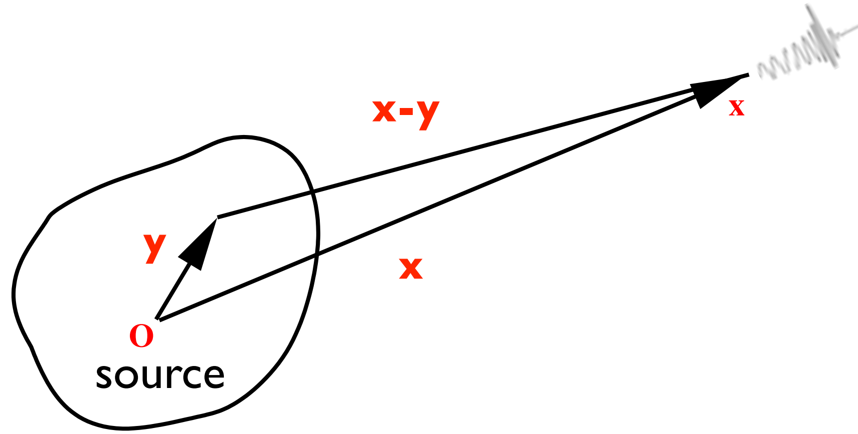
$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$$\begin{cases} \Box_F h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T\right) \\ \frac{\partial}{\partial x^\mu} h^\mu{}_\nu = \frac{1}{2}\frac{\partial}{\partial x^\nu} h^\mu{}_\mu , \end{cases} \quad \Rightarrow \quad \begin{cases} \Box_F \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \\ \frac{\partial}{\partial x^\mu} \bar{h}^\mu{}_\nu = 0 , \end{cases}$$

A perturbation of a flat spacetime propagates as a wave travelling at the speed og light

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$$\begin{cases} \Box_F \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \\ \frac{\partial}{\partial x^\mu} \bar{h}^\mu{}_\nu = 0\,, \end{cases}$$



$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int_V \frac{T_{\mu\nu}\left(t - \frac{|\mathbf{x}-\mathbf{y}|}{c}, \mathbf{y}\right)}{|\mathbf{x}-\mathbf{y}|}\, d^3y$$

The retarded potential solution automatically satisfies the harmonic gauge condition $\frac{\partial}{\partial x^\mu} \bar{h}^\mu{}_\nu = 0$

In vacuum $T_{\mu\nu} = 0$

plane wave solution

$$\begin{cases} \Box_F \bar{h}_{\mu\nu} = 0 \\ \frac{\partial}{\partial x^\mu} \bar{h}^\mu{}_\nu = 0 \end{cases}$$

$$\bar{h}_{\mu\nu} = \Re\left\{ A_{\mu\nu} e^{i k_\alpha x^\alpha} \right\}$$

$A_{\mu\nu}$ polarization tensor

k_α wave vector

$$k_\mu A^\mu{}_\nu = 0$$

from the gauge condition: wave vector and polarization tensor are orthogonal

Polarization degrees of gravitational waves

let us assume that a progressive wave is traveling along the x-direction

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)\bar{h}^{\mu}{}_{\nu}=0 \qquad \text{where} \qquad \bar{h}^{\mu}{}_{\nu}=\bar{h}^{\mu}{}_{\nu}(t-\frac{x}{c})$$

the harmonic gauge condition gives

$$\frac{\partial \bar{h}^{\mu}{}_{\nu}}{\partial x^{\mu}}=0 \quad \longrightarrow \quad \boxed{\bar{h}^{t}{}_{\nu}=\bar{h}^{x}{}_{\nu}}$$

i.e.

$$\begin{array}{ll} \bar{h}^{t}{}_{t}=\bar{h}^{x}{}_{t}, & \bar{h}^{t}{}_{y}=\bar{h}^{x}{}_{y}, \\ \bar{h}^{t}{}_{x}=\bar{h}^{x}{}_{x}, & \bar{h}^{t}{}_{z}=\bar{h}^{x}{}_{z} \end{array}$$

The harmonic gauge condition $\frac{\partial}{\partial x^\mu}\bar{h}^\mu{}_\nu = 0$ remains satisfied if we make an infinitesimal coordinate transformation

$$x^{\mu\prime} = x^\mu + \epsilon^\mu(x)$$

provided $\Box_F \epsilon^\mu = 0$

we can choose ε^μ to put
4 constraints on the components of $h_{\mu\nu}$

$$\bar{h}^t{}_x = \bar{h}^t{}_y = \bar{h}^t{}_z = 0, \qquad \bar{h}^y{}_y + \bar{h}^z{}_z = 0$$

this constraint + the relations among the components of $h_{\mu\nu}$ previously derived

$$\bar{h}^t{}_t = \bar{h}^x{}_t, \qquad \bar{h}^t{}_y = \bar{h}^x{}_y,$$
$$\bar{h}^t{}_x = \bar{h}^x{}_x, \qquad \bar{h}^t{}_z = \bar{h}^x{}_z$$

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{yy} & h_{yz} \\ 0 & 0 & h_{yz} & -h_{yy} \end{pmatrix}$$

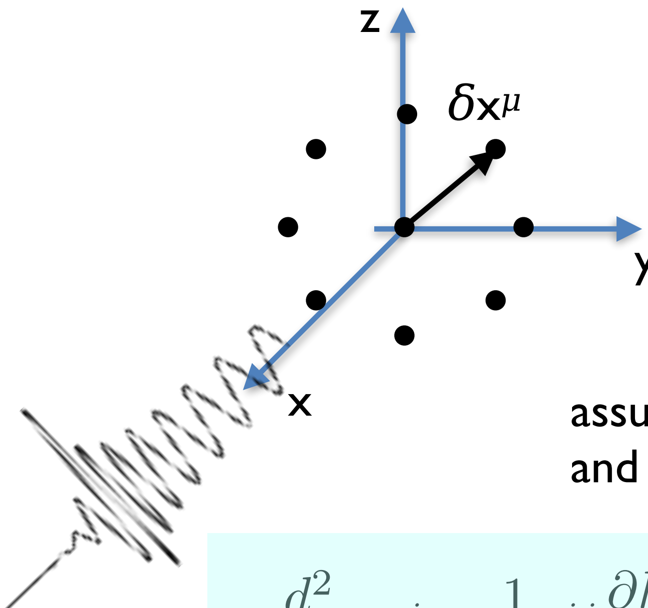
gravitational wave have only two degrees of freedom

Plus (+) polarization $h_{yy} = -h_{yy}$
Cross (x) polarization $h_{yz} = h_{zy}$

This gauge is said **TT**(transverse, traceless)- gauge

In the tt-gauge $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ and $h_{\mu\nu}$ are the same

how gravity affects the relative motion of neighbouring particles?



$$\frac{D^2 \delta x^\mu}{d\tau^2} = R^\mu_{\alpha\beta\gamma} u^\alpha u^\beta \delta x^\gamma$$ geodesic devitation

$R^\mu{}_{\alpha\beta\gamma}$ is the Riemann tensor

u^α is the particle four-velocity

assuming the particles are at rest before the wave arrives, and that the Riemann tensor is computed to first order in $h_{\mu\nu}$

