S-matrix approach to 2 and 3 hadron interactions

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Outline

• Opportunities and challenges in hadron spectroscopy

• Amplitudes from S-matrix analysis: 2-to-2 scattering 1-to-3 decays: how virtual exchanges become real

• Three particle scattering: the framework
How Hadrons Emerge from QCD

- Experimental or lattice signatures (real axis data: cross section bumps and dips, energy levels)
- Theoretical signatures (complex plane singularities: poles, cusps)
- What is the interpretation (constituent quarks, molecules, ...) ?

Reaction amplitudes
How Hadrons Emerge from QCD

• Experimental or lattice signatures (real axis data: cross section bumps and dips, energy levels)

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Reaction amplitudes
3pion challenge

Evolution in statistics: \( \pi^- p \rightarrow \pi^- \pi^+ \pi^- p \)

- **CERN ca. 1970**
  - \( O(10^2 / 10 \text{MeV}) \)

- **BNL (E852) ca 1995**
  - \( O(10^3 / 10 \text{MeV}) \)

- **COMPASS 2010**
  - \( O(10^5 / 10 \text{MeV}) \)
  - \( O(10^6 / 10 \text{MeV}) \)

Comparable statistics expected from JLab

- **96M events**
Signatures of new, unusual light resonances

- High precision PWA of 3π diffractive association yields a new $a_1(1460)$ incompatible with the quark model/Regge expectations.

- At low-t exotic wave production compatible with one pion exchange

- In photoproduction exotic mesons be produced via pion exchange

- Large exotic wave seen in $\eta(\pi)\pi$ production: FESR’s to constrain P-wave
Signatures of unusual heavy quark resonances


Virtual OPE

Real OPE

BESIII, PRL 112, 022001

EMARK ON ENERGY PEAKS IN MESON SYSTEMS
M. Nauenberg, A. Pais

If the width of particle $X$ is not very large we will stay close to the physical region. This almost singular behavior of $A(s)$ for certain physical $s$ causes the peaking effect to which we refer as an $(X,Y,Z)$ peak.
Amplitude signatures

New particles in the QCD spectrum

\[ \bar{D} D^* \rightarrow \bar{D} D^* \]

\[ X(3872) \]

Threshold

The “interesting stuff” happens on unphysical sheets. When singularities are close to the physical region rapid variations in amplitudes (cross sections) appear.

Other effects can be "generated" exchanges forces" (*)

("?) singularities because of confinement
Anatomy of resonances

When strength of interactions is reduced, bound states become resonances.

The only place for bound states pole to migrate is onto an unphysical sheet connected to the open channel branch point.

Violates Causality

Infinite amplitude (violates unitarity)
Properties of reaction amplitudes are determined by

**Causality**: Reaction amplitudes are smooth (analytical) functions of kinematical variables with singularities reflecting existence of constraints (laws)

**Unitarity**: Determines singularities.

**Crossing**: Dynamical relations: reaction amplitudes in the exchange channel (forces) are analogous to amplitude in the direct channel (resonance)
Old:
S matrix constraints -> “equations” (the Bootstrap program). Misses the specific “microscopics” e.g. Regge slope, CDD poles. Non unique solutions —> resonances appear not to be “self-generated”.

New:
Use the S-matrix principles to test QCD phenomenology.
2-to-2 scattering

\[ A(s, t) = \sum_l (2l + 1) a_l(s) P_l(z_s) \]

Partial waves test resonance hypothesis.

- Dispersion relation (Analyticity)

\[ a_l(s) = B_l(s) + \frac{1}{\pi} \int_{tr} ds' \frac{Ima_l(s')}{s' - s} \]

- Unitarity ...

|a(s' + i\epsilon)|^2 \rho(s') \]

...implies a relation between \( \text{Re } a_l \) and \( \text{Im } a_l \)

- \( B_l \) from cross channel interactions. Given \( B_l \) (e.g. for elastic \( \pi\pi \) scattering it can be related to \( a_l(s) \)'s. \text{Unitarity} “converts” a dispersion relation to an integral equation (eg Roy eq.) for the partial waves). \text{Solution} is non unique — there are resonances/bound states not constrained by “exchange forces”
Beyond 2 particle production Khuri-Treiman (KT) model

\[ A(s,t,M^2) = a(t,M^2) + a(s,M^2) + a(u,M^2) \]

- Isobars to be determined in terms of 2-to-2 scattering amplitude and a coupling, \( g \) to the production channel (\( g=1 \))

- Unitarity: t-channel process acts as a "driving" term for s-channel isobar

- Isobars are not partial waves. They are model amplitudes with direct channels branch points only

Or in decay channel

\[ \text{Diagrams of decay processes} \]

\[ f(s) \quad \text{and} \quad g f(s) \]
KT equations

Isobar amplitudes have only the normal threshold singularities

\[ a(s, M^2) = f(s) + f(s) \int_{tr} \frac{ds'}{\pi} \frac{\rho(s')b(s', M^2)}{s' - s} \]

\[ b(s, M^2) = \int_{-1}^{1} dz a(t(s, z, M^2), M^2) \]
Isobar amplitudes have only the normal threshold singularities

\[
a(s, M^2) = f(s) + f(s) \int_{\text{tr}} \frac{ds'}{\pi} \frac{\rho(s')b(s', M^2)}{s' - s}
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b(s, M^2) = \int_{-1}^{1} dz a(t(s, z, M^2), M^2)
\]
Pentaquark as a triangle singularity?
Have cusps been seen

