The Three Graces (Sandro Botticelli)

UNITARITY CONSTRAINTS ON 3→3 SCATTERING AMPLITUDE

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DFG

INTRODUCTION

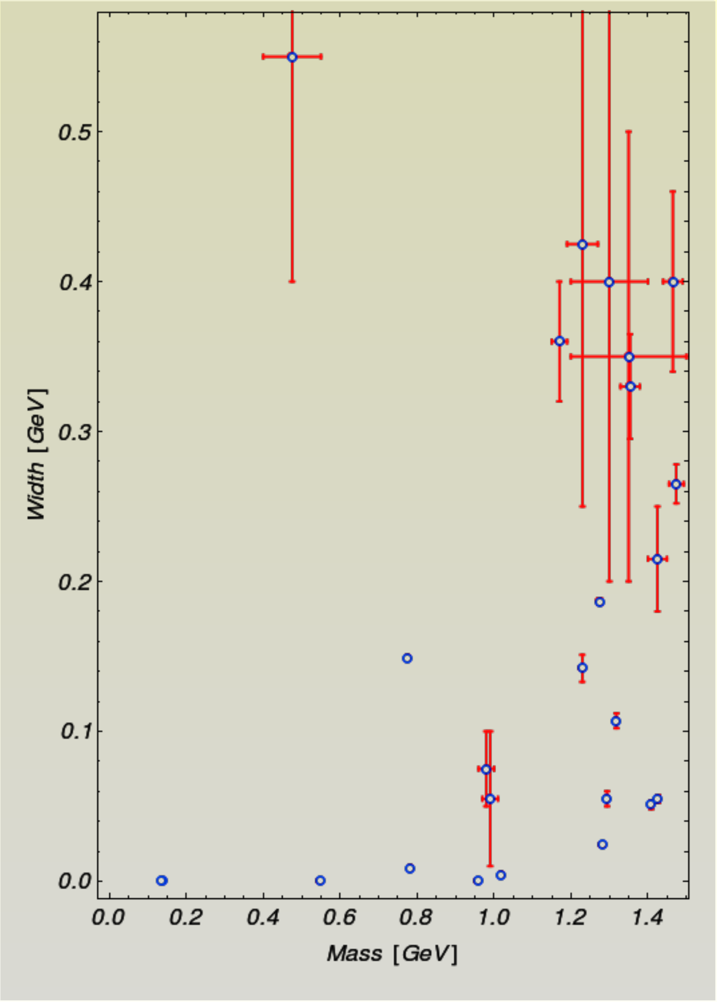
QCD at low energies → rich spectrum of excited states

Q1: how many are there?

→ missing resonance problem

Q2: what are they?

– quark-antiquark
– gluons
– hadron-hardon dynamics



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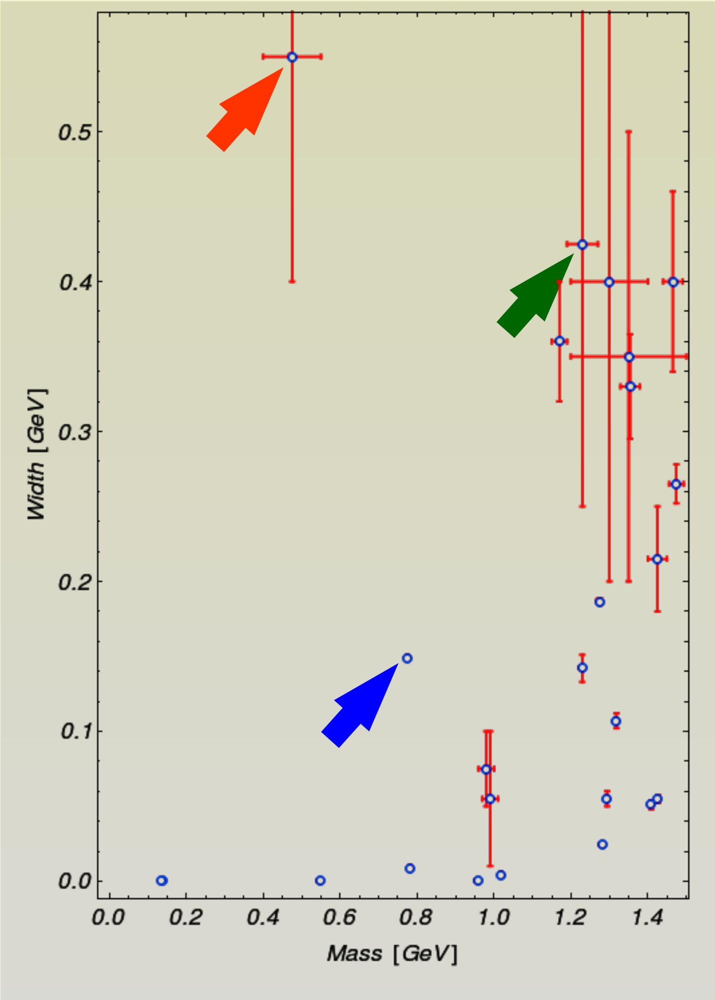


EXAMPLES:

– ***σ(500)*** , ***ϱ(770)***

couple dominantly to *2π,...*

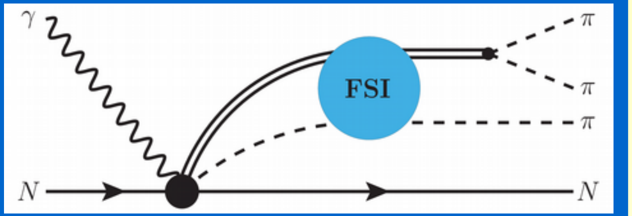
– ***a1(1260)***

couple dominantly to *3π,...*

Experiment

– Search for QCD exotics @ GlueX

* a1(1260)



– KL Beam @ GlueX

* *K*(892)* signature in $KN \to K\pi N$
* *$K\pi\pi$ channels(?)*

– Further applications:

* Roper puzzle ($\pi\pi N$)
* X(3872)

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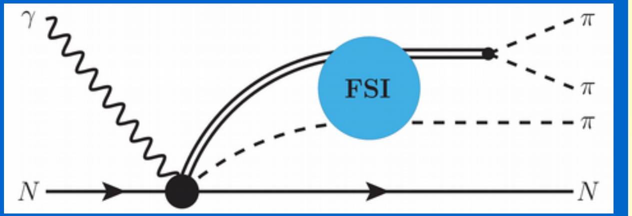

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Lattice QCD

Ab-initio numerical calculations

– Euclidean ST

– finite lattice spacing

– finite volume effects (THIS TALK)

→ 2-body **Q**uantization **C**ondition

[Lüscher (1986)]

→ 3-body **QC** not **yet** established

[Rusetsky, Polejaeva, Sharpe, Hansen, Briceno, Davoudi, Guo MM, Doring, ...]

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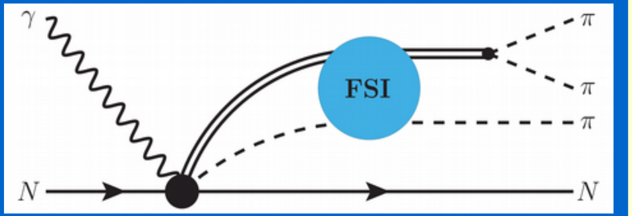

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THIS TALK: 3-BODY SCATTERING AMPLITUDE IN ISOBAR-FORMULATION

UNITARITY OF S-MATRIX

I. PART

IMAGINARY PARTS (INF. VOL.)

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II. PART

POWER LAW FIN. VOL. EFFECTS

I. PART

3→3 SCATTERING AMPLITUDE IN INFINITE VOLUME

[MM, Hu, Döring, Pilloni, Szczepaniak EPJ A53 (2017)]

T-MATRIX

- **3 asymptotic states (scalar particles of equal mass (*m*))**

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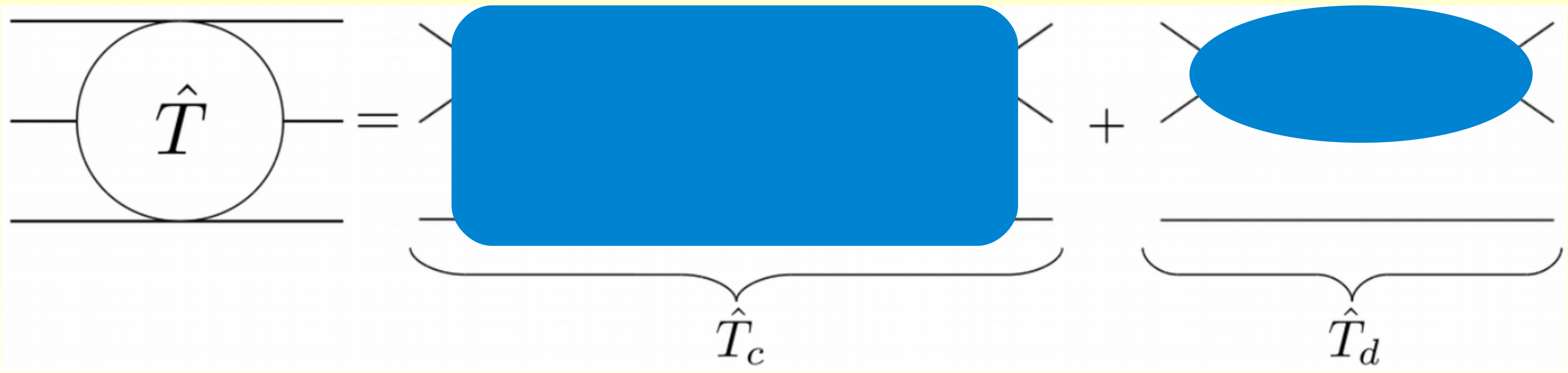

(all permutations considered)

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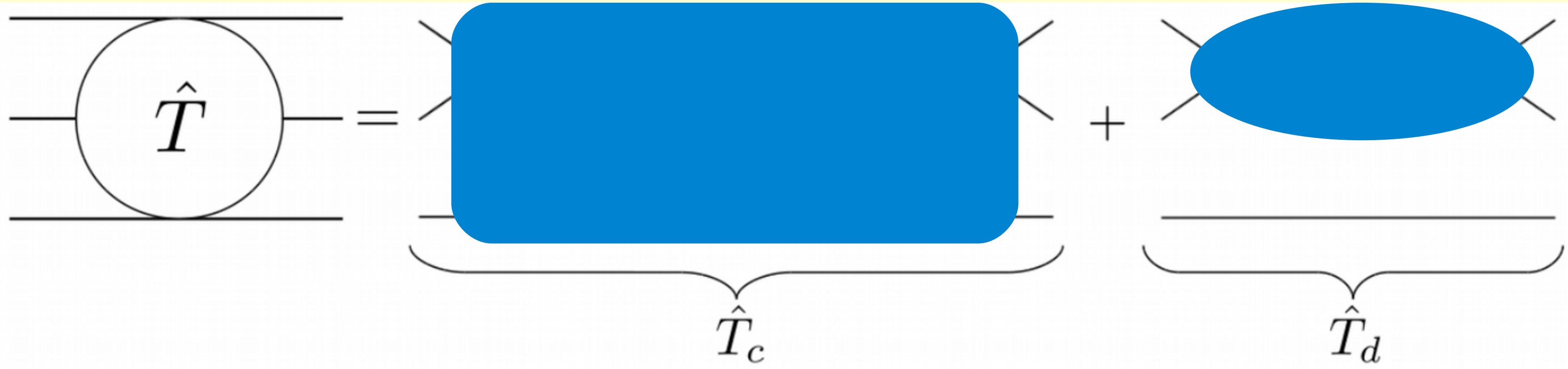