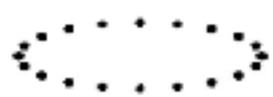
$$\frac{d^2}{dt^2}\delta x^i = \frac{1}{2}\eta^{ij}\frac{\partial h^{TT}_{jm}}{\partial t^2}\,\delta x^m \quad \longrightarrow \quad \delta x^i = \delta x^i_0 + \frac{1}{2}\,\eta^{ij}\,h^{TT}{}_{jk}\,\delta x^k_0$$

if the wave travels along x

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{yy} & h_{yz} \\ 0 & 0 & h_{yz} & -h_{yy} \end{pmatrix}$$

Plus (+) polarization

$h_{yy} = -h_{yy} \neq 0$



$$h_{yy} = h_+ e^{i\omega(t-\frac{x}{c})}$$

Cross (x) polarization

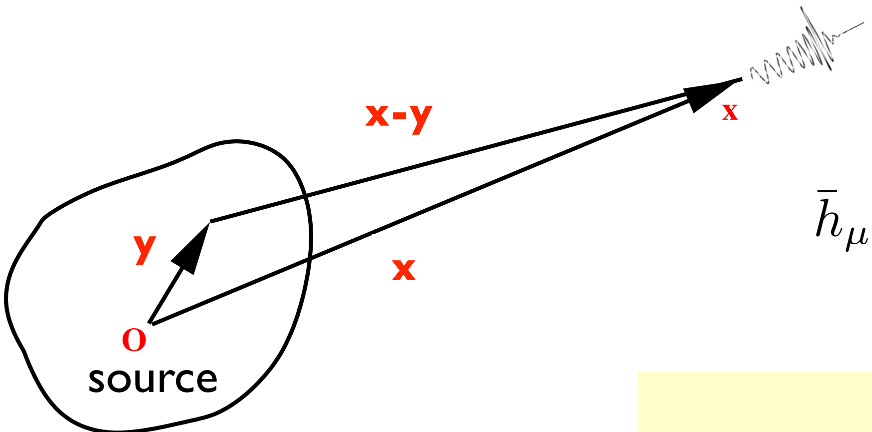
$h_{yz} = h_{zy} \neq 0$

$$h_{yz} = h_\times e^{i\omega(t-\frac{x}{c})}$$

Wave generation: the slow motion approximation

general solution $$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4}\int_V \frac{T_{\mu\nu}(t-\frac{|\mathbf{x}-\mathbf{y}|}{c},\mathbf{y}\ )}{|\mathbf{x}-\mathbf{y}\ |}\ d^3y$$

$$t_{ret} = t - \frac{|\mathbf{x}-\mathbf{y}|}{c}$$



Take the Fourier transform

$$\bar{h}_{\mu\nu}(\omega,\mathbf{x}) = \int_V d^3y\ T_{\mu\nu}(\omega,\mathbf{y})\ \frac{e^{i\omega\frac{|\mathbf{x}-\mathbf{y}|}{c}}}{|\mathbf{x}-\mathbf{y}\ |}$$

if the source is isolated

$|\mathbf{y}| < \epsilon, \quad \epsilon = \text{source size}$

λ is the wavelenght of the emitted radiation, and $\frac{\omega}{c} = \frac{1}{\lambda}$

if we assume that λ is much larger than the source size

$$\lambda >> \epsilon \ \rightarrow\ \frac{c}{\omega} >> \epsilon \ \rightarrow\ \frac{\omega\epsilon}{c} << 1 \quad \Longrightarrow \quad \boxed{\frac{\omega\mathbf{y}}{c} << 1}$$

i.e. the typical velocities are much smaller than the speed of light

$$\frac{e^{i\omega\frac{|\mathbf{x}-\mathbf{y}|}{c}}}{|\mathbf{x}-\mathbf{y}\ |} \sim \frac{e^{i\omega\frac{r}{c}}}{r} \qquad |\mathbf{x}| \equiv r$$

$$\bar{h}_{\mu\nu}(\omega,\mathbf{x}) = \frac{4G}{c^4}\frac{e^{i\omega\frac{r}{c}}}{r}\int_V T_{\mu\nu}(\omega,\mathbf{y})\ \ d^3y$$

finally, reverting to the time domain

$$\bar{h}_{\mu\nu}(t,r) = \frac{4G}{c^4 r}\int_V T_{\mu\nu}(t-\frac{r}{c},\mathbf{y})\ d^3y$$

However, to compute the emitted wave we do not need to know all the components of the stress-energy tensor:

Conservation of energy and momentum

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0 \qquad \int_V T^{\mu 0} d^3y = const \rightarrow \quad \bar{h}^{\mu 0} = const$$

Tensor Virial Theorem

$$\int_V T^{kn}(t,\mathbf{y})\ d^3y = \frac{1}{2}\ \frac{d^2}{dt^2} q^{kn}(t)$$

$$k,n = 1,3$$

$$q^{kn}(t) = \frac{1}{c^2}\int_V T^{00}(t,\mathbf{y})\ y^k\ y^n\ d^3y$$

q^{kn} quadrupole moment

T^{00} source energy density

NOTE THAT:

$$\begin{cases} \bar{h}^{\mu 0} = 0, \qquad \mu = 0..3 \\ \bar{h}^{ik}(t,r) = \frac{2G}{c^4 r}\left[\frac{d^2}{dt^2}\ q^{ik}(t-\frac{r}{c})\right] \end{cases}$$

$$\frac{4G}{c^4} \sim 8\cdot 10^{-50}\ s^2/g\ cm \quad !!!$$

finally, reverting to the time domain

$$\bar{h}_{\mu\nu}(t,r) = \frac{4G}{c^4 r}\int_V T_{\mu\nu}(t - \frac{r}{c}, \mathbf{y})\ d^3y$$

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NOTE THAT:

unlike em waves, no dipole radiation
(conservation of total momentum)

$$\vec{d}_{em} = \sum_i q_i \vec{r}_i \qquad \vec{d}_g = \sum_i m_i \vec{r}_i$$

In order to emit gravitational waves, a system must possess a certain degree of asymmetry

In conclusion

Gravitational waveform in the weak field, slow-motion approximation

$$\begin{cases} \bar{h}^{\mu 0} = 0, \qquad \mu = 0..3 \\ \bar{h}^{ik}(t,r) = \frac{2G}{c^4 r} \left[\frac{d^2}{dt^2} \, q^{ik}(t - \frac{r}{c}) \right] \end{cases}$$

$$q^{kn}(t) = \frac{1}{c^2} \int_V T^{00}(t, \mathbf{y}) \, y^k \, y^n \, d^3 y$$

q^{kn} quadrupole moment

T^{00} source energy density

Luminosity of a gravitational wave source

$$L_{GW} = \frac{dE_{GW}}{dt} = \frac{G}{5c^5} \left\langle \sum_{k,n=1}^{3} \dddot{Q}_{kn} \left(t - \frac{r}{c} \right) \dddot{Q}_{kn} \left(t - \frac{r}{c} \right) \right\rangle$$

