Dispersive Analysis of Mesonic 3-Body Decays

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Bound states in strongly coupled systems, Florence (Italy)

Outline

1. Introduction and Motivation
2. Dispersive analysis of $\eta' \to \eta\pi\pi$
3. Quark mass dependence of $\omega \to 3\pi$
4. Summary and Outlook

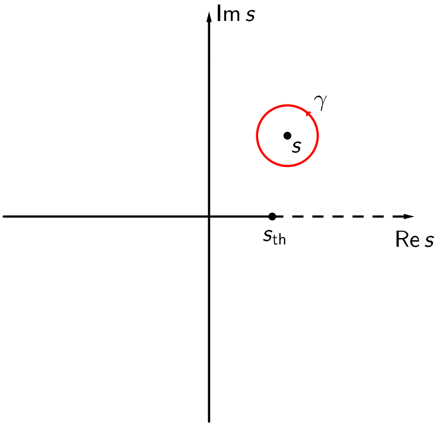
Advantages of dispersion relations

- based on fundamental properties of analyticity (causality), unitarity (probability conservation) and crossing
  $\Rightarrow$ model independence
- in contrast to effective field theories: dispersive methods describe the resummation of rescattering effects for considered particles
- Khuri-Treiman equations for 3-body decays: final state interaction (FSI) among all three decay products are fully taken into account

  [*Khuri and Treiman (1960)*]

Dispersion relation for a single-variable function

complex-valued function $f(s)$: analytic in the entire complex plane apart from a branch cut on the real axis for $s \geq s_{\text{th}}$



- Cauchy's theorem:

$$f(s) = \frac{1}{2\pi i} \oint_\gamma \frac{f(s')}{s' - s} ds'$$

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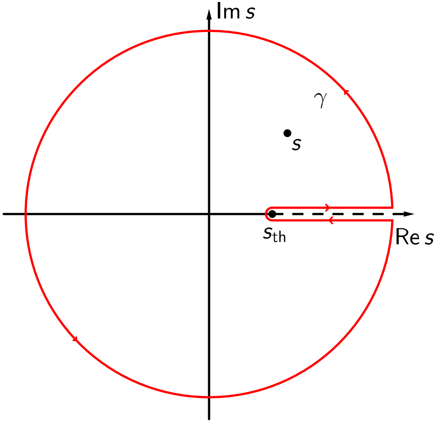

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- extend γ around the cut

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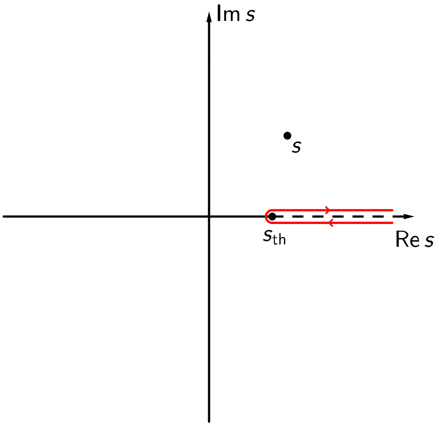

- Cauchy's theorem:

$$f(s) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(s')}{s' - s} \mathrm{d}s'$$

- extend γ around the cut
- $\lim_{|s|\to\infty} f(s) \to 0$: forces the integral along complex arc to vanish

$$f(s) = \frac{1}{2\pi \mathrm{i}} \int_{s_{\text{th}}}^{\infty} \frac{\mathrm{disc} f(s')}{s' - s} \mathrm{d}s'$$

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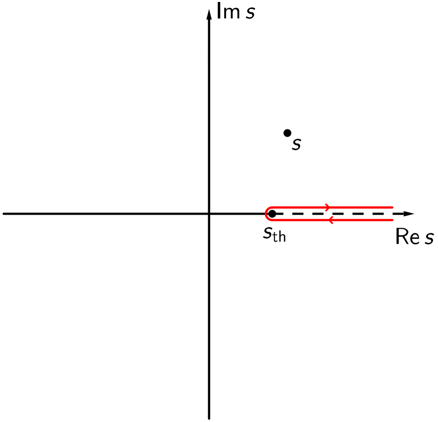

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less restrictive high energy behaviour of $f(s) \Rightarrow$ subtraction polynomial

Applications for dispersion relations in 3-body decays

analysis of $\eta \to 3\pi$: small phase space and far below any resonances, but FSI play already an important role

- problematic to describe with ChPT
- dispersive framework leads to a feasible description
- isospin-breaking process: extraction of the quark-mass ratio

[*Kambor et al. (1996), Anisovich and Leutwyler (1996), Kampf et al. (2011), Guo et al. (2015), Colangelo et al. (2017), Albaladejo and Moussallam (2017)*]

powerful tool to analyse decays outside the regime of validity for ChPT

- $\eta' \to \eta\pi\pi$ and $\eta' \to 3\pi$ decays [*this talk, in preparation*]
- $\omega/\phi \to 3\pi$ decays [*Niecknig, Kubis and Schneider (2012), Danilkin et al. (2014)*]
- heavy flavour decays like $D \to K\pi\pi$ [*Niecknig and Kubis (2015, 2017)*]

Part 1

Dispersive analysis of $\eta' \to \eta\pi\pi$

Eur. Phys. J. **C77**, 489 (2017) [arXiv:1705.04339 [hep-ph]]

in collaboration with B. Kubis, S. P. Schneider and P. Stoffer

Why of interest?