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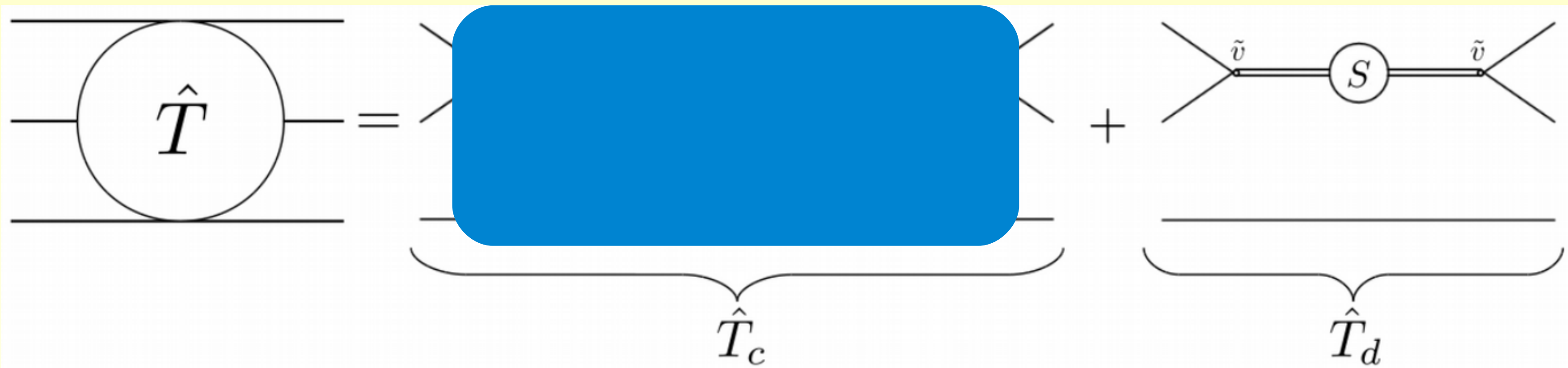

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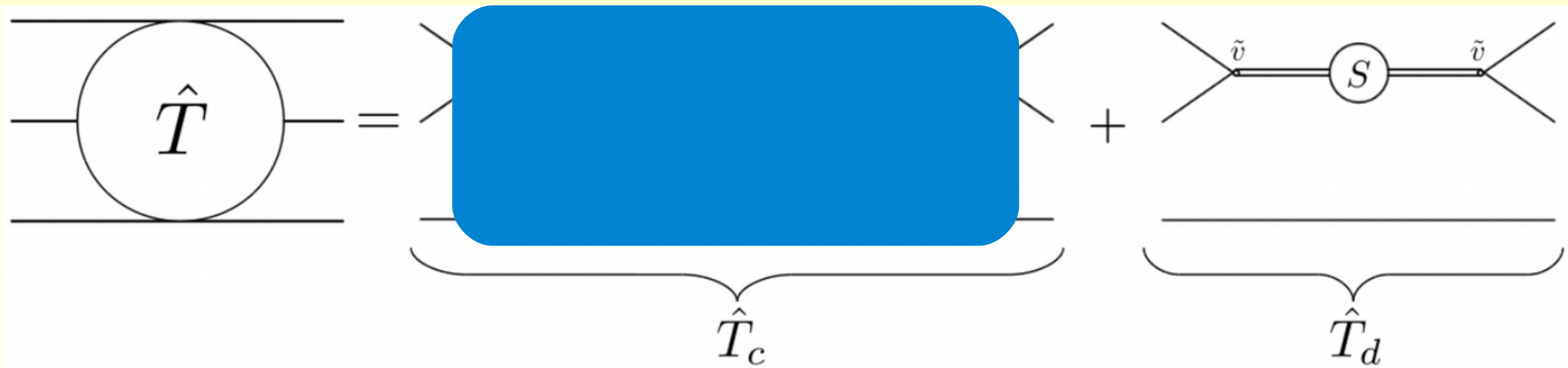

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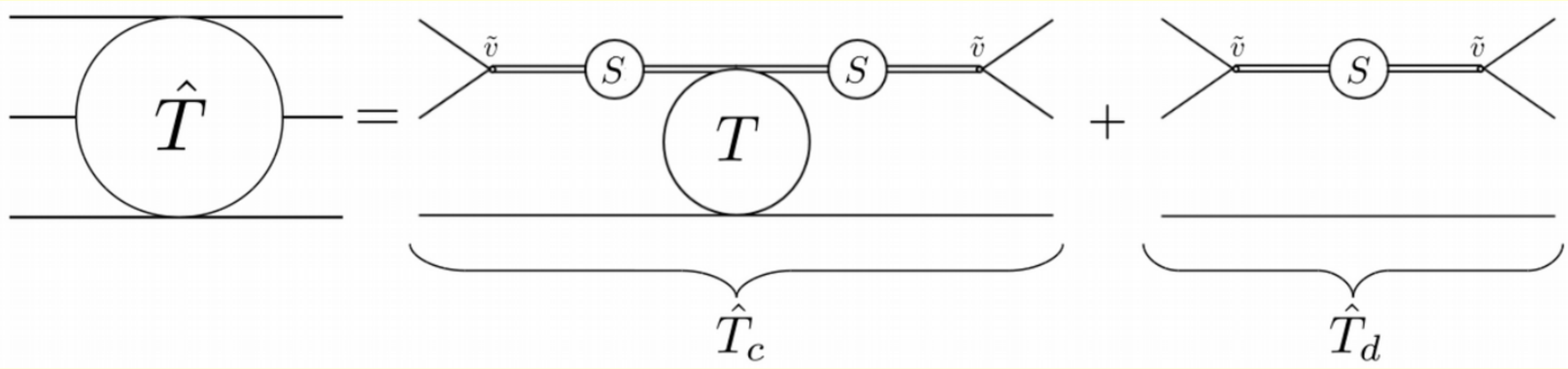

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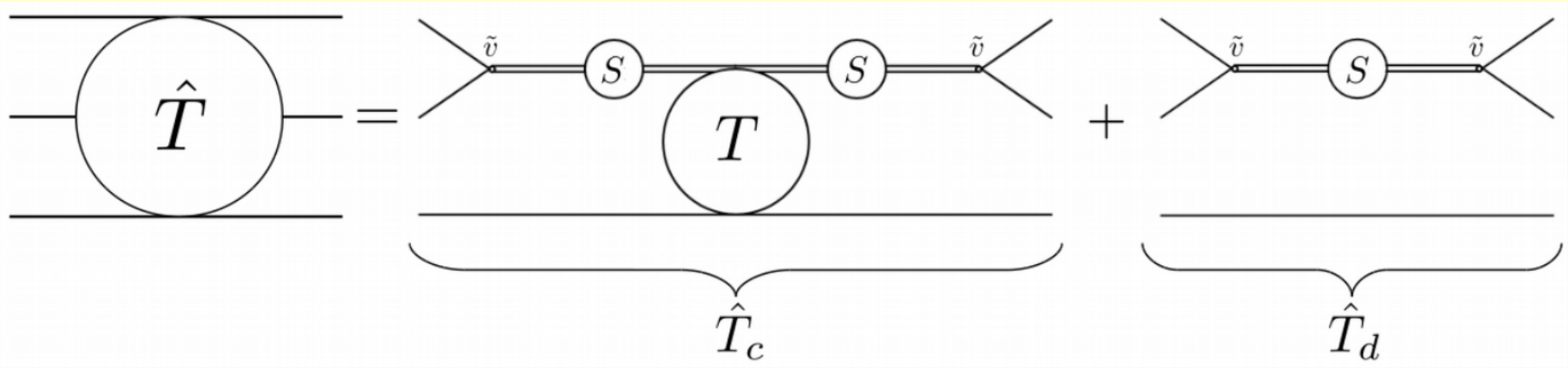

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3 unknown functions

8 kinematic variables

UNITARITY

3-body Unitarity (normalization condition ↔ phase space integral)

$$\boxed{\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle \quad = \quad i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle}$$

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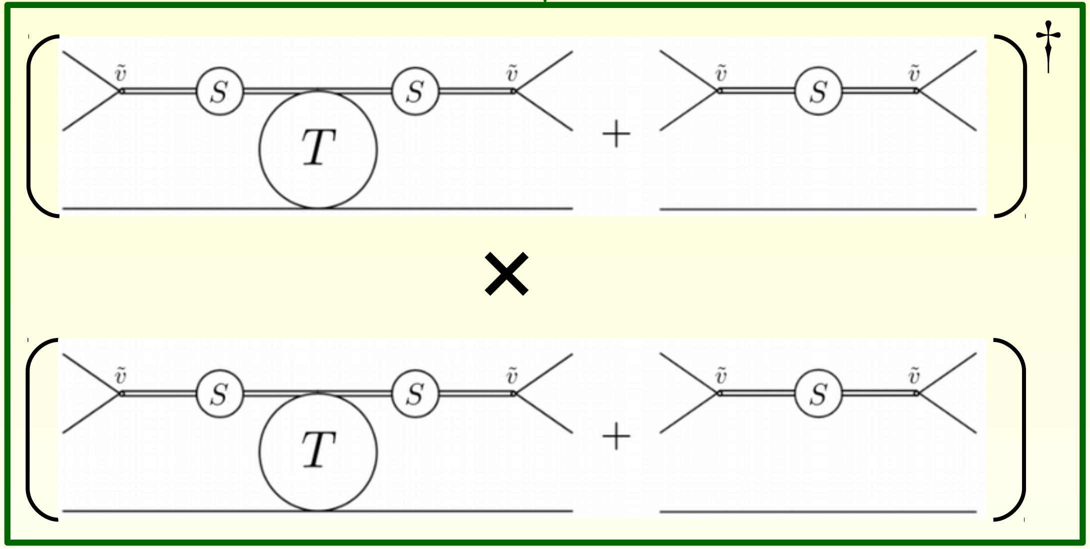

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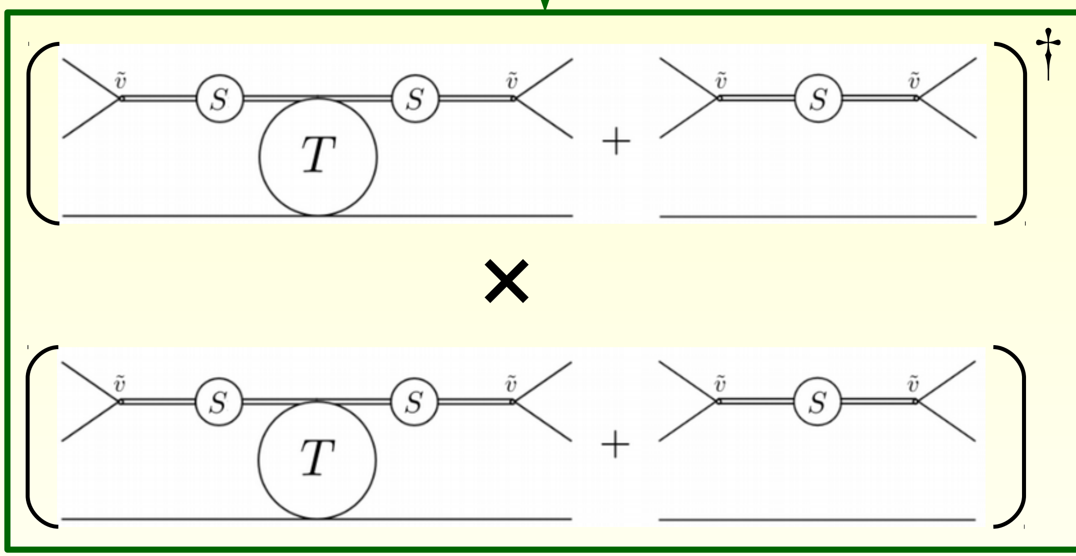

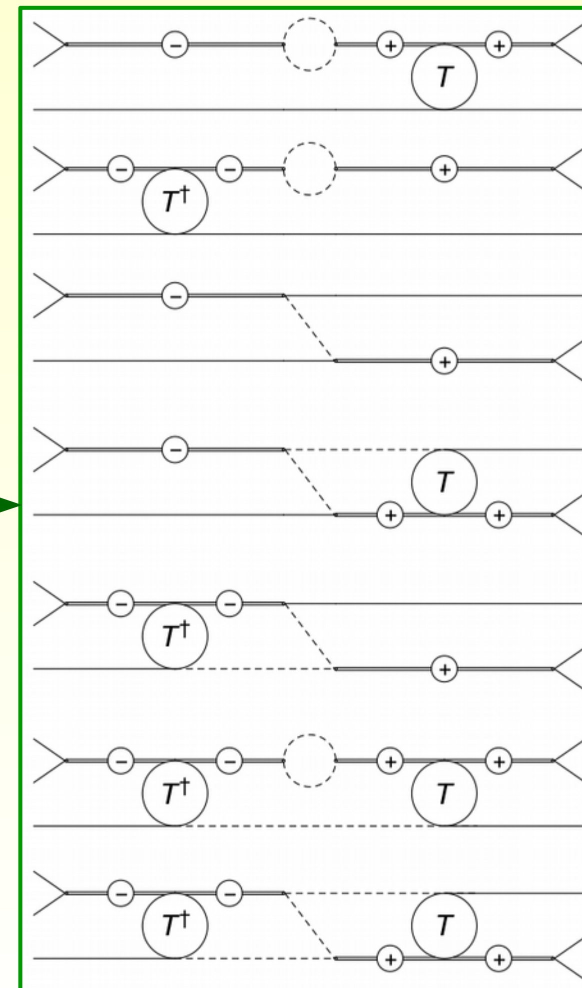

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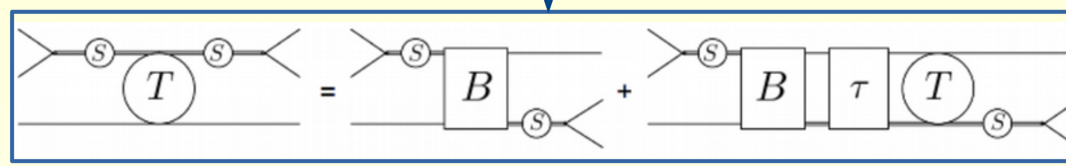

General Ansatz for the isobar-spectator interaction

→ B & τ are unknown!!!