$$Q_{jk} \equiv q_{jk} - \frac{1}{3} \delta_{jk} q$$

reduced quadrupole moment

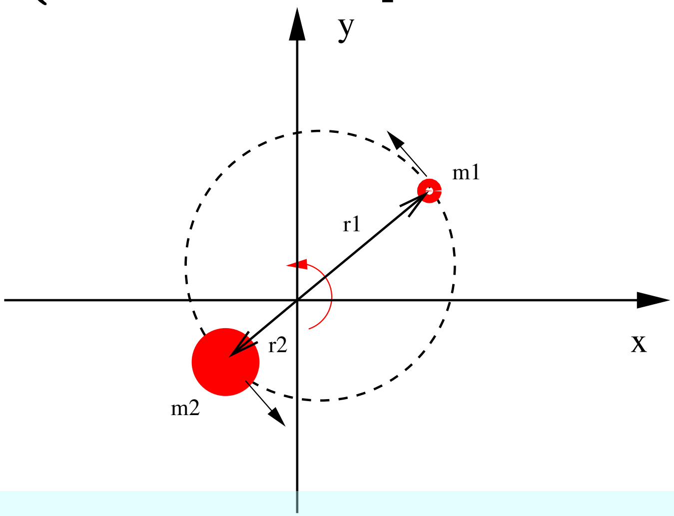
$$\begin{cases} \bar{h}^{\mu 0} = 0, \qquad \mu = 0..3 \\ \bar{h}^{ik}(t,r) = \frac{2G}{c^4 r}\left[\frac{d^2}{dt^2}\ q^{ik}(t-\frac{r}{c})\right] \end{cases}$$

binary system in circular orbit

$$q^{kn}(t) = \frac{1}{c^2}\int_V T^{00}(t,\mathbf{y})\ y^k\ y^n\ d^3y$$

q^{kn} quadrupole moment

T^{00} source energy density



1) the equation of motion

$$x_1 = \frac{m_2}{M} l_0 \cos\omega_K t \qquad x_2 = -\frac{m_1}{M} l_0 \cos\omega_K t$$

$$y_1 = \frac{m_2}{M} l_0 \sin\omega_K \qquad y_2 = -\frac{m_1}{M} l_0 \sin\omega_K t$$

$$\omega_K = \sqrt{\frac{GM}{l_0^3}}$$

$$l_0 = r_1 + r_2$$

$$M = m_1 + m_2$$

2) compute the quadrupole components

$$T^{00} = c^2 \sum_{n=1}^{2} m_n\ \delta(x-x_n)\ \delta(y-y_n)\ \delta(z)$$

$$q_{xx} = \frac{\mu}{2}\ l_0^2\ \cos 2\omega_K t + cost$$

$$q_{yy} = -\frac{\mu}{2}\ l_0^2\ \cos 2\omega_K t + cost1 \qquad \mu = \frac{m_1 m_2}{M}$$

$$q_{xy} = \frac{\mu}{2}\ l_0^2\ \sin 2\omega_K t$$

3) compute the waveform

$$h_{ij}(t,r) = -\frac{h_0}{r}\ A_{ij}(t-\frac{r}{c})$$

$$A_{ij}(t) = \begin{pmatrix} \cos\ 2\omega_K t & \sin\ 2\omega_K t & 0 \\ \sin\ 2\omega_K t & -\cos\ 2\omega_K t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h_0 = \frac{4\ \mu\ M\ G^2}{l_0\ c^4}$$

instantaneous wave amplitude

waves are emitted at twice the orbital frequency

The binary radiates energy in gravitational waves

$$L_{GW} = \frac{dE_{GW}}{dt} = \frac{G}{5c^5}\left\langle \sum_{k,n=1}^{3} \dddot{Q}_{kn}\left(t-\frac{r}{c}\right)\dddot{Q}_{kn}\left(t-\frac{r}{c}\right)\right\rangle$$

$$Q_{jk} \equiv q_{jk} - \frac{1}{3}\delta_{jk} q$$

reduced quadrupole moment

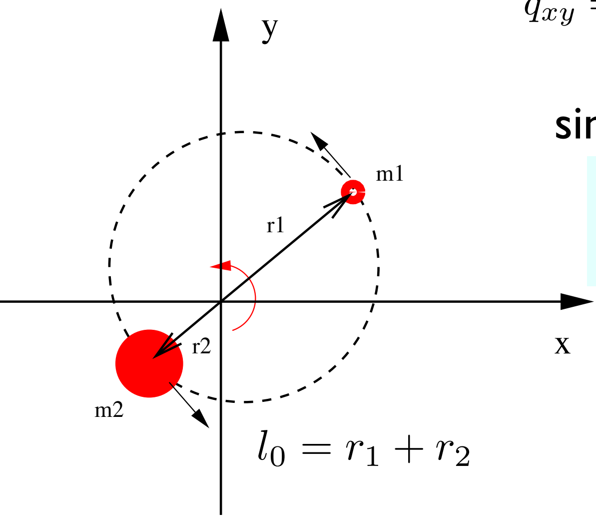
using the components of the quadrupole moment we just computed

$$q_{xx} = \frac{\mu}{2}\, l_0^2\, \cos 2\omega_K t + cost$$

$$q_{yy} = -\frac{\mu}{2}\, l_0^2\, \cos 2\omega_K t + cost1$$

$$q_{xy} = \frac{\mu}{2}\, l_0^2\, \sin 2\omega_K t$$

$$L_{GW} = \frac{32}{5}\frac{G^4}{c^5}\frac{\mu^2 M^3}{l_0^5} \qquad \mu = \frac{m_1 m_2}{M}$$



$$l_0 = r_1 + r_2$$

since total energy is conserved

$$\frac{dE_{orb}}{dt} + L_{GW} = 0$$

the orbital energy is

$$E_{orb} = E_K + U = -\frac{1}{2}\frac{G\mu M}{l_0}$$

$$l_0^3\,\frac{dl_0}{dt} = -\left[\frac{64}{5}\,\frac{G^3}{c^5}\,\mu\, M^2\right]$$

$$l_0(t) = l_0^{in}\left[1 - \frac{t}{t_c}\right]^{1/4}$$

$$t_c = \frac{5}{256}\frac{c^5}{G^3}\frac{\left(l_0^{in}\right)^4}{\mu M^2}$$ time to coalescence

$l_0^{in} =$ orbital distance at t=0

due to gravitational wave emission, the orbital distance decreases in time

$$l_0(t) = l_0^{in}\left[1 - \frac{t}{t_c}\right]^{1/4}$$

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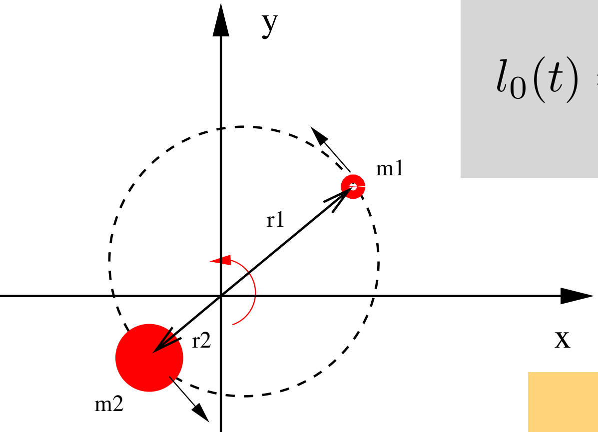


$$l_0 = r_1 + r_2$$

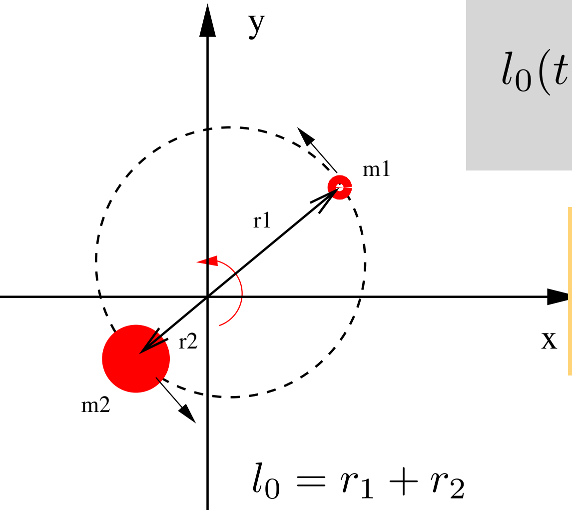
$$\omega_K = \sqrt{\frac{GM}{l_0^3}} \quad \longrightarrow \quad \omega_K(t) = \omega_K^{in}\left[1 - \frac{t}{t_c}\right]^{-3/8}, \qquad \omega_K^{in} = \sqrt{\frac{GM}{(l_0^{in})^3}}$$

the orbital frequency increases in time

and consequently, the orbital period $T_K = \frac{2\pi}{\omega_K}$ decreases

First indirect proof of the existence of gravitational waves:
Nobel Prize in 1993