- main (hadronic) contribution to the total decay width
  BR($\eta' \to \eta\pi\pi$) = 0.652(11) [*PDG (2016)*]

- due to $U(1)_A$ anomaly of QCD η' is not a Goldstone boson:
  $\Rightarrow$ standard ChPT breaks down for processes involving an η'

- potentially clean access to constrain $\eta\pi$ scattering (energy far below $K\bar{K}$ inelastic threshold)

- experimental measurements of decay spectra available
  - $\eta' \to \eta\pi^+\pi^-$: BES-III (2011, 2017) & VES
  - $\eta' \to \eta\pi^0\pi^0$: recent data from A2 & BES-III (2017)

  [*VES (2007), BES-III (2011, 2017), A2 (2017)*]

- $\eta' \to \eta\pi^0\pi^0$ shows a cusp effect at $\pi^+\pi^-$-threshold

  [*Kubis and Schneider (2009)*]

Kinematics of the $\eta' \to \eta\pi\pi$ decay

transition amplitude:

$$\langle \pi^i(p_1)\pi^j(p_2)\eta(p_3)|T|\eta'(P)\rangle = (2\pi)^4\delta^{(4)}(P - p_1 - p_2 - p_3)\,\delta^{ij}\,\mathcal{A}(s,t,u)$$

Mandelstam variables:

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_2 + p_3)^2$$

(charged decay channel $\eta' \to \eta\pi^+\pi^-$ and neutral channel $\eta' \to \eta\pi^0\pi^0$ differ only by isospin breaking effects)

Analytic properties of the $\eta' \to \eta\pi\pi$ amplitude

- $\mathcal{A}(s,t,u)$ has a right-hand branch cut in the complex s-plane, starting at the $\pi\pi$-threshold
- similar situation in t- and u-planes, branch cuts starting at the $\eta\pi$-threshold
- left-hand cuts present due to crossing

Reconstruction theorem

[*Stern et al. (1993), Ananthanarayan et al. (2001), Zdráhal et al. (2008)*]

- $\mathcal{A}(s,t,u)$ can be decomposed into single-variable functions that possess just a right-hand cut

$$\mathcal{A}(s,t,u) = \mathcal{A}_0(s) + \mathcal{A}_1(t) + \mathcal{A}_1(u)$$

  $\mathcal{A}_0$ contains $\pi\pi$-FSI effects (I=0, S-wave)

  $\mathcal{A}_1$ contains $\eta\pi$-FSI effects (I=1, S-wave)

- neglect discontinuities of P- and higher partial waves:

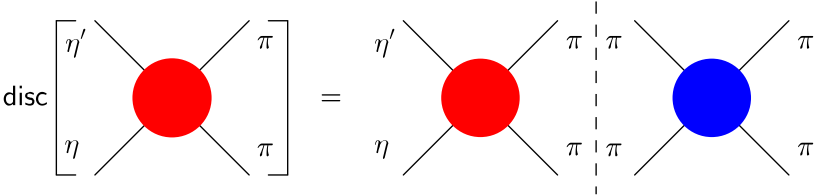
  $\pi\pi$ P-wave forbidden by C-Parity

  $\eta\pi$ P-wave has exotic quantum numbers

  D- and higher partial waves neglected due to small phase space

Unitarity condition

s-channel:



(analogous for t- & u-channel)

discontinuity equations for the single-variable functions:

$$\mathrm{disc}\mathcal{A}_0(s) = 2i\,\theta\big(s - 4M_\pi^2\big)\,\big[\mathcal{A}_0(s) \qquad\qquad\big]\, e^{-i\delta_0(s)}\,\sin\delta_0(s)$$