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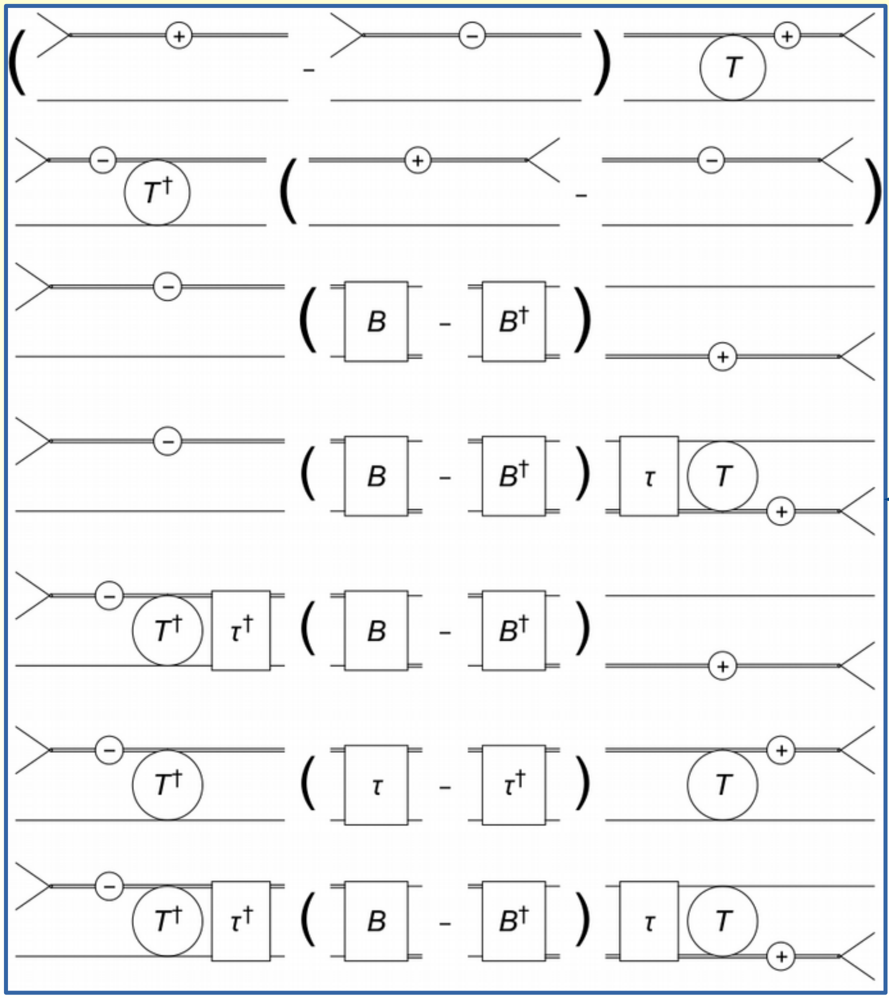

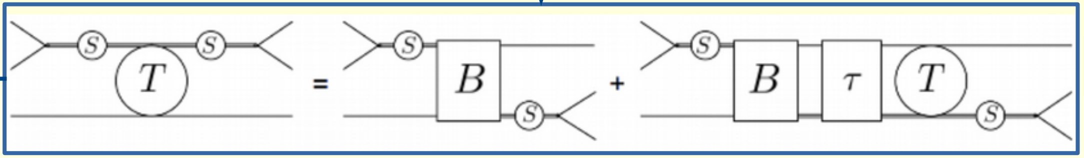

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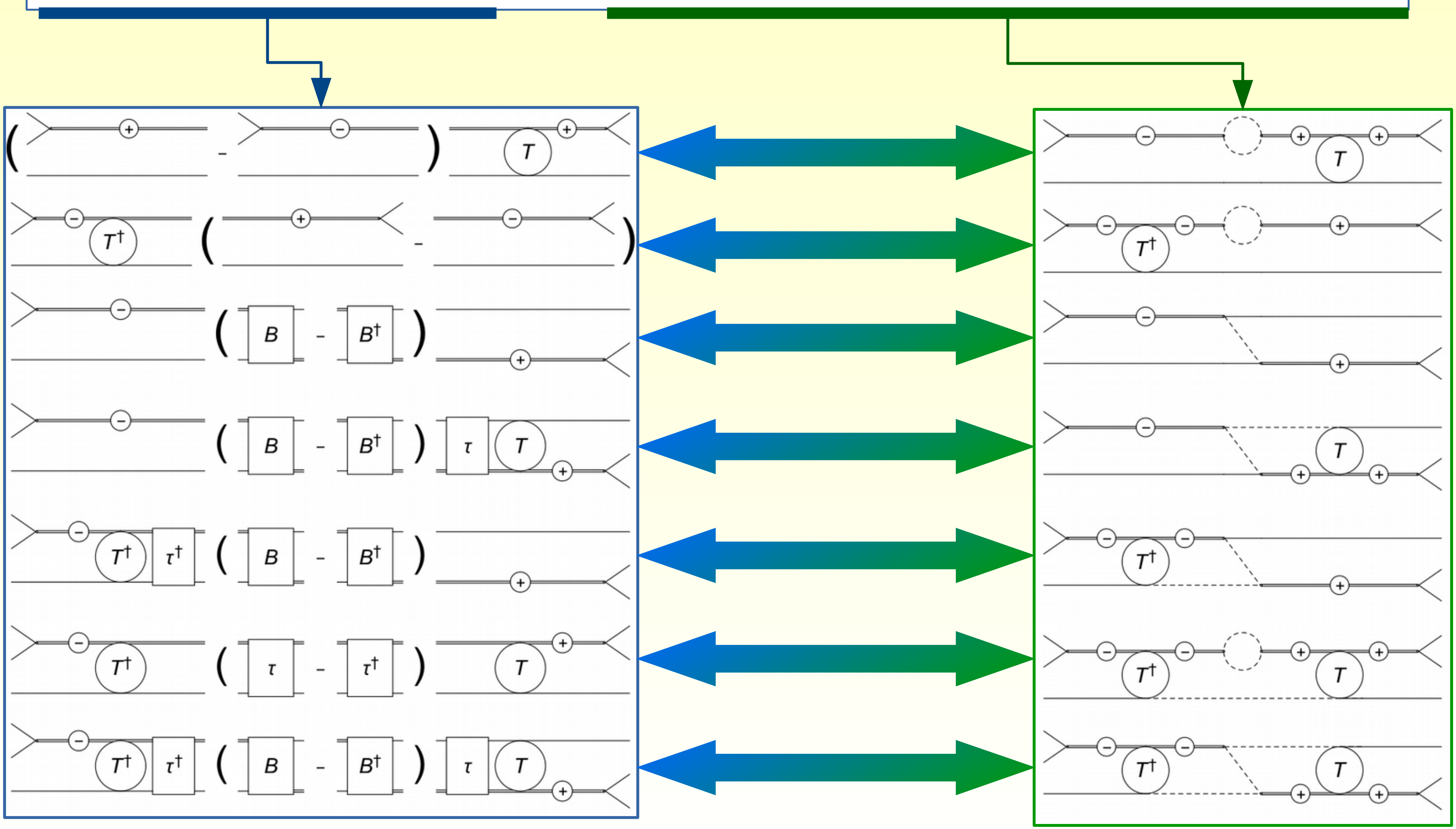
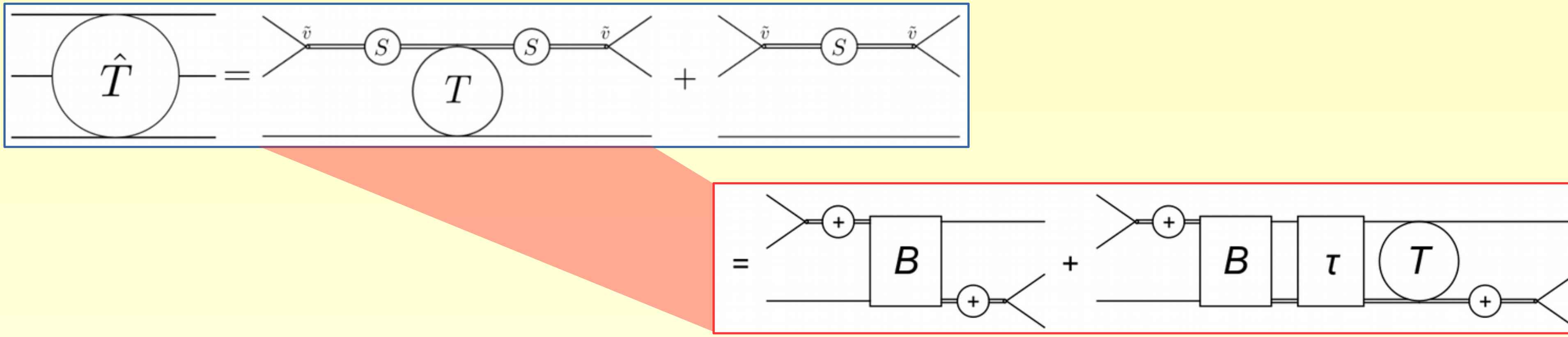


INTEGRAL EQUATION

3 → 3 scattering amplitude as a 3-dimensional integral equation



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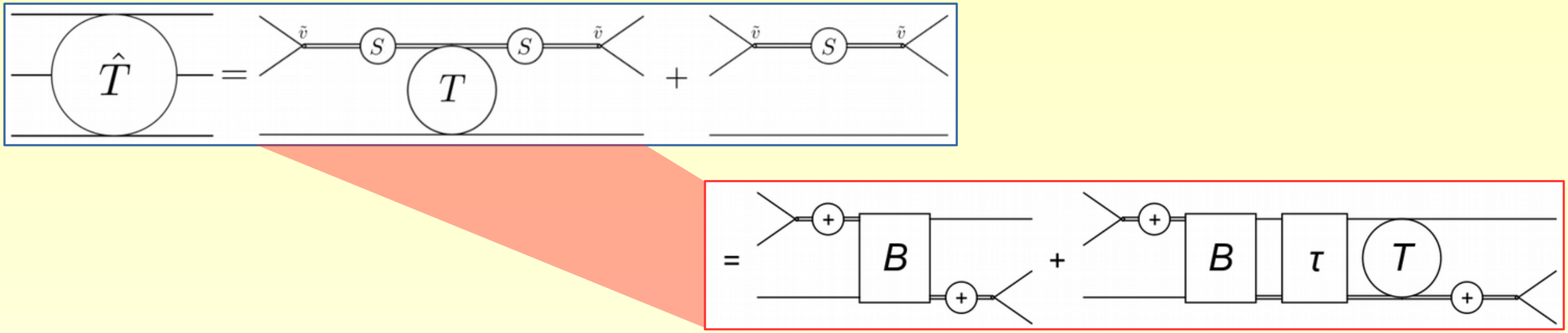