J.M.Weisberg, J.H.Taylor
Relativistic Binary Pulsar PSR1913+16:Thirty Years of Observation in Binary Radio Pulsars ASP Conference series, 2005
eds. F.A.A. Rasio, I.H.Stairs



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$l_0^{in} =$ orbital distance at t=0

$$\omega_K = \sqrt{\frac{GM}{l_0^3}} \quad \rightarrow \quad \omega_K(t) = \omega_K^{in}\left[1 - \frac{t}{t_c}\right]^{-3/8}, \qquad \omega_K^{in} = \sqrt{\frac{GM}{(l_0^{in})^3}}$$

The wave amplitude increase as well

$$h_0(t) = \frac{4\pi^{2/3}\ G^{5/3}\ \mathcal{M}^{5/3}}{c^4}\ \nu_{GW}^{2/3}(t)$$

$$\nu_{GW}(t) = \frac{2\omega_K}{2\pi} = \frac{1}{\pi}\sqrt{\frac{GM}{l_0^3(t)}}$$

gravitational wave frequency

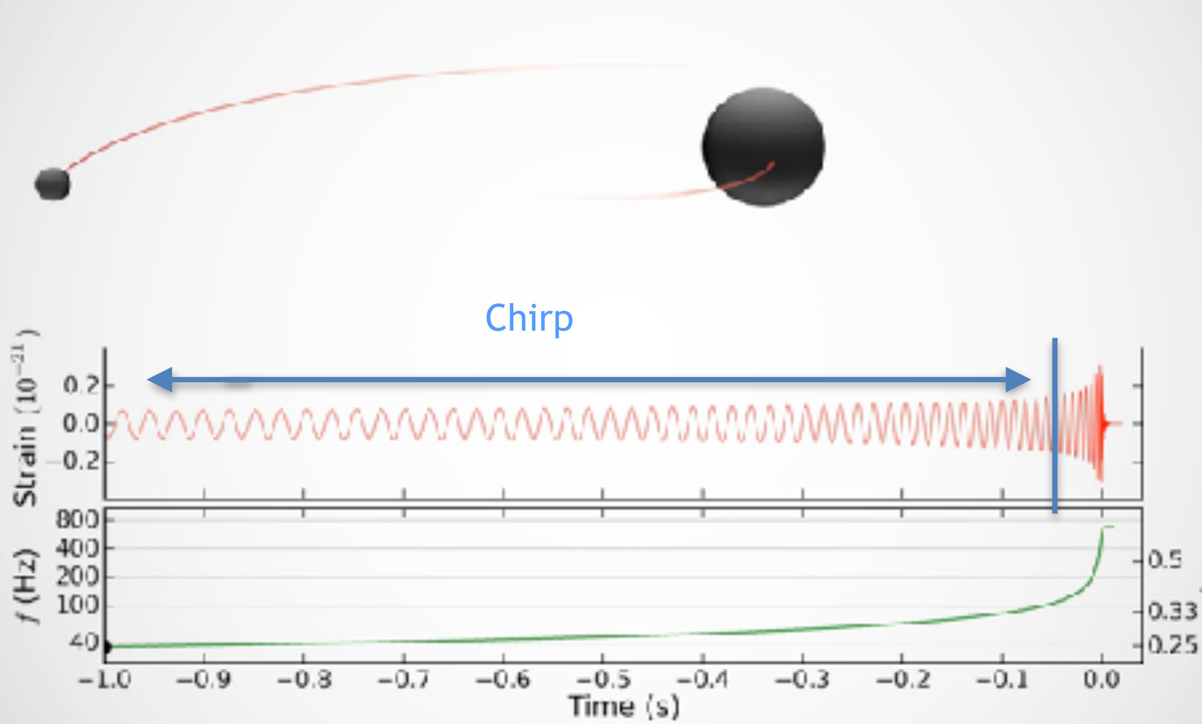
$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$ chirp mass

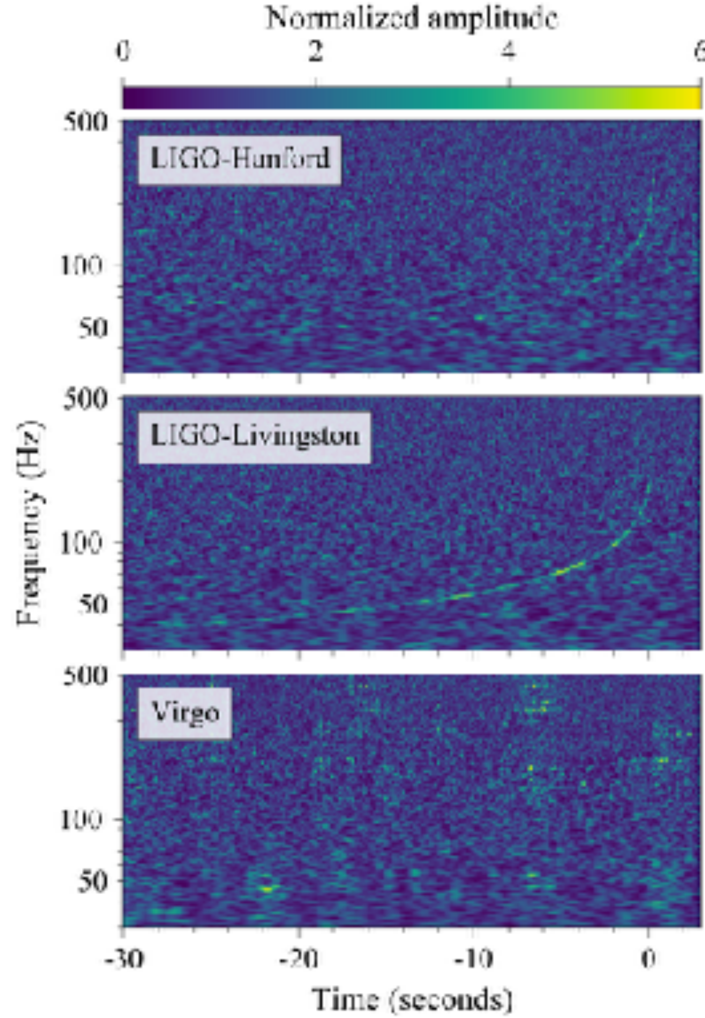
$$h_{ij}(t,r) = -\frac{h_0}{r}\ A_{ij}(t - \frac{r}{c})$$

$$A_{ij}(t) = \begin{pmatrix} \cos\ 2\omega_K t & \sin\ 2\omega_K t & 0 \\ \sin\ 2\omega_K t & -\cos\ 2\omega_K t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h_0 = \frac{4\ \mu\ M\ G^2}{l_0\ c^4}$$