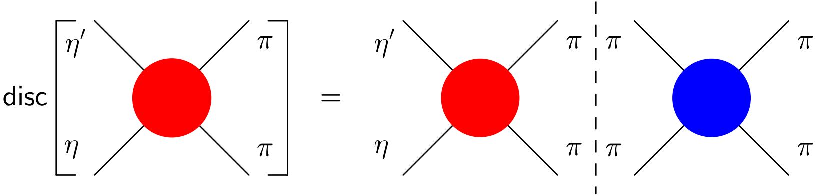
$$\mathrm{disc}\mathcal{A}_1(t) = 2i\,\theta\big(t - (M_\eta + M_\pi)^2\big)\,\big[\mathcal{A}_1(t) \qquad\qquad\big]\, e^{-i\delta_1(t)}\,\sin\delta_1(t)$$

$\Rightarrow$ Omnès problem (neglecting crossed-channel interactions)

$\delta_0(s)$, $\delta_1(t)$: S-wave $\pi\pi$ and $\eta\pi$ scattering phase shifts

Unitarity condition

s-channel:



(analogous for t- & u-channel)

discontinuity equations for the single-variable functions:

$$\mathrm{disc}\mathcal{A}_0(s) = 2\mathrm{i}\,\theta\big(s-4M_\pi^2\big)\,\big[\mathcal{A}_0(s)+\hat{\mathcal{A}}_0(s)\big]\,e^{-\mathrm{i}\delta_0(s)}\,\sin\delta_0(s)$$

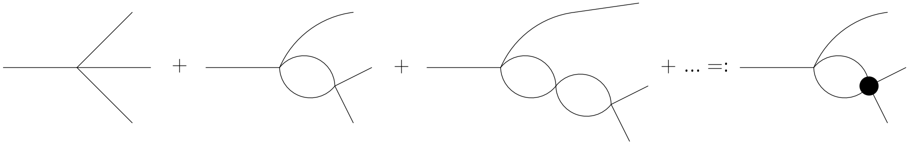
$$\mathrm{disc}\mathcal{A}_1(t) = 2\mathrm{i}\,\theta\big(t-(M_\eta+M_\pi)^2\big)\,\big[\mathcal{A}_1(t)+\hat{\mathcal{A}}_1(t)\big]\,e^{-\mathrm{i}\delta_1(t)}\,\sin\delta_1(t)$$

$\Rightarrow$ inhomogeneous Omnès problem

$\hat{\mathcal{A}}_I$: inhomogeneities, angular averages of crossed-channel $\mathcal{A}_I$ functions

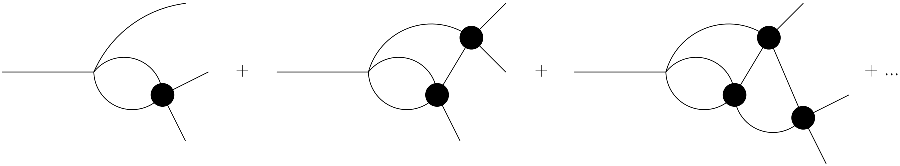
Physical interpretation

Omnès function:



⇒ iteration of two-particle bubble diagrams

inhomogeneities:



⇒ account for crossed-channel interactions

Khuri-Treiman equations in Omnès representation

dispersive representation for the functions $\mathcal{A}_I$ in Omnès form:

$$\mathcal{A}_0(s) = \Omega_0(s)\left\{\alpha + \beta s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\hat{\mathcal{A}}_0(s')\sin\delta_0(s')}{|\Omega_0(s')|(s'-s)}\right\}$$

$$\mathcal{A}_1(t) = \Omega_1(t)\left\{\gamma t + \frac{t^2}{\pi}\int_{(M_\eta+M_\pi)^2}^{\infty}\frac{dt'}{t'^2}\frac{\hat{\mathcal{A}}_1(s')\sin\delta_1(t')}{|\Omega_1(t')|(t'-t)}\right\}$$

Omnès function: [*Omnès (1958)*]

$$\Omega_I(s) = \exp\left\{\frac{s}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{ds'}{s}\frac{\delta_I(s')}{(s'-s)}\right\}$$

$\Rightarrow$ 3 (real) subtraction constants α, β, γ: free parameters in the dispersion relation, not fixed by unitarity

Intermediate summary

- set of coupled integral equations:

  $\Rightarrow \mathcal{A}_0(s)$, $\mathcal{A}_1(t)$: DR involving $\hat{\mathcal{A}}_0(s)$, $\hat{\mathcal{A}}_1(t)$

  $\Rightarrow \hat{\mathcal{A}}_0(s)$, $\hat{\mathcal{A}}_1(t)$: angular integrals over $\mathcal{A}_0(s)$, $\mathcal{A}_1(t)$