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$$\mathrm{Disc}\, B(u) = 2\pi i \, \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}} v^2$$

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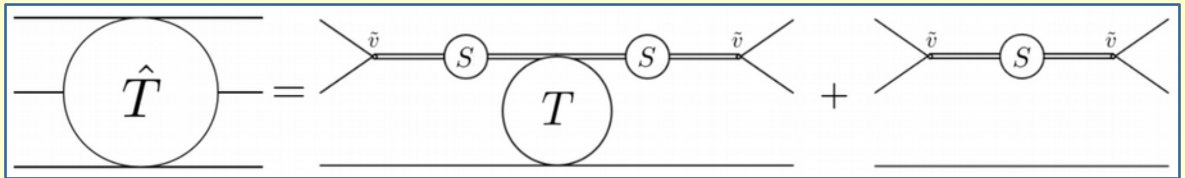

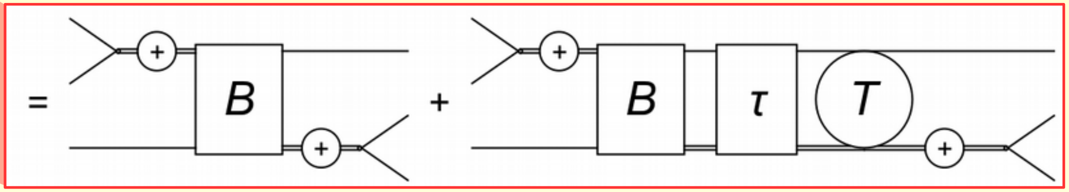

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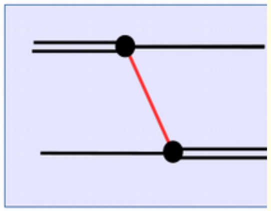

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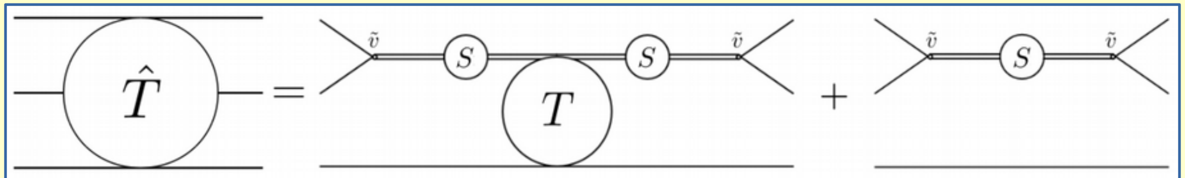

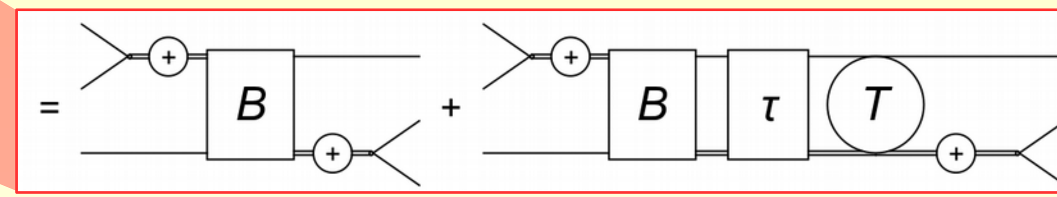

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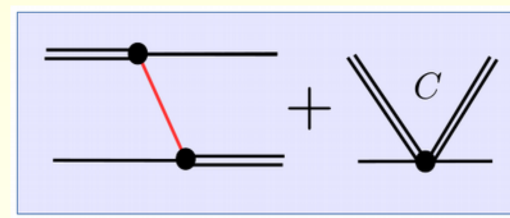

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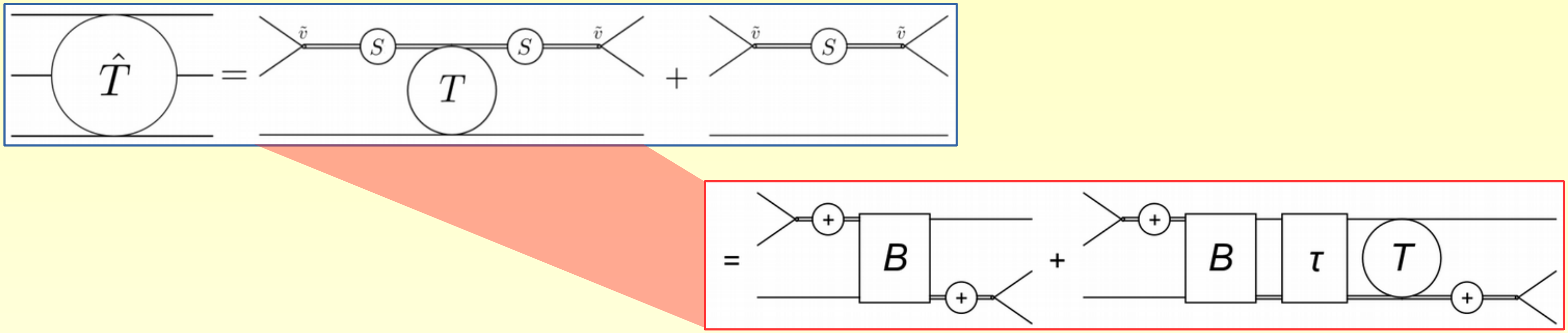

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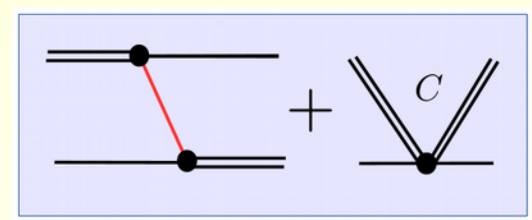

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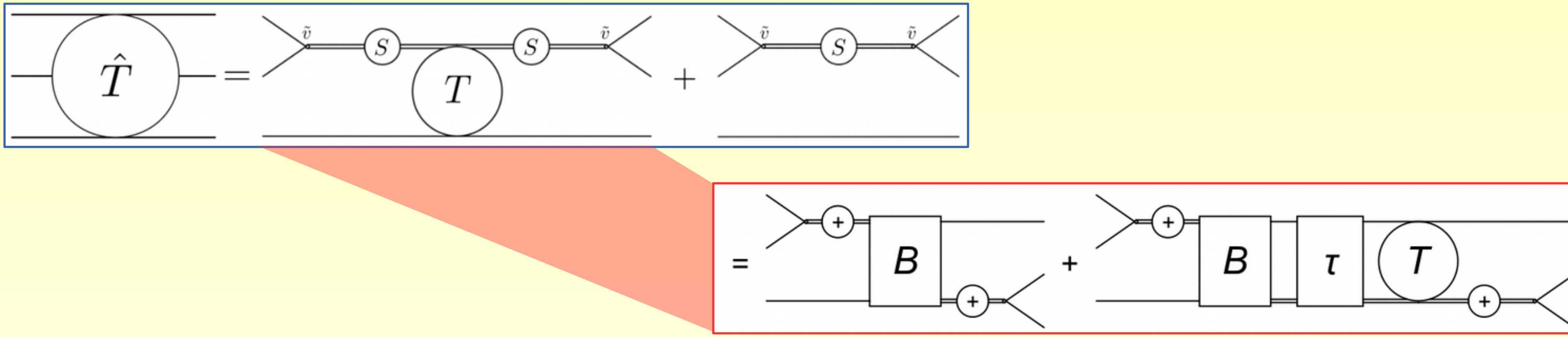

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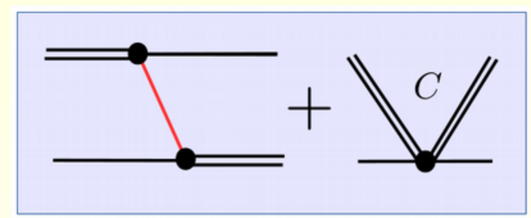

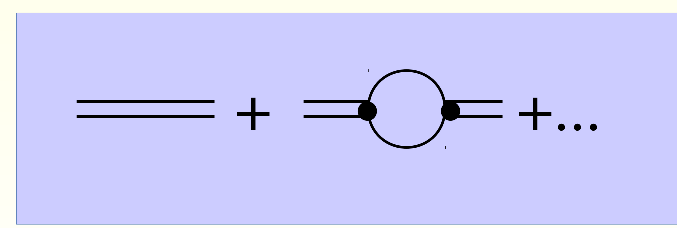

THE ONLY IMAGINARY PARTS REQUIRED BY 3b UNITARITY

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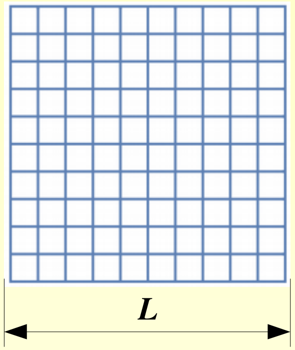
Lattice QCD calculations are performed in finite volume



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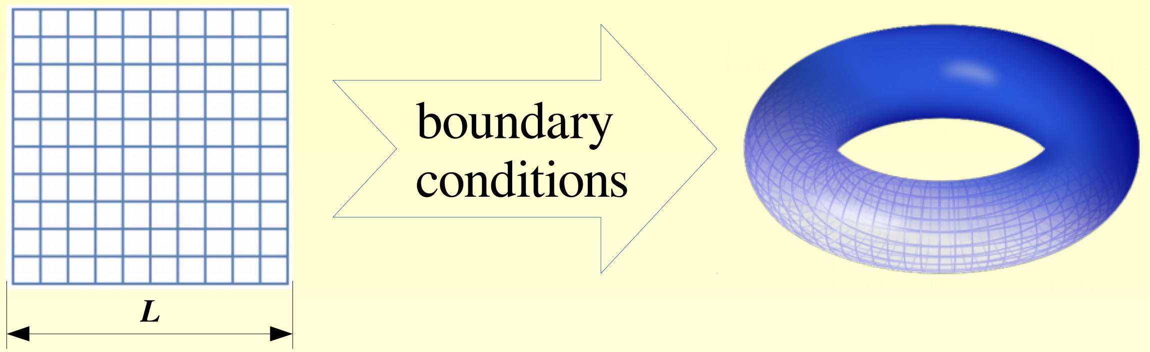