instantaneous wave amplitude

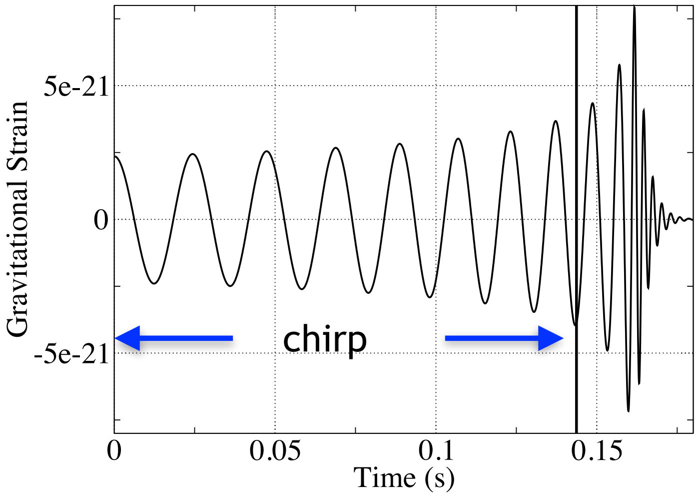




$$\nu_{GW}(t) = \frac{2\omega_K}{2\pi} = \frac{1}{\pi}\sqrt{\frac{GM}{l_0^3(t)}}$$

$$h_0(t) = \frac{4\pi^{2/3}\ G^{5/3}\ \mathcal{M}^{5/3}}{c^4}\ \nu_{GW}^{2/3}(t)$$

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$ chirp mass



wave frequency

$$\nu_{GW}(t) = \frac{1}{\pi}\sqrt{\frac{GM}{l_0^3(t)}} = \frac{5^{3/8}}{8\pi} \left(\frac{c^3}{G\mathcal{M}}\right)^{5/8} \left[\frac{1}{t_c - t}\right]^{3/8}$$

chirp mass

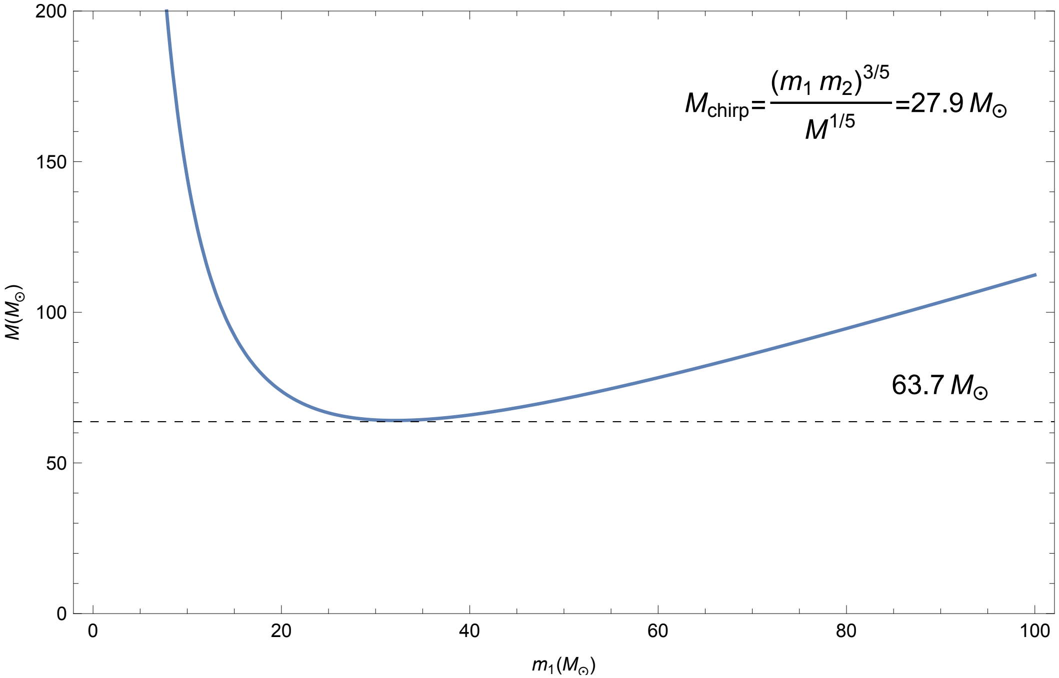
$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} = \frac{c^3}{G}\left[\frac{5}{96}\pi^{-8/3}\ \nu^{-11/3}\ \dot{\nu}\right]^{3/5}$$

measuring the wave frequency and its time derivative, we measure the chirp mass

What do we infer from the chirp mass?

GW150914

$M=m_1+m_2$



$$\mathcal{M} \simeq 28 M_\odot \rightarrow (m_1+m_2) \gtrsim 63.7 M_\odot$$

Too large to be two neutron stars

$$\nu_{GW}(t) = \frac{1}{\pi}\sqrt{\frac{GM}{l_0^3(t)}}$$

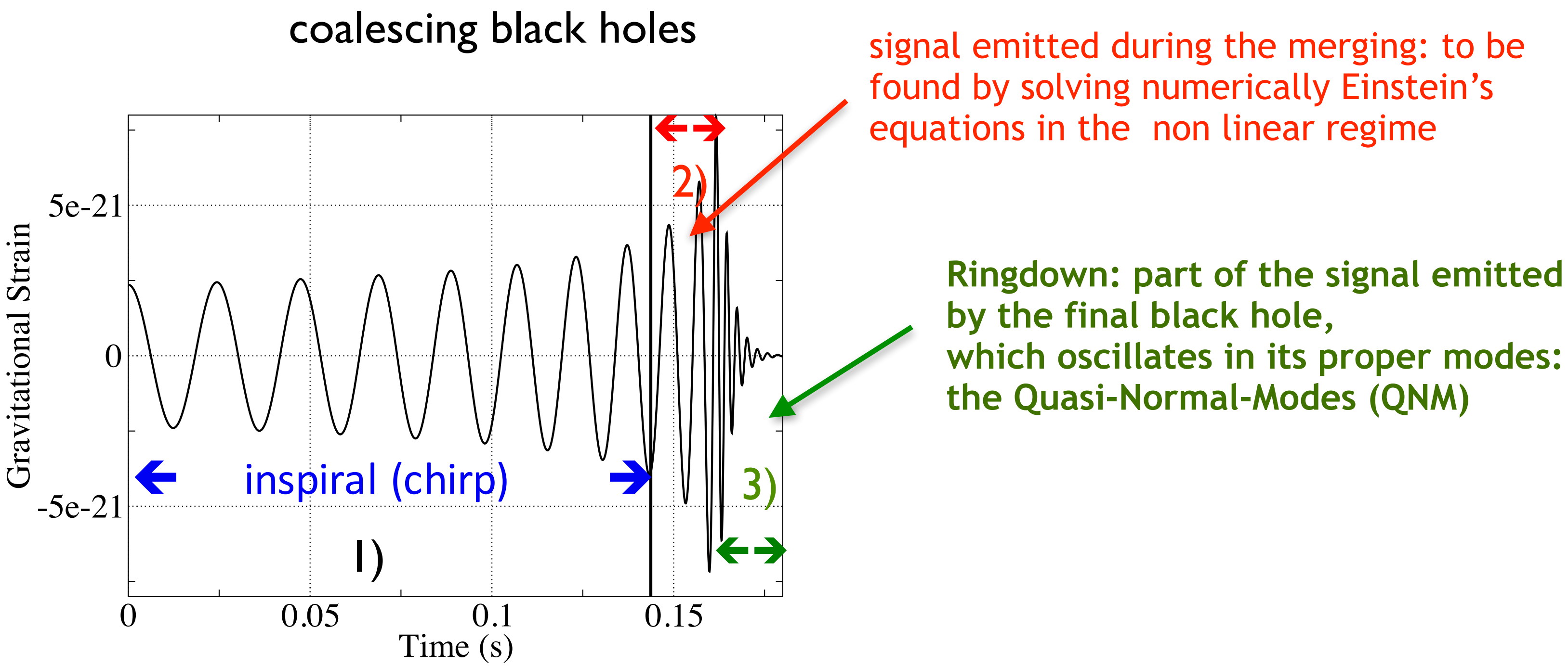
$M \gtrsim 63.7\ M_\odot$

over 0.2 s the wave frequency increases from 35 to 150 Hz, from which we infer that, just before merging, the distance bewteen the two masses was

$$d_{orb}(150\ Hz) \simeq 339\ km$$

The two objects must be extremely compact!