- input: $\pi\pi$- and $\eta\pi$-scattering phase shifts

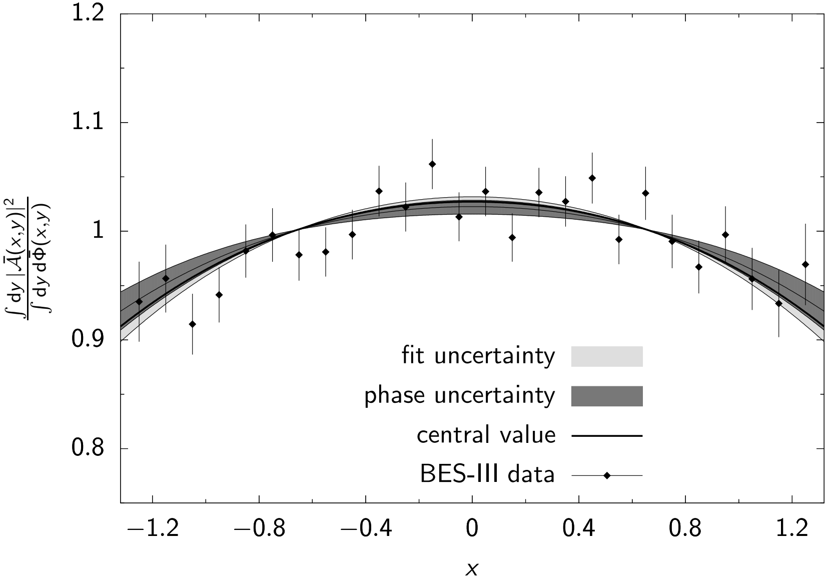
- problem linear in the 3 subtraction constants

  $\Rightarrow$ construct 3 basis solutions

- system solved numerically by iteration

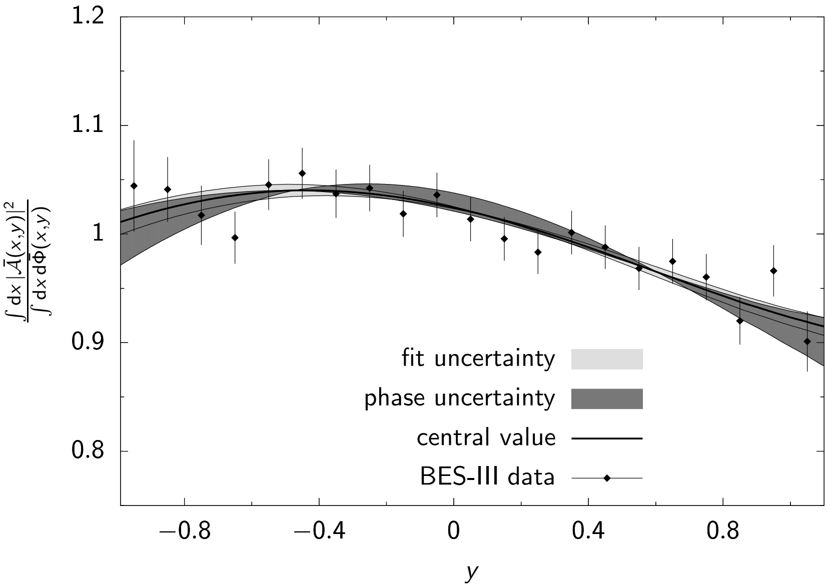
- subtraction constants up to now undetermined

Dalitz-plot x-projection



Dalitz-plot variables: $x \propto (t - u)$ and $y \propto -s$

Dalitz-plot y-projection



Isospin breaking effects in $\eta' \to \eta\pi^0\pi^0$: the $\pi^+\pi^-$ cusp

isospin breaking due to the π mass difference:

- correction for phase space is straightforward
- amplitude must have all thresholds at the right places

  $\Rightarrow$ difficult: $\pi\pi$-phase shifts derived in formalism relying on isospin symmetry

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constructing an effective $\pi^0\pi^0$-phase shift based on the neutral-pion scalar form factor $F_0(s)$ [*Colangelo et al. (2009)*]

correct analytic structure near the $\pi\pi$-thresholds:

- isospin breaking $\propto \sqrt{M_{\pi^+}^2 - M_{\pi^0}^2}$ (nonanalytic) retained
- isospin breaking $\mathcal{O}\big(M_{\pi^+}^2 - M_{\pi^0}^2\big)$ (analytic) neglected

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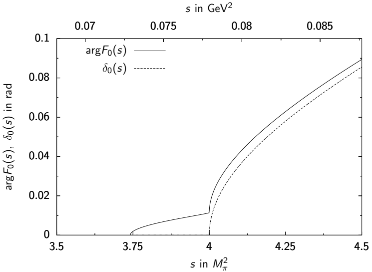