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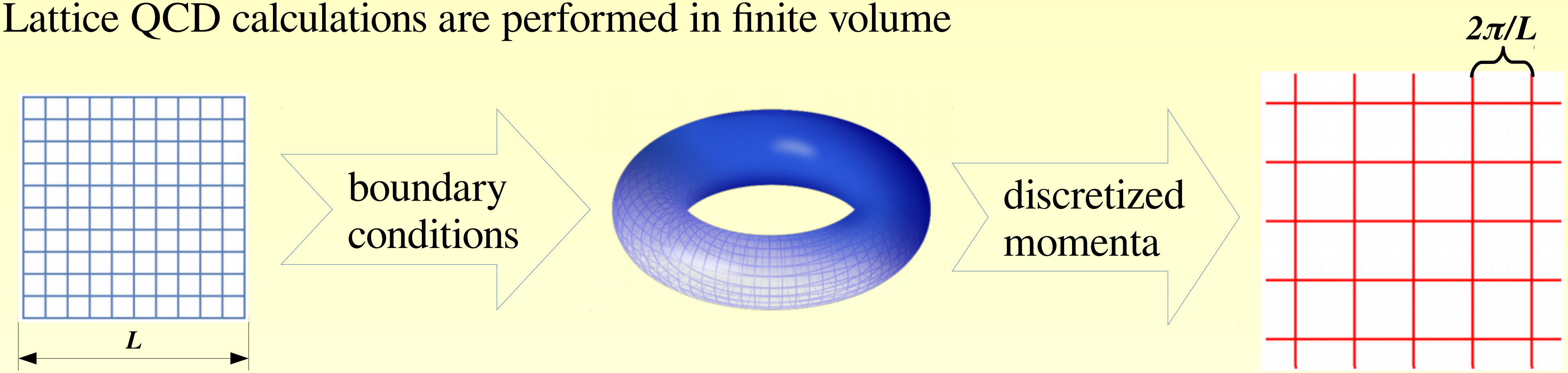

momenta & spectra are discretized

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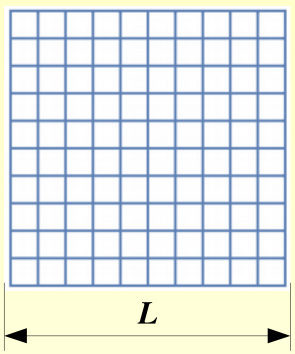
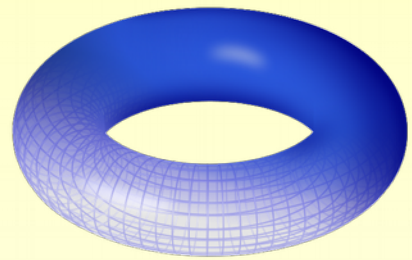
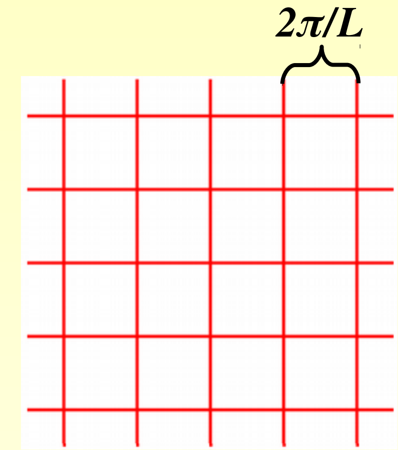


boundary conditions

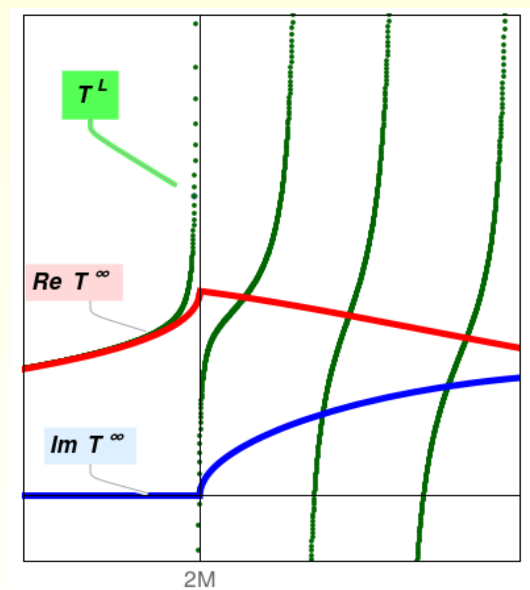


discretized momenta



momenta & spectra are discretized

– LSZ formalism relates Greens fct. & S-matrix



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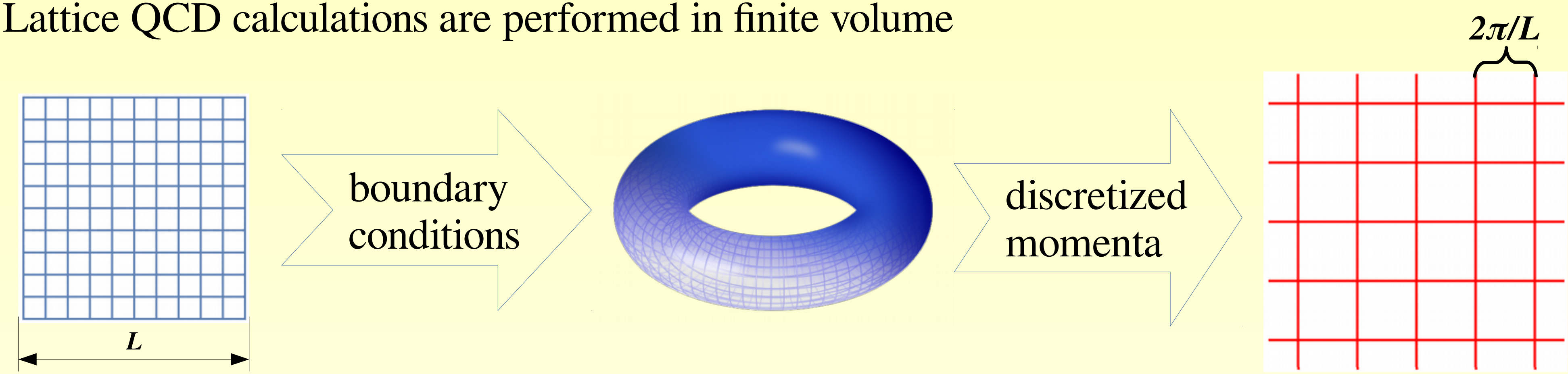

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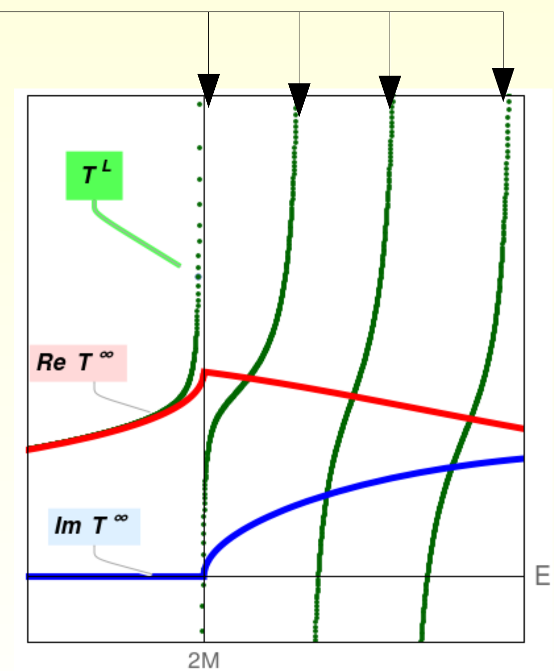
– LSZ formalism relates Greens fct. & S-matrix

$\rightarrow$ ***$T(E^*) = \infty$ $\leftrightarrow$ $E^* \in$ Energy-Eigenvalues***



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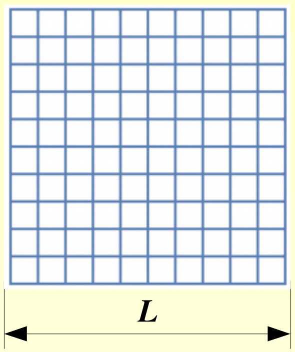

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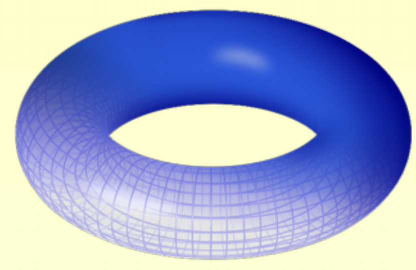

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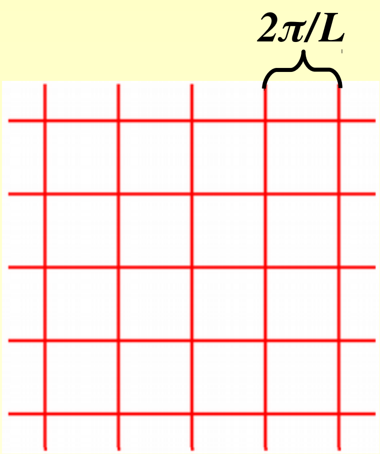

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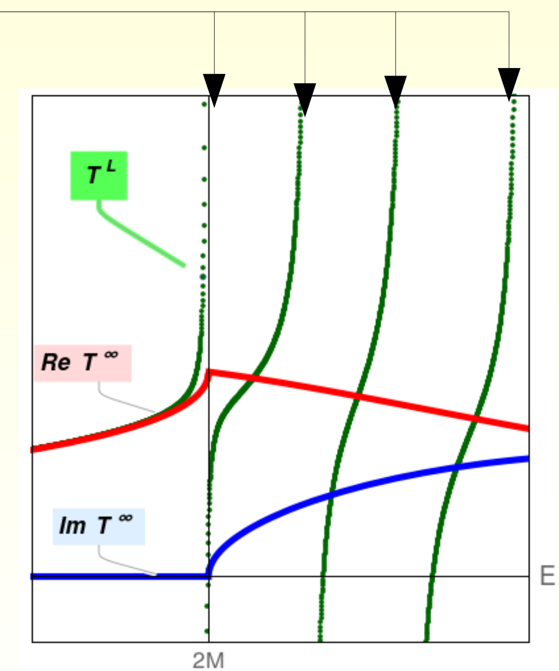
  → ***T(E*) = ∞*** ↔ ***E**** ∈ *Energy-Eigenvalues*

- well established in 2-body

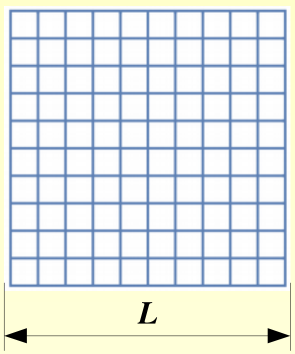
[Lüscher (1986)]

...

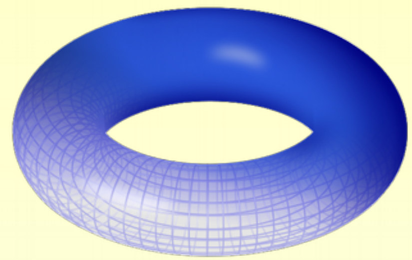


LATTICE QCD SETUP

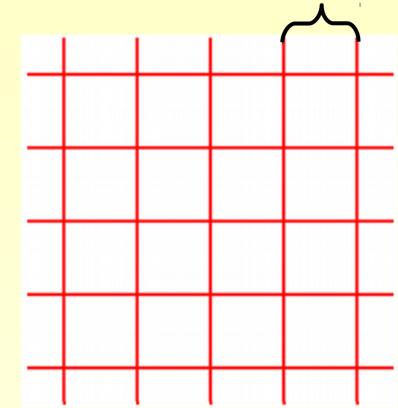
Lattice QCD calculations are performed in finite volume



boundary conditions



discretized momenta



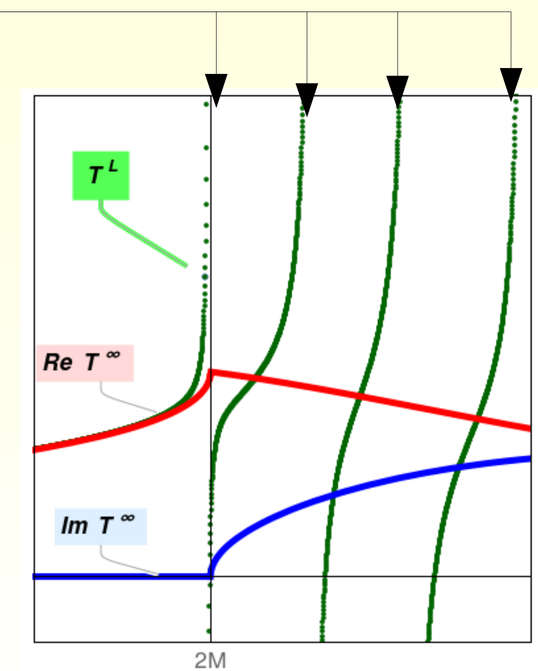
momenta & spectra are discretized

– LSZ formalism relates Greens fct. & S-matrix

$\rightarrow$ ***T(E*) = ∞*** $\leftrightarrow$ ***E**** ∈ *Energy-Eigenvalues*

– well established in 2-body [Lüscher (1986)] ...