DIRECT DETERMINATION OF A SHORT NUCLEAR LIFETIME ($\approx 10^{-20}$ s) 
BY THE PROXIMITY SCATTERING METHOD

J. LANG, R. MÜLLER, W. WÖLFLI, R. BÖSCH and P. MARMIER
Laboratorium für Kernphysik, Eidg. Techn. Hochschule, Zürich

Received 4 February 1966

$$b + t \rightarrow 1 + 2 + 3 \quad [d + ^{12}C \rightarrow n + p + ^{12}C],$$

$$t = 7 \cdot 10^{-21} \text{s}$$

$$E_d = 5.39 \text{ MeV}$$
$$\theta_r = \theta_p = 90^\circ$$
$$\phi_{rp} = 10^\circ$$

$$b + t \rightarrow 1 + R, \quad \text{with } Q\text{-value } Q_1 \quad [d + ^{12}C \rightarrow n + ^{13}\text{N}^*, \quad Q_1 = -3.82 \text{ MeV}],$$

$$R \rightarrow 2 + 3 \quad \text{with } Q\text{-value } Q_2 \quad [^{13}\text{N}^* \rightarrow p + ^{12}C, \quad Q_2 = 1.59 \text{ MeV}],$$
Amplitude analysis for $Z_c(3900)$

One can test different parametrizations of the amplitude, which correspond to different singularities $\rightarrow$ different natures

Triangle rescattering, logarithmic branching point

Triangle rescattering, logarithmic branching point

(anti)bound state, II/IV sheet pole («molecule»)

Resonance, III sheet pole («compact state»)

Szczepaniak, PLB747, 410-416
Szczepaniak, PLB757, 61-64
Guo et al. PRD92, 071502

Tornqvist, Z.Phys. C61, 525
Swanson, Phys.Rept. 429
Hanhart et al. PRL111, 132003

Maiani et al., PRD71, 014028
Facci et al., PRD87, 111102
Esposito et al., Phys.Rept. 668

A. Pilloni
Fit: III
Fit: III
Two models for cusps

(2nd sheet singularities near physical region)

\[ A_l(s) = \int d\zeta_s A(s, t(s, \zeta_s), u(s, \zeta_s)) P_l(\cos \theta) \]

\[ t = -\frac{(s - 4m^2)}{2}(1 - \zeta_s) \]

\[ A_0(s) \sim \int_{-1}^{1} d\zeta_s \frac{1}{m_e^2 + \frac{(s-4m^2)}{2}(1 - \zeta_s)} \]

\[ \propto (m_e^2 - t(s, \zeta_s))^{-1} \]

\[ t/u \text{ channel exchange close to the s-channel physical region} \]

(Leads to a cusp in the s-channel)
A cusp from a vertex

\[
\int_{s_0}^{s_1} ds' \sqrt{1 - \frac{4 N(s')}{s' s' - s}}
\]

\[N = \frac{1}{s} \text{ vs exp}(-s)\]

A big difference
Scattering through resonances

\[ f(s) = \frac{1}{K^{-1}(s) - i\Gamma(s)} \]

\( \infty \) number of poles = confinement

\( K^{-1}(s) \) needs to have \( \infty \) number of poles (and zeros)

Quadratically spaced radial trajectories

\[ K(s) = \sum_{r=1}^{\infty} \frac{g_r^2}{m_r^2 - s} \rightarrow \sum_{r} \frac{1}{r^2 - s} \sim \frac{\cos(\pi \sqrt{s})}{\sin(\pi \sqrt{s})} \]

Linearly spaced radial trajectories (Veneziano)

\[ K(s) \sim \frac{\Gamma(a-s)}{\Gamma(b-s)} \quad \text{Exponential form factors related to infinite number of particles (confinement)!} \]
3-to-3 amplitude from a 2-to-2 (1-to-3) KT model

\[ T(\sigma', M^2, \sigma) = B(\sigma', M^2, \sigma) \]

\[ + \int d\sigma'' B(\sigma', M^2, \sigma'') \tau(\sigma'', M^2) T(\sigma'', M^2, \sigma) \]

(Aitchison 1965)
Relation to other approaches

\[
T(q, p; s) = B(q, p; s) - \int \frac{\alpha t}{(2\pi)^3} B(q, l; s) \frac{\tau(\sigma(l))}{2E(l)} T(l, p; s)
\]

\[
T_{3\rightarrow 3}(\sigma', s, \sigma) = \sum_{\sigma''} \left[ \frac{1}{1 - \tau(s) B(s)} \right]_{\sigma', \sigma''} \big[ B(s) \big]_{\sigma'', \sigma}
\]

- M. Mai et al. (JPAC)
- I. Aitchison (Khuri-Treiman)
- H. Hammer et al. (EFT)

“\( \varrho \) vs Chew-Madelstam” phase space (properly removes unphysical singularities from \( \varrho \))

- Reproduces 3-to-3 unitarity on the real axis only
- Analyticity in sub-channel variables?

Next step is to derive the proper dispersive representation

\[
T = \frac{1}{K^{-1}(s) - i\rho(s)}
\]

\[
T = \frac{1}{K^{-1}(s) - \frac{1}{\pi} \int_{tr} ds' \frac{\rho(s')}{s' - s}}
\]
3-to-3 scattering from dispersion relations

Special thanks to Andrew Jackura
Summary

• Existing 3 particle scattering amplitudes may have spurious singularities — analyticity not enforced.

• Proposed framework: unitarity + analyticity (only normal thresholds in particle-isobar amplitudes) → dispersion relations + short range inputs → N/D equations → amplitudes!