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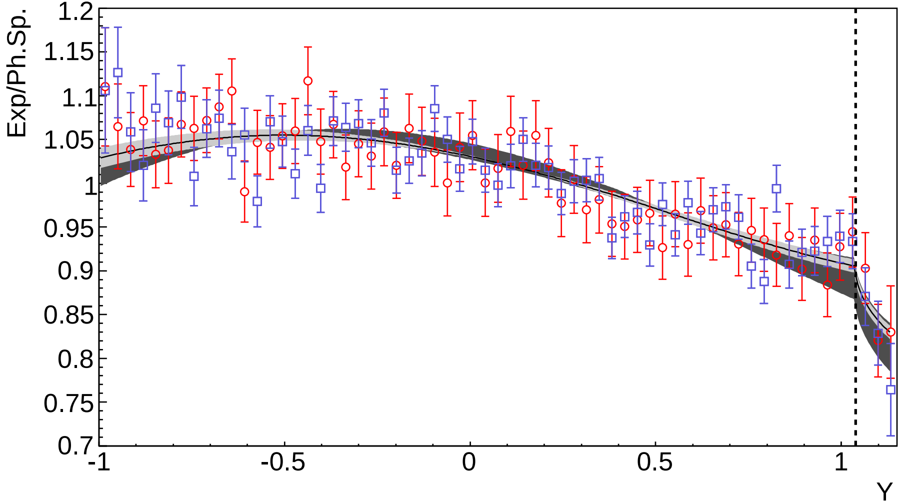
Prediction: Dalitz-plot y-projection for $\eta' \to \eta\pi^0\pi^0$



[*Figure from A2 (2017)*]

A2 data in good agreement with the dispersive representation
(α, β, γ taken from fit to BES-III (2011) data)

Part 2

Quark mass dependence of $\omega \to 3\pi$

in collaboration with M. Dax and B. Kubis

Why of interest?

- lattice QCD calculations are often carried out at unphysically high quark masses m_q $\Rightarrow$ tools for extrapolation needed
- ChPT allows us to study the m_q dependence in the low energy regime (relates $M_\pi^2 \propto \hat{m},\ \hat{m} = m_u = m_d$)
- unitarised ChPT: allows for a description of 2-body resonances (e.g. ρ in $\pi\pi \to \pi\pi$ scattering)
- 3-body resonances (e.g. $\omega \to 3\pi$) are more difficult

  $\Rightarrow$ dispersive approach
- construct dispersive representation for $\omega \to 3\pi$ and keep track of all M_π dependences

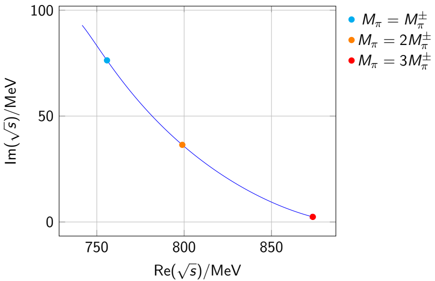
P-wave $\pi\pi \to \pi\pi$ scattering in unitarised ChPT

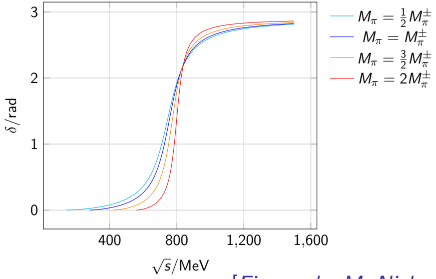
[*Truong (1991), Dobado and Peláez (1997), Peláez and Ríos (2010)*]

calculate the P-wave T-matrix for $\pi\pi \to \pi\pi$ in ChPT and use the inverse amplitude method (IAM) for unitarisation: yields an analytic structure that allows for resonance poles

$\Rightarrow$ ρ pole position and $\pi\pi$ phase shift as function of M_π

[*Hanhart, Peláez and Ríos (2008)*]





[*Figures by M. Niehus*]

effect has been confirmed in lattice simulations

[*Bolton, Briceño and Wilson (2016)*]

Decomposition of the $\omega \to 3\pi$ amplitude

[*Niecknig, Kubis and Schneider (2012)*]

amplitude $\omega(P,n) \to \pi(p_1)\,\pi(p_2)\,\pi(p_3)$ is of odd intrinsic parity:

$$\mathcal{A}(s,t,u) = \mathrm{i}\epsilon_{\mu\nu\alpha\beta}\, n^\mu\, p_1^\nu\, p_2^\alpha\, p_3^\beta\, \mathcal{F}(s,t,u)$$

Bose-symmetry allows for odd partial waves only in $\mathcal{F}(s,t,u)$

reconstruction theorem:

$$\mathcal{F}(s,t,u) = \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)$$

approximation: neglecting discontinuities from F- and higher partial waves

unitarity condition:

$$\mathrm{disc}\mathcal{F}(s) = 2\mathrm{i}\,\theta\big(s - 4M_\pi^2\big)\,[\mathcal{F}(s) + \hat{\mathcal{F}}(s)]\, e^{-\mathrm{i}\delta(s)} \sin\delta(s)$$