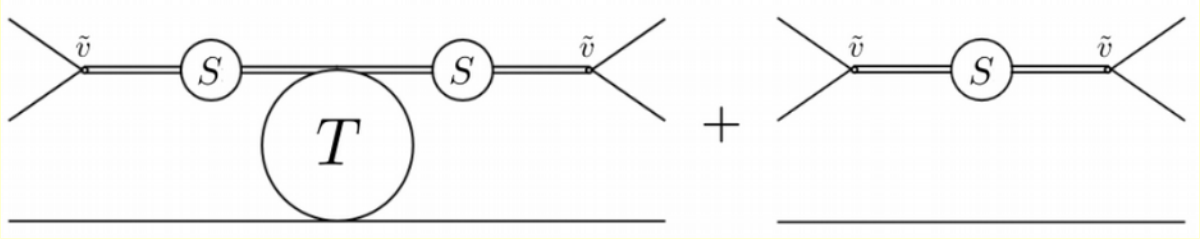
– 3-body analog under investigation Sharpe, Rusetsky, Hansen, Polejaeva, Briceno, Davoudi, Guo, Pang, MM, Doring



DISCRETIZATION

Discretize 3b-scattering amplitude → 3b Quantization Condition

- ***v*** **is cut-free**



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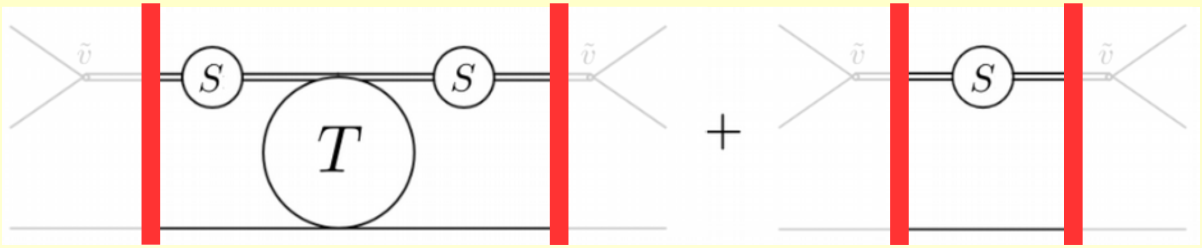

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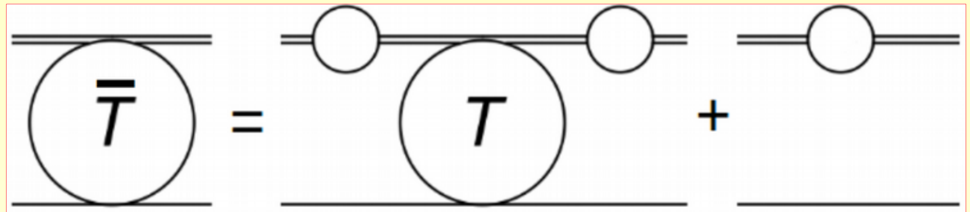
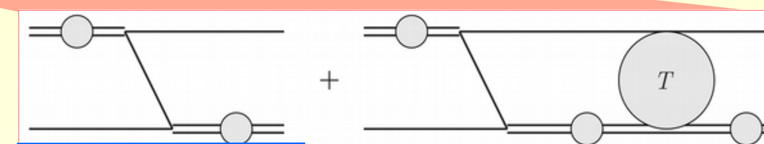
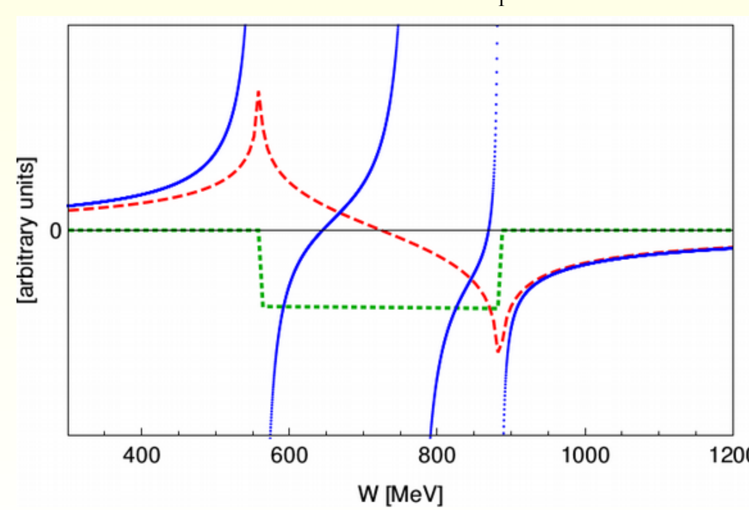




– High-dimensional problem

– B (ope potential) is singular!

S-wave infinite volume vs. A_1^+ finite volume



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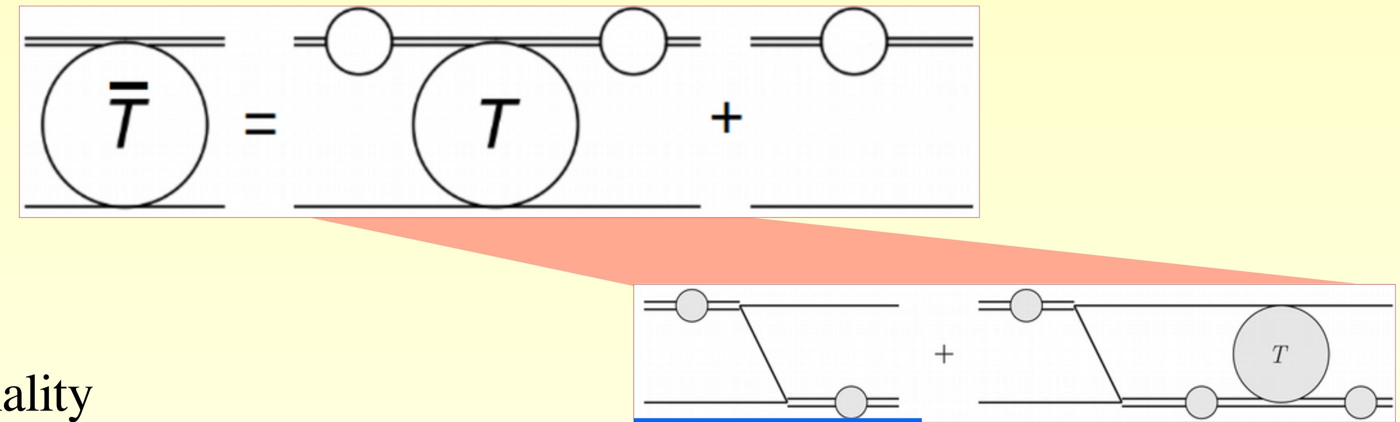


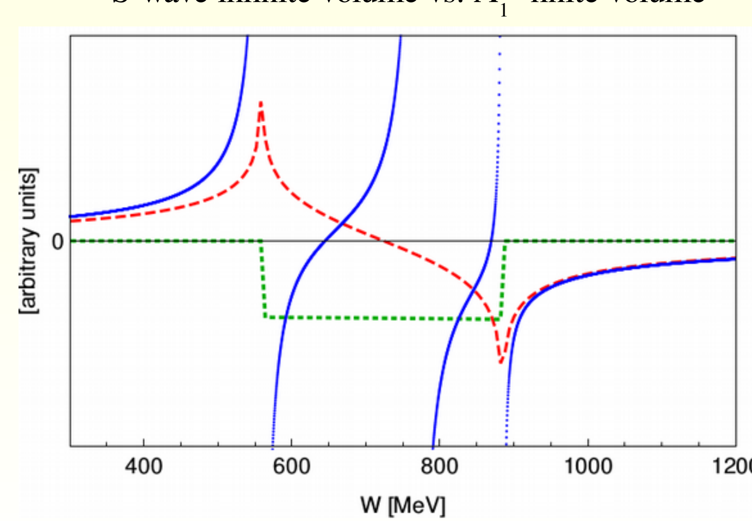
– High dimensionality

– B (ope potential) is singular!

→ Project to irreps of cubic group:

$\{A_1|A_2|E|T_1|T_2\}$

S-wave infinite volume vs. A_1^+ finite volume

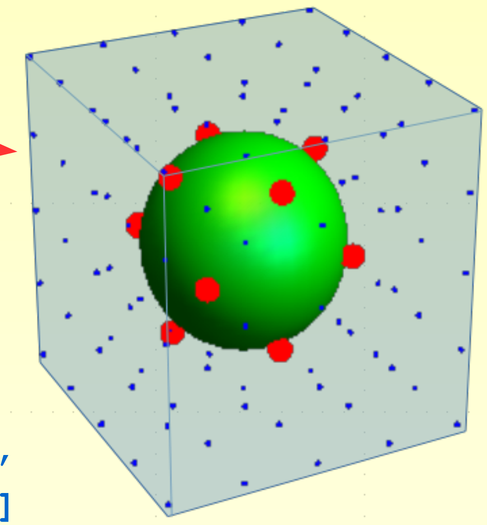


PROJECTION TO IRREPS

1) Separation of variables

– shells = sets of points related by $\boldsymbol{O_h}$

– *inf. vol. analog:* radii and angles



[Döring,Hammer,MM,Pang, Rusetsky,Wu(2018)]

PROJECTION TO IRREPS

1) Separation of variables

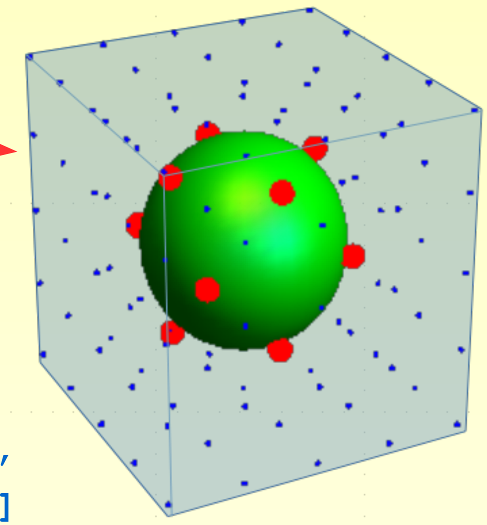
– shells = sets of points related by $\boldsymbol{O_h}$

– *inf. vol. analog:* radii and angles

2) Find the ONB of functions on each shell

– $$f^s(\hat{\mathbf{p}}_j) = \sqrt{4\pi}\sum_{\Gamma\alpha}\sum_u f_u^{\Gamma\alpha s}\chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}_j)$$

– *inf. vol. analog:* PWA



[Döring,Hammer,MM,Pang, Rusetsky,Wu(2018)]

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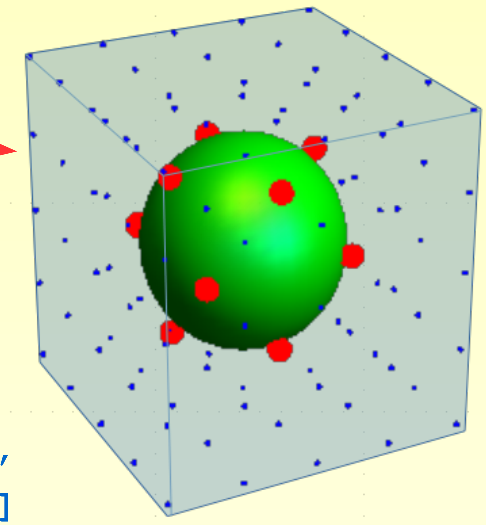

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[Döring,Hammer,MM,Pang, Rusetsky,Wu(2018)]

Projection of 3-body-Quantization-Condition = FINAL RESULT

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma\mathbf{ss}'}_{\mathbf{uu}'}(\mathbf{W}^2) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W}^2)^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right) = \mathbf{0}$$

[MM,Döring]

QUANTIZATION CONDITION

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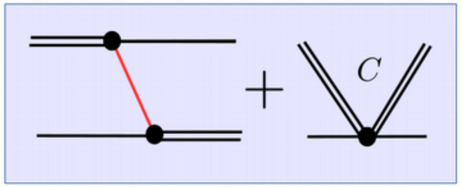


W – total energy

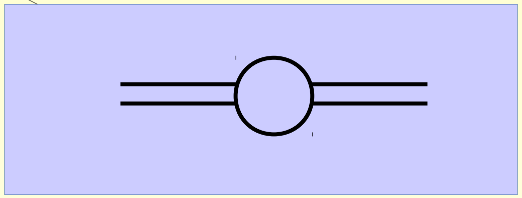
s/s' - shell index

u/u' - basis index

ϑ – multiplicity

L – lattice volume

Es – 1p. energy



QUANTIZATION CONDITION

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{ss'}}_{\mathbf{uu'}}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

