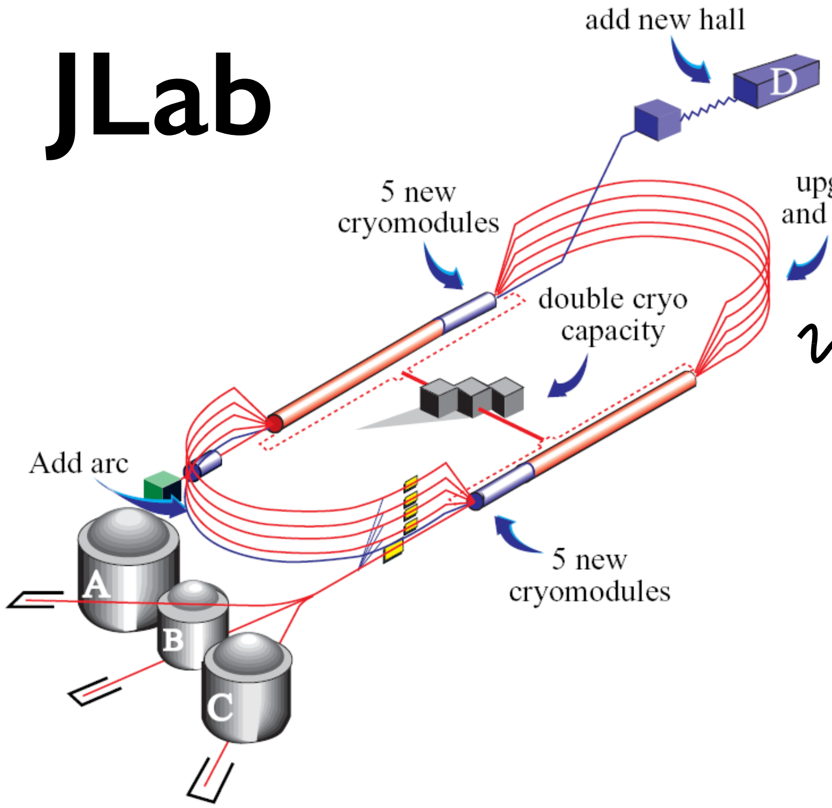
Relativistic three-particle states in a box