P-wave $\pi\pi$ scattering phase shift

M_π dependences of the amplitude

absolute value squared of the $\omega \to 3\pi$ amplitude

$$|\mathcal{A}(s,t,u)|^2 = \frac{1}{4}\left[stu - M_\pi^2(M_\omega^2 - M_\pi^2)^2\right] \times |\mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)|^2$$

kinematic factor: trivial M_π dependence and $u(s,t,M_\pi^2,M_\omega^2)$

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M_ω expected to depend on M_π: $M_\omega(M_\pi^2) \simeq M_\rho(M_\pi^2)$

[*Bijnens and Godzinsky (1996)*]

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[*Bijnens and Godzinsky (1996)*]

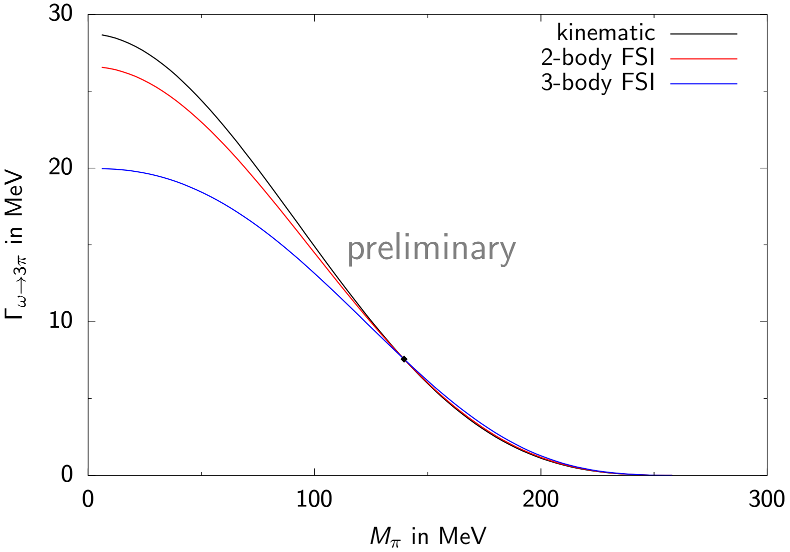
dynamic part: several non-trivial M_π dependences

$$\mathcal{F}(s) = \Omega(s)\left\{\alpha + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'}\frac{\hat{\mathcal{F}}(s')\sin\delta(s')}{|\Omega(s')|(s'-s)}\right\}$$

$$\hat{\mathcal{F}}(s) = \frac{3}{2}\int_{-1}^{1}\mathrm{d}z\left(1-z^2\right)\mathcal{F}\left(t(s,z,M_\pi^2,M_\omega^2)\right)$$

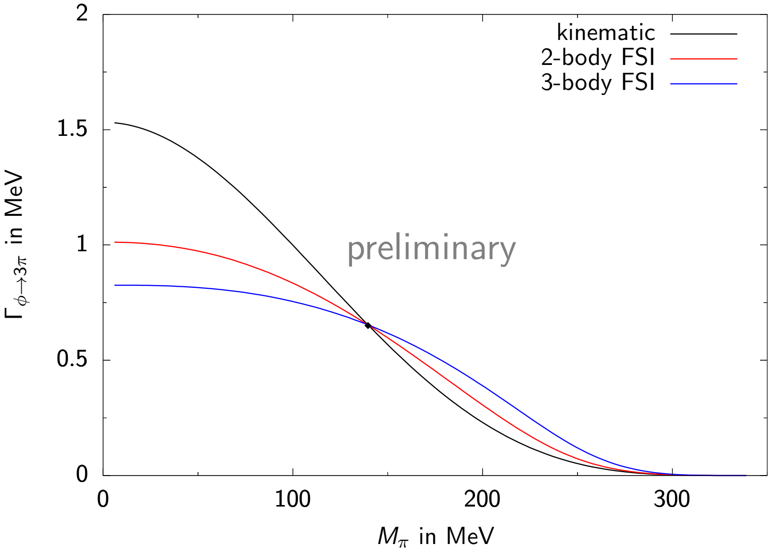
use M_π-dependent $\pi\pi$ phase shifts as input (unitarised ChPT/lattice QCD)

$\omega \to 3\pi$ decay width vs. M_π



subtraction constant α is fixed at the physical point (black rectangle)

$\phi \to 3\pi$ decay width vs. M_π



similar to $\omega \to 3\pi$, but $M_\phi > M_\omega$ $\Rightarrow$ larger phase space