Are they Black Holes?



To identify the source we need:

1) improve the description of the inspiralling part of the signal near merging
2) compute the signal emitted during the merging and match it with the inspiralling part
3) compute the ringing tail and match it with the merging part of the signal

1) Modelling the inspiral: Post-Newtonian expansion beyond the quadrupole approximation

Systems with (relatively) weak gravitational fields & low velocities: dynamics of GR expressed as Newton's laws + corrections, using quantities and concepts of Newtonian physics!

if $g^{\alpha\beta}$ is a solution of Einstein's eqs. $G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}$ then the tensor

$$h^{\alpha\beta} = \eta^{\alpha\beta} - \sqrt{-g} g^{\alpha\beta}$$

satisfies the equations

$$\Box_F h^{\alpha\beta} = -\frac{16\pi G}{c^4} \tau^{\alpha\beta}(T, h)$$

$\tau^{\alpha\beta}$ is an effective energy-momentum pseudo-tensor

WARNING: $h^{\alpha\beta}$ is not a perturbation!

we expand the solutions as

$$h^{\alpha\beta} = \sum_n \epsilon^n h_n^{\alpha\beta}$$

Expansion parameter: $\varepsilon \sim \frac{v}{c} \sim \sqrt{\frac{GM}{rc^2}}$

and find the expansion coefficients iteratively

$$h_0^{\alpha\beta} = 0, \quad \Box_F h_n^{\alpha\beta} = -\frac{16\pi G}{c^4} \tau^{\alpha\beta}[T, h_{n-1}]$$

several mathematical subtleties: different expansions in near zone and wave zone to be matched, regularization procedures (some approaches use techniques similar to field theory), etc….

if we use this approach, compute the waveform for the inspiralling going beyond the quadrupole approximation, and take the Fourier transform of the signal

$$h(f) = \mathcal{A}(f) e^{i\psi(f)} \qquad \psi(f) = \psi_{PP} + \psi_{\bar{Q}} + \psi_{\bar{\lambda}}$$

$x = (v/c)^2 \to \frac{1}{c^2}(Gm\pi f)^{2/3}$

PN expansion parameter

point-particle contribution

$$\psi_{PP}(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3}{128}(\mathcal{M}\pi f)^{-5/3}\left\{1 + \left(\frac{3715}{756} + \frac{55}{9}\eta\right)x - (16\pi - 4\beta)x^{3/2} + \left(\frac{15293365}{508032} + \frac{27145}{504}\eta + \frac{3085}{72}\eta^2 - 10\sigma\right)x^2 + \mathcal{O}(x^{5/2})\right\}$$

☐ contains information on the mass ratio of the two coalescing bodies: combining this with the measured chirp mass, the individual masses can be resolved

$m = m_1 + m_2$

$\eta = m_1 m_2 / m^2$

☐ $$\beta = \frac{1}{(m_1+m_2)^2}\frac{\mathbf{L}}{|L|}\left[\left(\frac{113}{12} + \frac{25}{4}\frac{m_2}{m_1}\right)\mathbf{S_1} + \left(\frac{113}{12} + \frac{25}{4}\frac{m_1}{m_2}\right)\mathbf{S_2}\right]$$

$\mathbf{L}$ = orbital angular momentum

$\mathbf{S_1}, \mathbf{S_2}$ individual spin angular momenta

σ contains spin-spin and spin-orbit terms. Note that it appears in the 2-PN term (x^2)

The quantity which is actually measured is

$$\chi_{eff} = \frac{c}{G}\left(\frac{\mathbf{S_1}}{m_1} + \frac{\mathbf{S_2}}{m_2}\right)\cdot\frac{\mathbf{L}}{M}$$

which shows the degree of alignments of the individual spins with the orbital angular momentum (0^o=aligned, 180^o antialgned) if we use this approach, compute the waveform for the inspiralling going beyond the quadrupole approximation, and take the Fourier transform of the signal

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Quadrupole induced by rotation

$$\psi_{\bar{Q}} = \frac{3}{128}(\mathcal{M}\pi f)^{-5/3}\left\{-50\left[\left(\frac{m_1^2}{m^2}\chi_1^2 + \frac{m_2^2}{m^2}\chi_2^2\right)(Q_S - 1) + \left(\frac{m_1^2}{m^2}\chi_1^2 - \frac{m_2^2}{m^2}\chi_2^2\right)Q_a\right]x^2\right\}$$

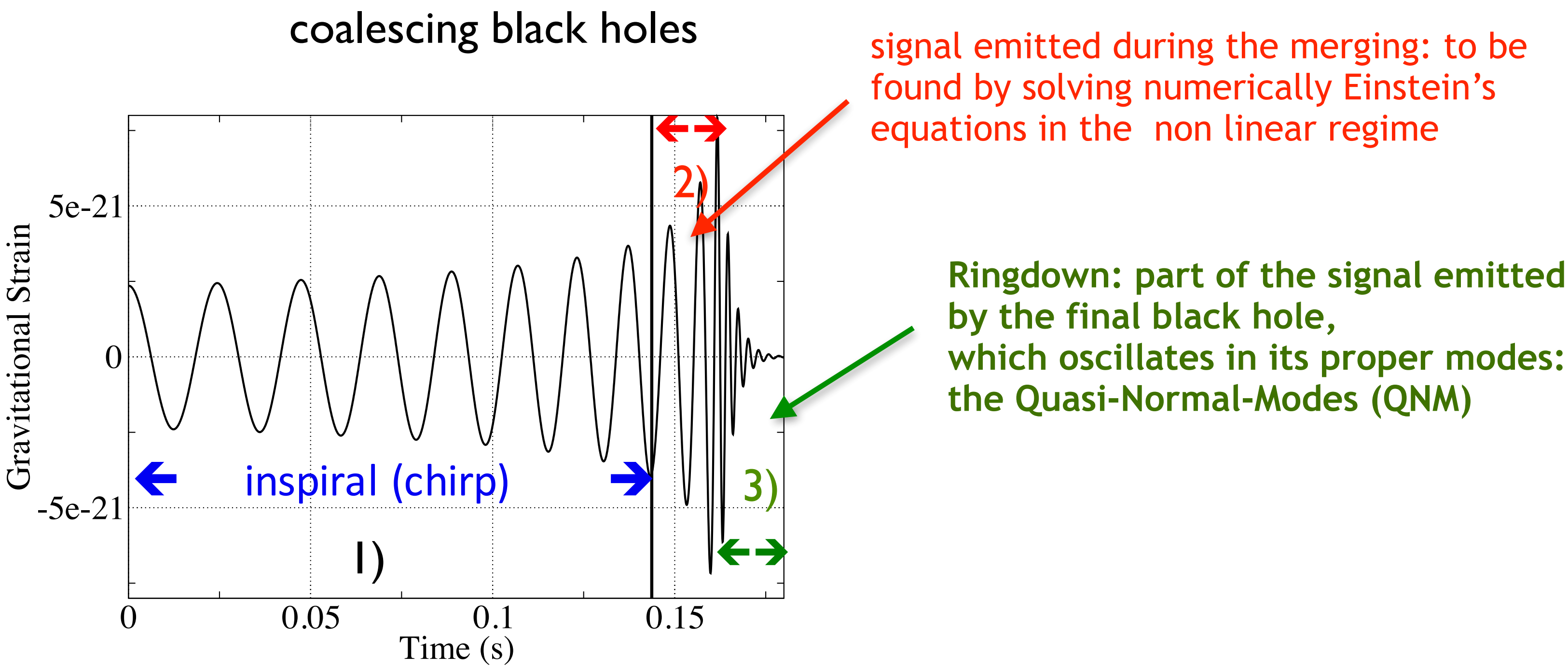
$$Q_S = \frac{\bar{Q}_1 + \bar{Q}_2}{2}, \quad Q_a = \frac{\bar{Q}_1 - \bar{Q}_2}{2}$$