- **3 particles in finite volume:** ***m=138 MeV, L=3 fm***

QUANTIZATION CONDITION

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- 3 particles in finite volume: ***m=138 MeV, L=3 fm***
- **one S-wave isobar → two unknowns:**
  - vertex(Isobar→2 stable particles)
  - subtraction constant (~mass)

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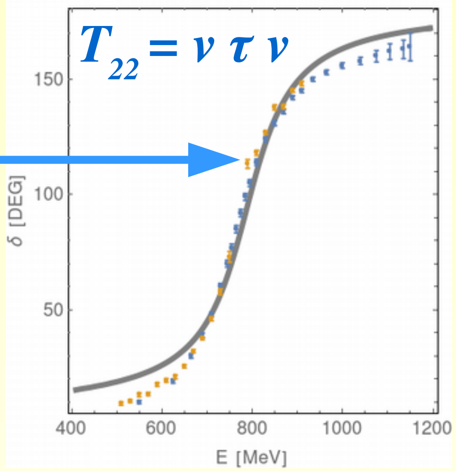

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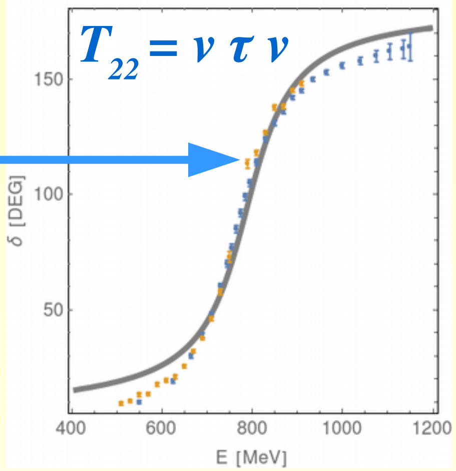
- **Project to $\Gamma = A^{1+}$**

  → prediction of 3body energy-eigenlevels



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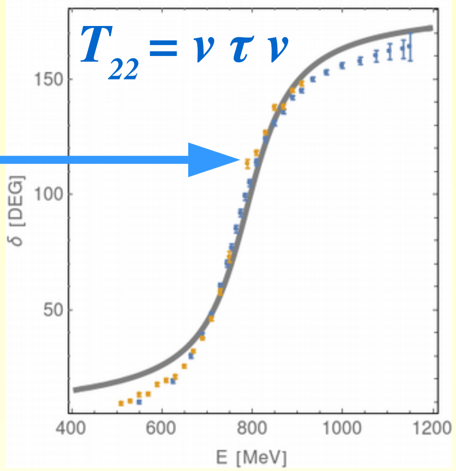

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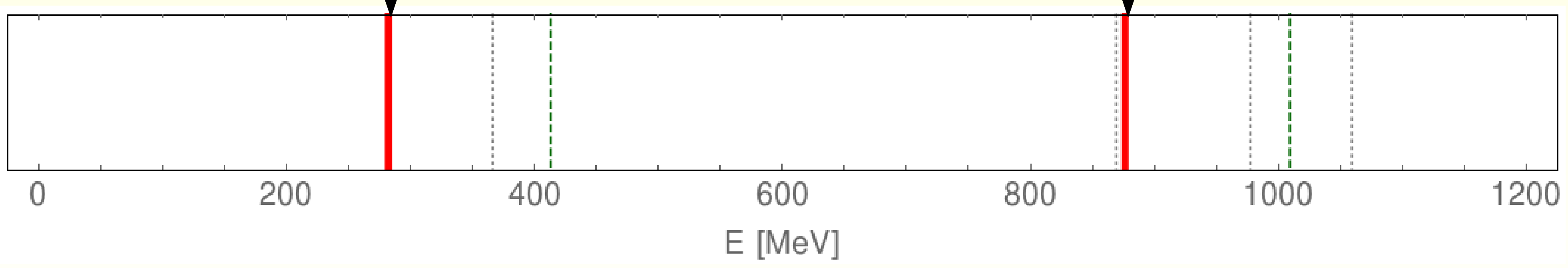

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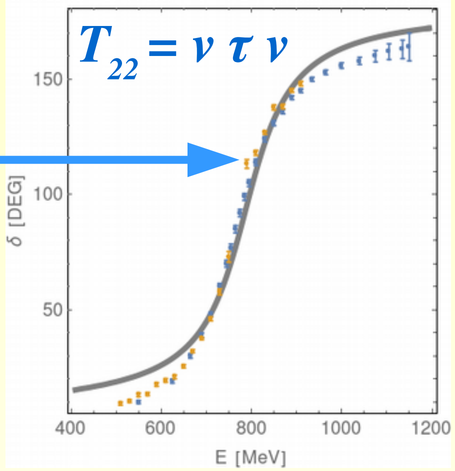

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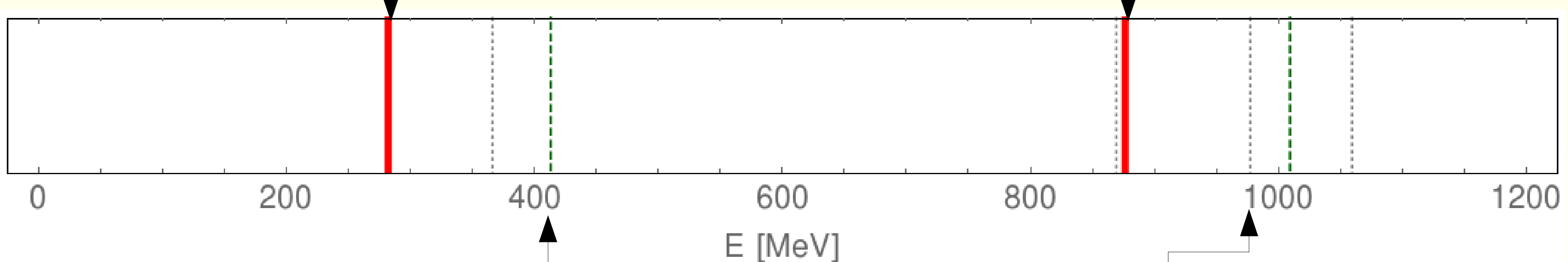


unphysical lvls cancel out (exact proof available)

SUMMARY/OUTLOOK

3-body scattering amplitude from 2&3 body Unitarity

– interaction kernel = one-particle-exchange

– single approximation: number of isobars

– flexible parametrization: real contributions can be added to the kernel

3-body Quantization Condition in fin. vol. derived

– cancellations of unphysical poles revealed analytically

– projection to irreps done

– technical feasibility on a numerical example

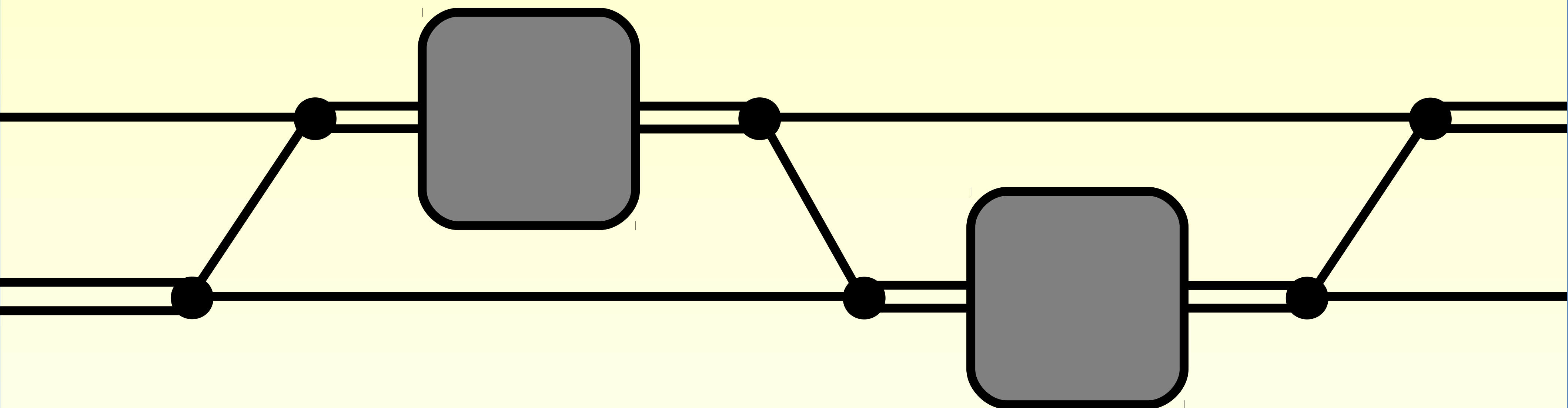
TBD: analysis of physical systems

TBD: multiple channels

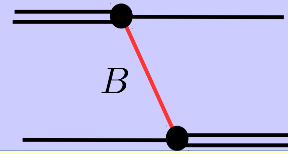
TBD: inclusion of isospin & angular momentum

The Three Graces (Sandro Botticelli)

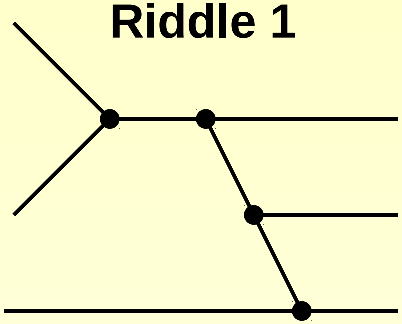
THANK YOU!

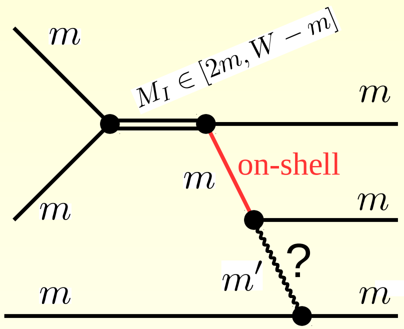


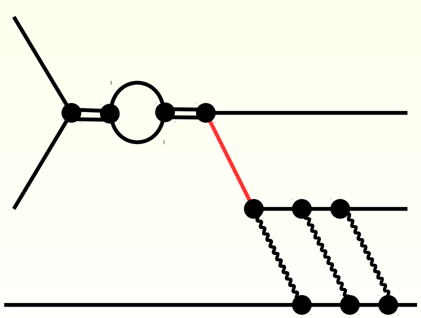
The Power of Unitarity

Question: Does  provide full imaginary part of all possible 3→ 3 transitions?