Summary and Outlook

- dispersion relations are a powerful analysation tool
- derived a Khuri-Treiman dispersive representation for 3-body decays to describe the 3-particle FSI
  - based on analyticity, unitarity and crossing
  - input: S-wave $\pi\pi$- and $\eta\pi$/P-wave $\pi\pi$-scattering phase shifts
  - 3 subtraction constants/one subtraction constant
- dispersive representation for $\eta' \to \eta\pi\pi$ in good agreement with experimental data
- allows for book keeping of all M_π dependences in the $\omega \to 3\pi$ decay
- ω serves as paradigm case $\Rightarrow$ framework can be generalised (already done for $\phi \to 3\pi$: more dynamics due to larger phase space)

Spares

M_π dependence of the subtraction constant

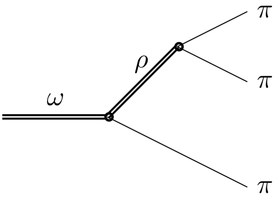
dispersive representation of $\mathcal{F}(s)$:

$$\mathcal{F}(s) = \Omega(s)\left\{\alpha + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'}\frac{\hat{\mathcal{F}}(s)\,\sin\delta(s')}{|\Omega(s')|(s'-s)}\right\}$$

in general: α can be M_π dependent

study $\rho \to \pi\pi$ in unitarised ChPT vs. VMD model with M_π-independent coupling $g_{\rho\pi\pi}$ $\Rightarrow$ good agreement for $\Gamma_{\rho\to\pi\pi}$ [*Hanhart, Peláez and Ríos (2014)*]

transfer $g_{\rho\pi\pi}(M_\pi) = const.$ to $g_{\omega\rho\pi}(M_\pi) = const.$ in analogy for:

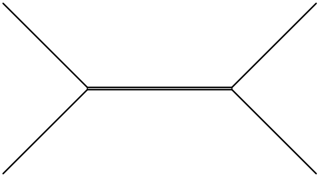


$\Rightarrow$ subtraction constant assumed to be M_π independent

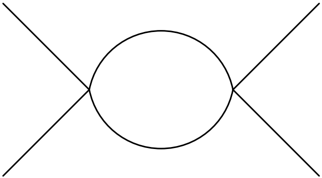
Final state interactions in $\eta' \to \eta\pi\pi$

consider to $2 \to 2$ FSI: $\pi\pi \to \pi\pi$ in s- and $\eta\pi \to \eta\pi$ in t- & u-channel

processes that may contribute to $2 \to 2$ scattering:



resonance-exchange



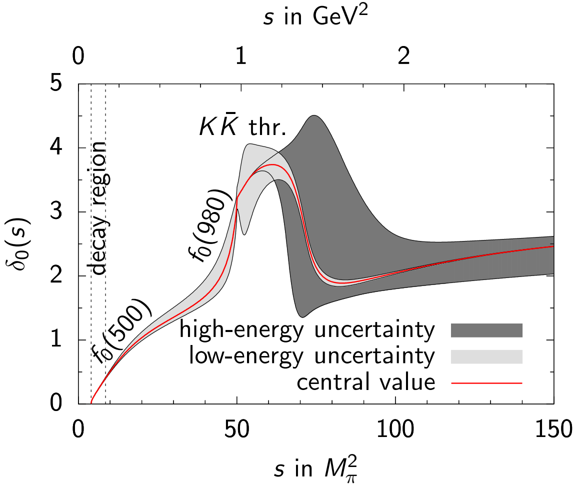
intermediate loop

restrict to S-wave processes:

	resonances	intermediate states
$\pi\pi$-system:	$f_0(500)$, $f_0(980)$	$\pi\pi$, $(K\bar{K})$
$\eta\pi$-system:	$a_0(980)$, $a_0(1450)$	$\eta\pi$, $(K\bar{K})$

(mimic intermediate $K\bar{K}$ contributions in an effective manner)

$\pi\pi$-scattering phase shift (S-wave, $I = 0$)

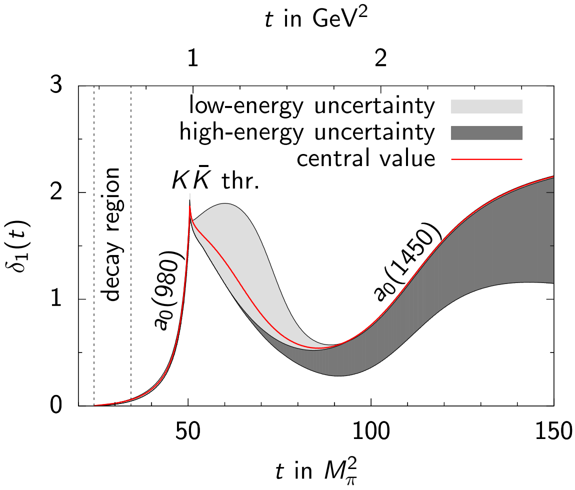


elastic regime: Roy equation analyses [*Caprini et al. (2012)*]

inelastic regime: study of a coupled channel Omnès matrix [*Daub et al. (2016)*]

and large-N_c ChPT constraints on $\eta'\eta \to (\pi\pi/K\bar{K})$

$\eta\pi$-scattering phase shift (S-wave, $I = 1$)



phase of the scalar form factor $F_S^{\eta\pi}(t)$ calculated out of a coupled channel T-matrix ($\eta\pi/K\bar{K}$) [*Albaladejo and Moussallam (2015)*]