Tidal contribution:

$$\psi_{\bar{\lambda}} = -\frac{3}{128}(\mathcal{M}\pi f)^{-5/3}\left\{24[(1 + 7\eta - 31\eta^2)\lambda_S + (1 + 9\eta - 11\eta^2)\lambda_a \delta m]x^5 + \right\} + \mathcal{O}(x^6)$$

$$\lambda_S = \frac{\bar{\lambda}_1 + \bar{\lambda}_2}{2}, \quad \lambda_a = \frac{\bar{\lambda}_1 - \bar{\lambda}_2}{2} \qquad \delta m = \frac{m_1 - m_2}{m}$$

Tidal contributions become relevant when the NS velocities are high, i.e. before merging



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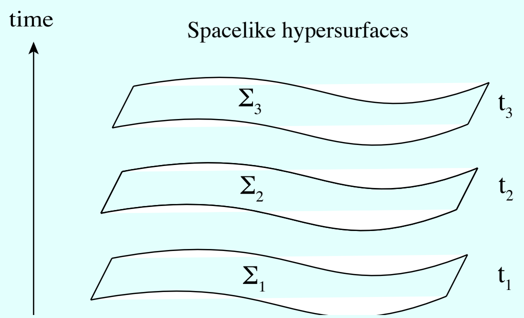
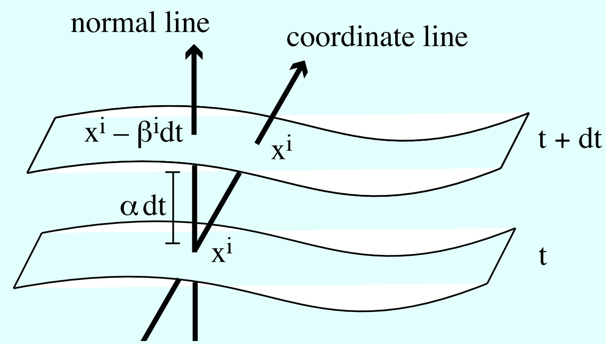
Modelling the merger: Numerical Relativity

When no approximation scheme can be applied (e.g. binary plunge & merger, supernova explosion, etc.)
Einstein's equations have to be solved in their full non-linear form.
This requires heavy computational resources (parallel computing) but also refined mathematics!

In GR there is no global time coordinate, but we need an *evolution* for numerical impementation: give up general covariance!

Initial value formulation of Einstein's equations: 3+1 split of spacetime





$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j$$

where α : lapse function; β^i : shift vector ; γ_{ij} : spatial metric ; n_α : normal 1-form to hypersurface

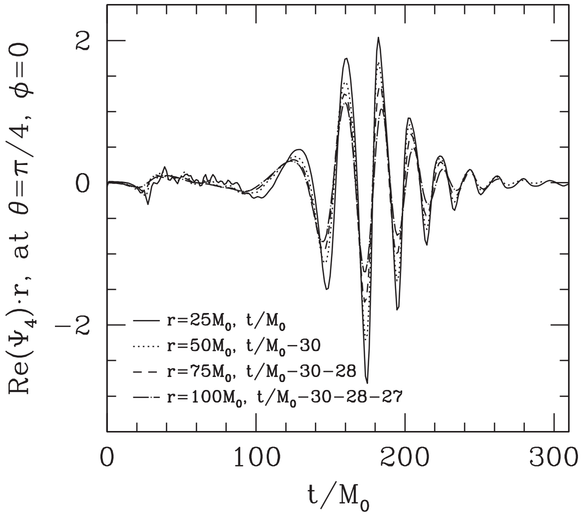
Using the tools of differential geometry, Einstein's equations decompose in elliptic equations (consistency conditions on a given t=const. hypersurface), and hyperbolic equations for the time evolution of the spacetime metric.
Large numerical facilities are needed to solve these equations

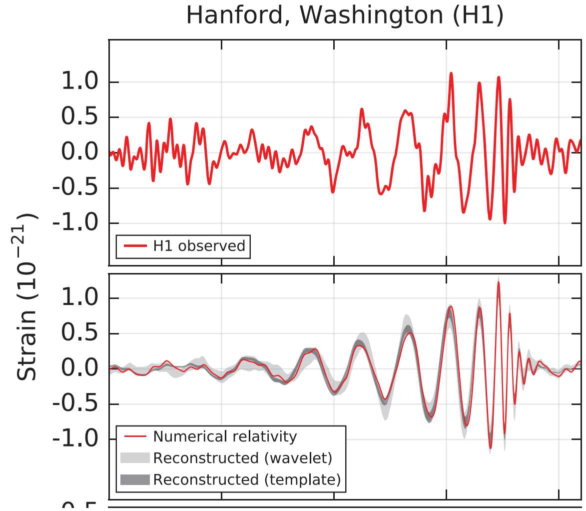
Numerical studies of BH-BH coalescence started in the late 1990s with the Grand Challenge project. Many problems had to be solved:

- excision of singularity (how to teach the machine that there is no spacetime there?)
- different scales of the problems (multi-grid mesh refinement)
- gauge choice is a long-standing problem
- great care needed to avoid numerical errors to grow
- etc….