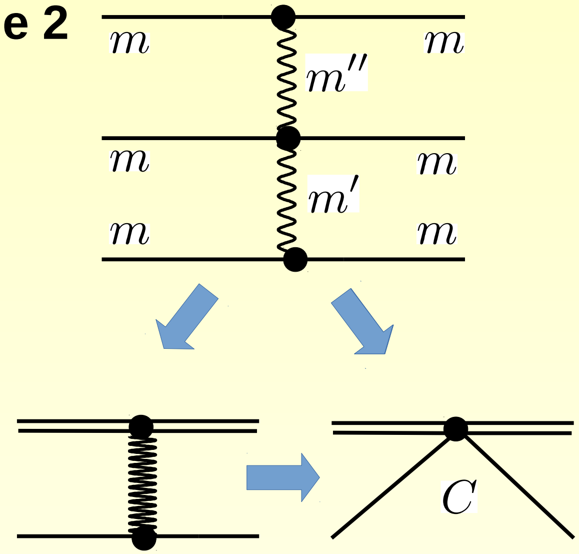
Riddle 1

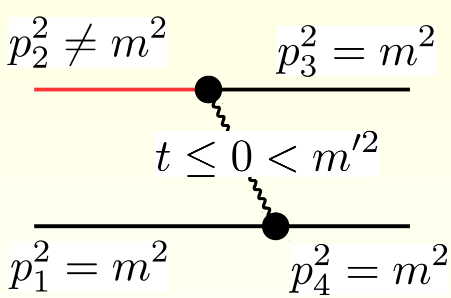


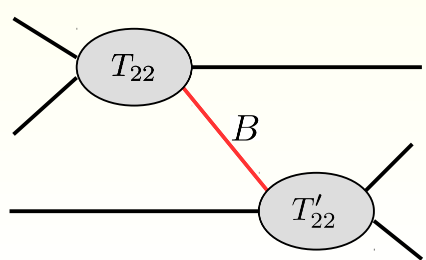




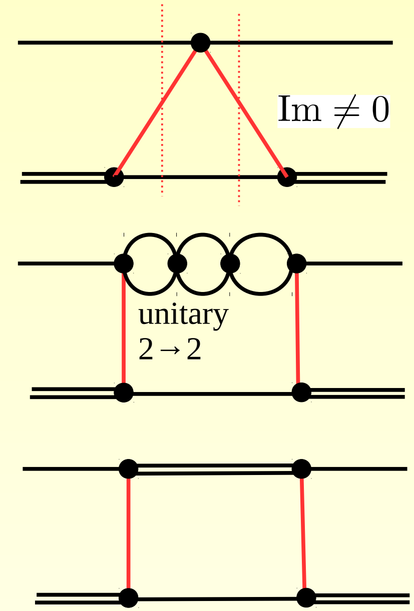
Riddle 2



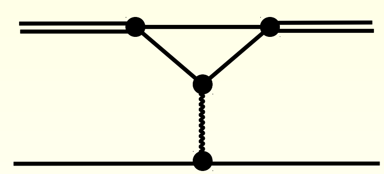




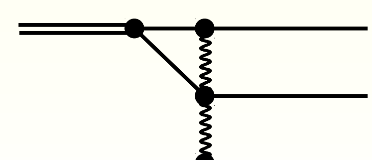
Riddle 3



Riddle 4



Riddle 5



- Projection of T

$$T^{ss'}(\hat{\mathbf{p}}_j,\hat{\mathbf{p}}_{j'}) = 4\pi \sum_{\Gamma\alpha}\sum_{uu'} \chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}_j) T_{uu'}^{\Gamma ss'} \chi_{u'}^{\Gamma\alpha s'}(\hat{\mathbf{p}}_{j'})\,,$$

$$T_{uu'}^{\Gamma ss'} = \frac{4\pi}{\vartheta(s)\vartheta(s')} \sum_{j=1}^{\vartheta(s)}\sum_{j'=1}^{\vartheta(s')} \chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}_j) T^{ss'}(\hat{\mathbf{p}}_j,\hat{\mathbf{p}}_{j'}) \chi_{u'}^{\Gamma\alpha s'}(\hat{\mathbf{p}}_{j'})$$

Cancellations:

→ fin. vol. normalization of δ-distribution!

$$\bar{T}^{A_1^+}_{nm}(s) = \tau_n(s) T^{A_1^+}_{nm}(s) \tau_m(s) - 2E_n \tau_n(s) \frac{L^3}{\vartheta(n)} \delta_{nm}$$

$$T^{A_1^+}_{nm}(s) = B^{A_1^+}_{nm}(s) - \frac{1}{L^3} \sum_{x \in sets_8} \vartheta(x) B^{A_1^+}_{nx}(s) \frac{\tau_x(s)}{2E_x} T^{A_1^+}_{xm}(s)$$

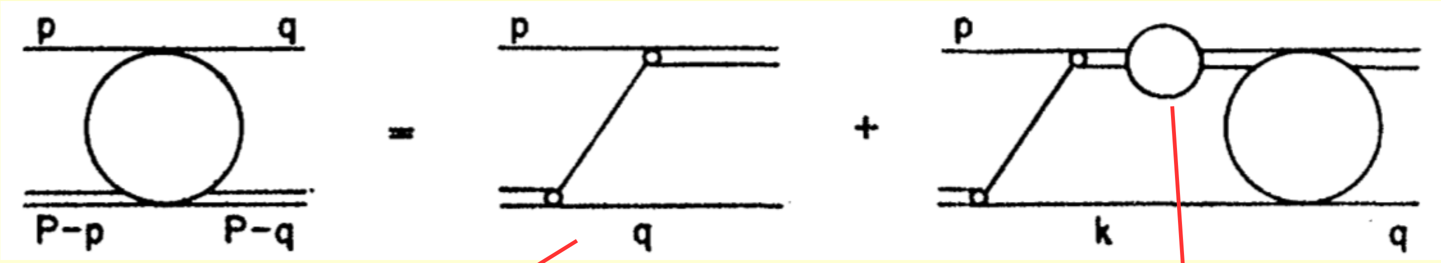
$B^{A_1^+}$ singular at $W^+ = E_m + E_n + E(\boldsymbol{q}_{nj} + \boldsymbol{p}_{mi})$

τ_m^{-1} singular at $W^{\pm\pm} = E_m \pm E((2\pi/L)\boldsymbol{y}) \pm E((2\pi/L)\boldsymbol{y} + \boldsymbol{p}_{mi})$ for $\boldsymbol{y} \in \mathbb{Z}^3$

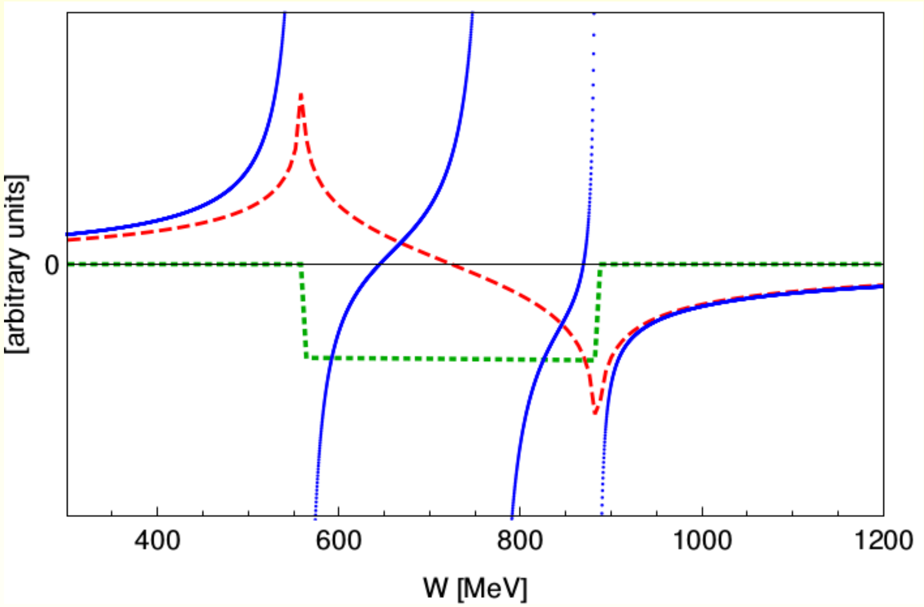
– when isobar-momenta are discretized in the 3-body cms momenta

$$\tau = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3} \int d^3\boldsymbol{\ell} \frac{\lambda^2}{2E_\ell(\sigma(k) - 4E_\ell^2 + i\epsilon)}$$

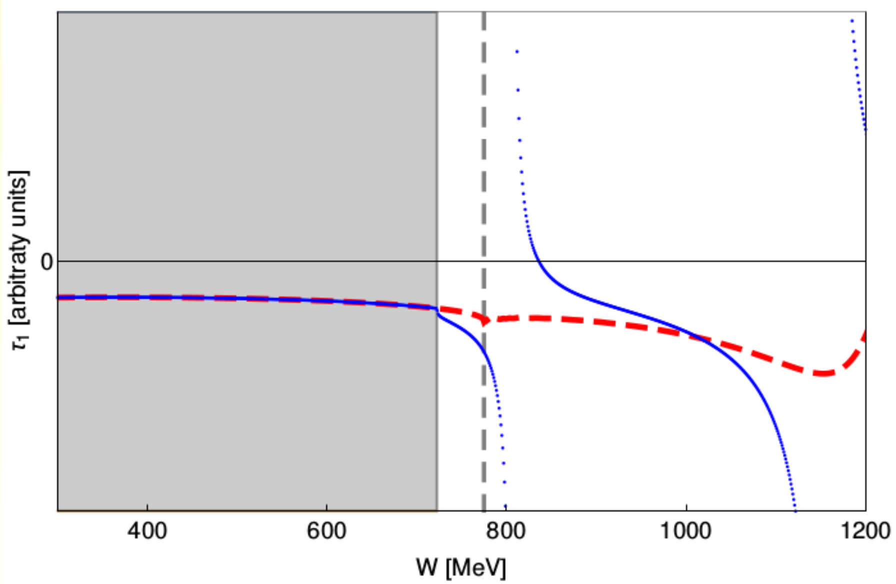
Power-law finite-volume effects dictated by three-body unitarity



S-wave infinite volume vs. A_1^+ finite volume
shell 1 to shell 1

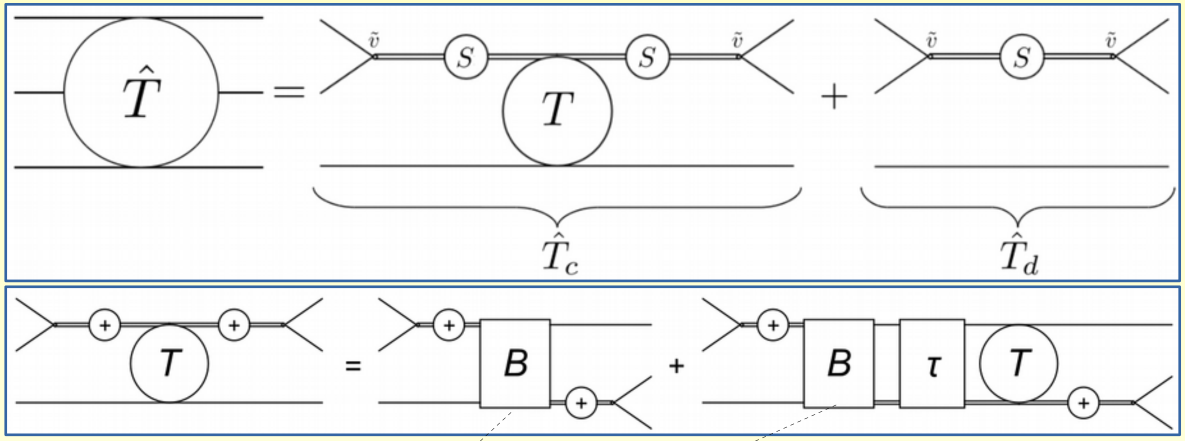


Tower of boosted 2→2 amplitudes
to implement 3-body quantization condition



SCATTERING AMPLITUDE

$3 \to 3$ scattering amplitude is a 3-dimensional integral equation



– Imaginary parts of ***B, S*** are fixed by **unitarity/matching**

– For simplicity ***υ=λ*** (full relations available)

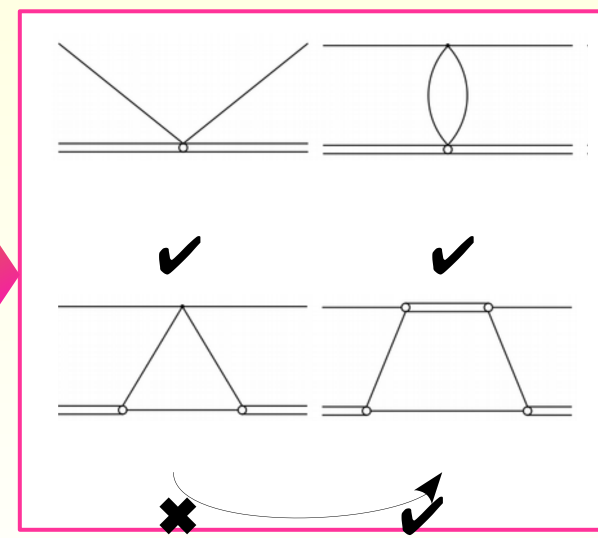
$$\text{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}}$$

- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q-\sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)}$$

- one-**π** exchange in TOPT → ***RESULT!***

?



Unitarity & Matching

- 3-body Unitarity (normalization condition ↔ phase space integral)

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$