Fit to experimental data

known from experiment:

- partial decay width [PDG (2016)]

$$\Gamma_{\eta'\to\eta\pi\pi} = \frac{1}{256\pi^3 M_{\eta'}^3} \int \mathrm{d}s\,\mathrm{d}t\,|\mathcal{A}(s,t,u)|^2$$

- measurements of the Dalitz-plot parameters

[VES (2007), BES-III (2011, 2017), A2 (2017)]

$$|\mathcal{A}(s,t,u)|^2 \approx |\mathcal{N}|^2(1 + ay + by^2 + \cancel{cx} + dx^2 + \ldots),$$

$$x \propto (t-u), \qquad y \propto -s$$

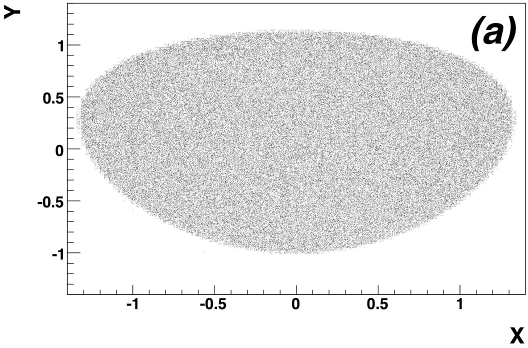
terms odd in x violate C-parity (not considered in DR)

$\Rightarrow$ can fix the subtraction constants through a fit to data

Dalitz plot for $\eta' \to \eta\pi\pi$

representation of the physical decay region (allowed phase space) for three-body decays

experimental data: scatter plot of $y \propto -s$ vs $x \propto (t-u)$



$\eta' \to \eta\pi^+\pi^-$ Dalitz plot with 43800 events [BES-III (2011)]

Dalitz-plot parameters

- extract Dalitz-plot parameters from the Taylor expansion of our amplitude
- parameters are well reproduced
- allows us to extract even higher coefficients of the expansion
- higher coefficients extremely tiny

in 10^{-3}	BES-III (2011) data	DR fit
a	$-47 \pm 11 \pm 3$	$-41 \pm 9 \pm 1$
b	$-69 \pm 19 \pm 9$	$-88 \pm 7 \pm 11$
d	$-73 \pm 12 \pm 3$	$-68 \pm 11 \pm 2$
$\kappa_{03}[y^3]$	–	$8 \pm 1 \pm 2$
$\kappa_{21}[yx^2]$	–	$-12 \pm 2 \pm 1$
$\kappa_{04}[y^4]$	–	$3 \pm 1 \pm 1$
$\kappa_{22}[y^2x^2]$	–	$3 \pm 1 \pm 1$
$\kappa_{40}[x^4]$	–	$0 \pm 1 \pm 0$

Soft-pion theorem for $\eta' \to \eta\pi\pi$

[Riazuddin and Oneda 1971, Adler 1965]

current algebra statement for amplitudes involving π's valid in the limit of $p_\pi \to 0$:

- SPT predicts two zeros (crossing symmetry) in $\mathcal{A}(s,t,u)$ at

$$s_1 = 0, \quad t_1 = M_{\eta'}^2, \quad u_1 = M_\eta^2 \quad \& \quad s_2 = 0, \quad t_2 = M_\eta^2, \quad u_2 = M_{\eta'}^2$$

- protected by chiral SU(2)×SU(2) symmetry $\Rightarrow$ Adler zeros

removed in models with explicit inclusion of scalar resonance $a_0(980)$

[Deshpande and Truong 1978]

study $\mathcal{A}(s,t,u)$ in our dispersive framework:

- encounter zeros close to soft-π points (slightly smaller $|t-u|$)
- at resonance positions: observe peak in $\mathrm{Im}\,\mathcal{A}$ & zero in $\mathrm{Re}\,\mathcal{A}$

$\Rightarrow$ although corrections at soft-π points are $\mathcal{O}\big(M_\pi^2/(M_{\eta'}^2 - M_{a_0}^2)\big)$:

DR refutes the resonance argument, zeros of the amplitude survive

Adler-zeros for $\eta' \to \eta\pi\pi$