Breakthrough:

F. Pretorius Phys.Rev.Lett. 95 (2005) 121101





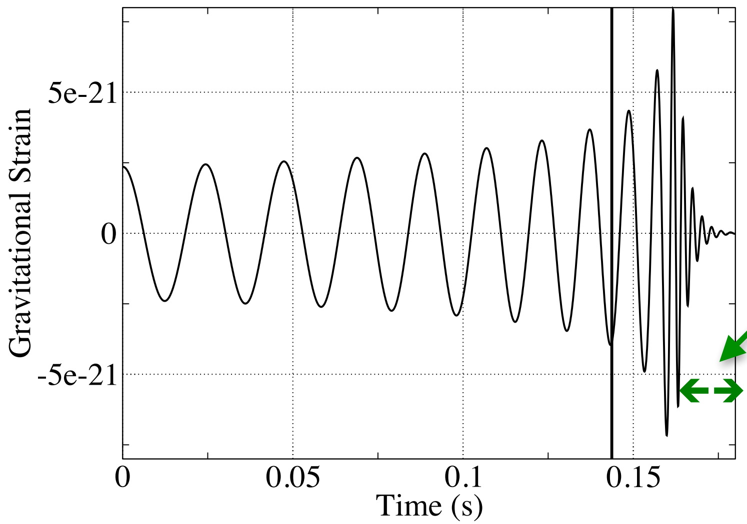
LIGO first detection

After decades of numerical studies on BH coalescence, a bank of templates has been set up

Fitting formulae based on numerical simulations of BH merging have been found which, compared to the meging part of the detected signal, allow to estimate:

individual masses ans spins

mass and angular momentum of the final black hole



Ringdown: part of the signal emitted by the final black hole, which oscillates in its proper modes: the Quasi-Normal-Modes (QNM)

How do we compute the frequencies of the Quasi-Normal Modes of stars and black holes?
Again using perturbation theory $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$ $\qquad |h_{\mu\nu}| << |g^0_{\mu\nu}|$

if $g^0_{\mu\nu} = \eta_{\mu\nu}$

Einstein's eqs. reduce to

$$\Box_F \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \quad \rightarrow \quad \left[\nabla^2 + \omega^2\right] \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

if $g^0_{\mu\nu}$ is a solution describing a black hole (rotating or non rotating), or a non rotating star, Einstein's equations can be reduced to a Schroedinger-like equation for the radial part of appropriately defined perturbation functions

$$\frac{d\Psi_{\ell m}}{dr_*^2} + \left[\omega^2 - V_{\ell m}(r_*)\right] \Psi_{\ell m} = 0$$

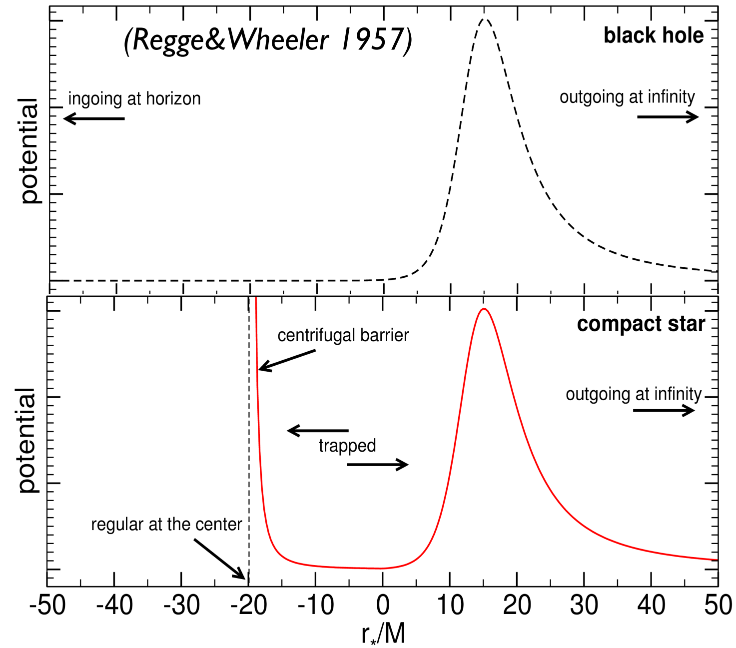
A Schroedinger-like equation for BH perturbations

$$\frac{d\Psi_{\ell m}}{dr_*^2} + \left[\omega^2 - V_{\ell m}(r_*)\right]\Psi_{\ell m} = 0$$

$$r_* = r + 2M\log\left(\frac{r}{2M} - 1\right)$$

non rotating BH: the potential depends only on the BH mass
rotating BH: the potential is complex and depends also on the angular momentum and on the frequency

non rotating star: the potential depends on how the energy density and the pressure are distributed inside the star, i.e. it depends on the equation of state of matter



(Chandrasekhar & Ferrari, 1991)

The spacetime curvature act as a potential

The quasi-normal modes are complex frequency solutions of the wave equation, such that

for BHs

$\Psi_{\ell m} \sim e^{i\omega r_*} \quad r_* \to \infty$ pure outgoing wave at infinity

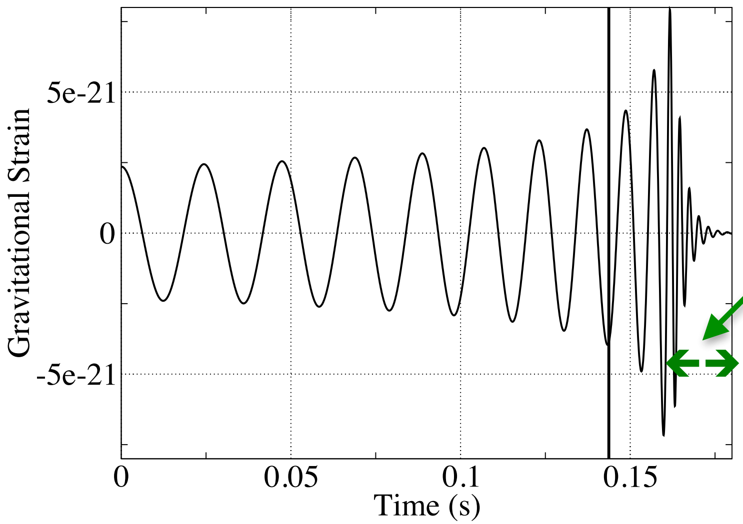
$\Psi_{\ell m} \sim e^{-i\omega r_*} \quad r_* \to -\infty$ pure ingoing wave at the horizon

for stars

$\Psi_{\ell m} \sim e^{i\omega r_*} \quad r_* \to \infty$ pure outgoing wave at infinity

$\Psi_{\ell m}$ regular at $r = 0$

more details in Paolo & Leonardo's talks



Ringdown: part of the signal emitted by the final black hole, which oscillates in its proper modes: the Quasi-Normal-Modes (QNM)

the ringdown is a superposition of damped sinusoids at the frequencies and with the damping times of the QNMs

In General Relativity the QNM frequencies depends only on the black hole mass and the angular momentum **(no hair theorem)**

$$\mathbf{M} = \mathbf{n}\mathbf{M}_\odot \qquad \nu_0 \sim (\mathbf{12}/\mathbf{n})\mathbf{kHz} \qquad \tau \sim \mathbf{n} \cdot \mathbf{5.5} \times \mathbf{10}^{-5}\ \mathbf{s}$$

frequency increases up to 30% if the BH rotates

for $\mathbf{M} = \mathbf{60}\ \mathbf{M}_\odot \qquad \nu_0 = \mathbf{200\ Hz}, \qquad \tau_0 = \mathbf{3.3\ ms}$

in the non rotating case

The frequency of the lowest quasi-normal mode has been extracted from the detected ringdown of the firts event GW150914. The black hole **mass and angular momentum** agree with the values found from the merging

To find the waveform of the gravitational wave signal emitted in the coalescence of two black holes is a very very complex problem

1) the inspiral: we can apply approximation schemes (PN formalism), but we need many terms in the expansion to extract physical information

2) the merging: Einstein's equations have to be integrated in their full non-linear complexity

3) ringing tail : the frequencies of oscillation of the final black hole are found using perturbation theory

4) the three part of the signal must be matched, and this is quite difficult: when does the inspiral end and the merging start? When the ringing tail sets in?

If the merging bodies are neutron stars 2) and 3) have to include the dinamics of matter and its coupling with the gravitational field