Maxwell T. Hansen
CERN

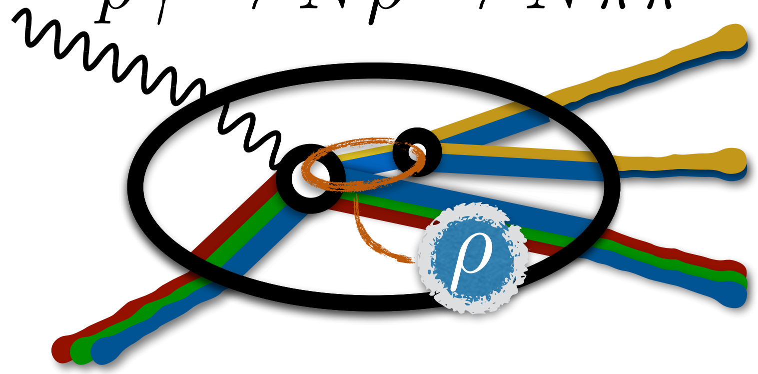
March 12th, 2018

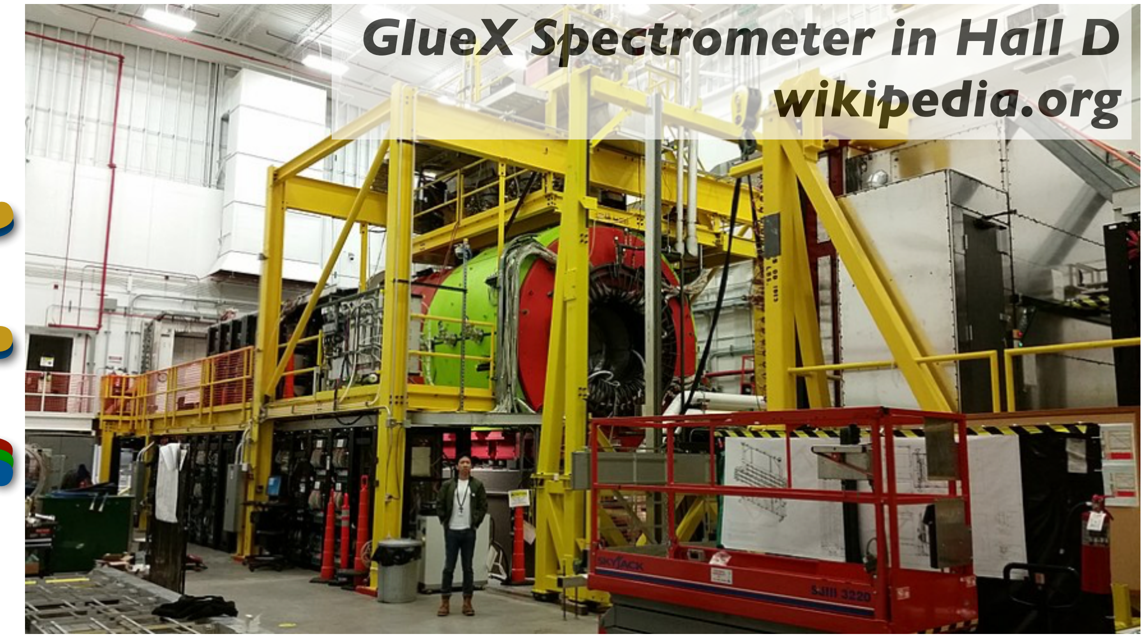
JLab



Hall D (GlueX)

$$p\gamma \to N\rho \to N\pi\pi$$





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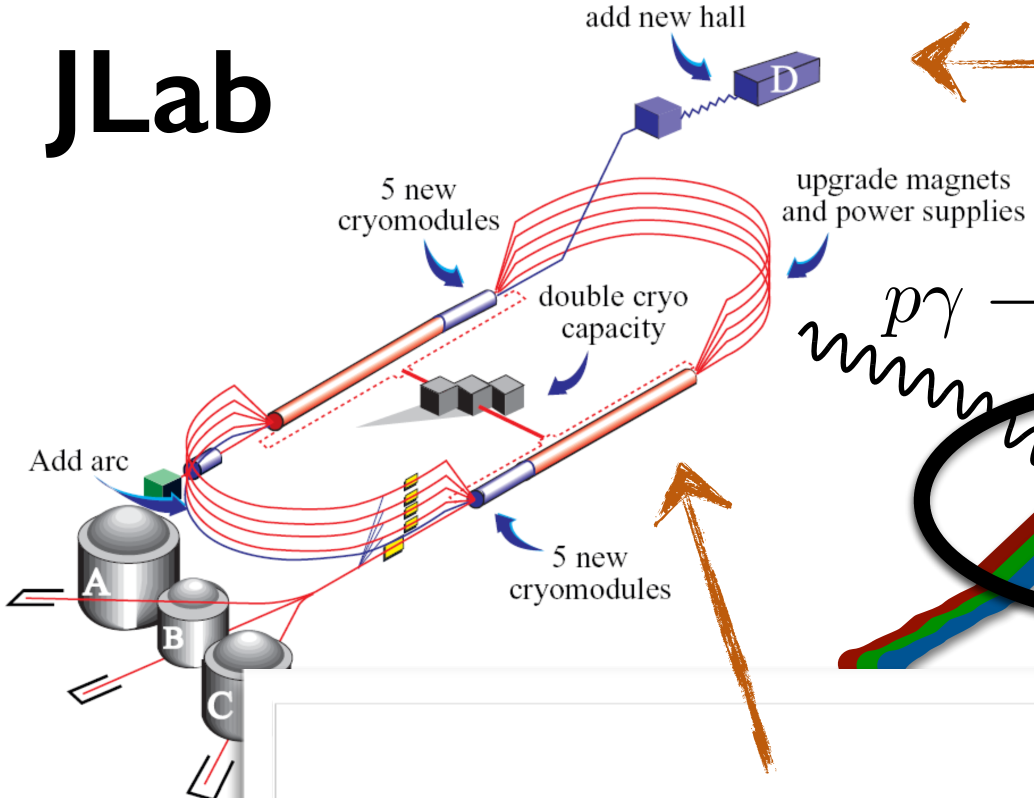
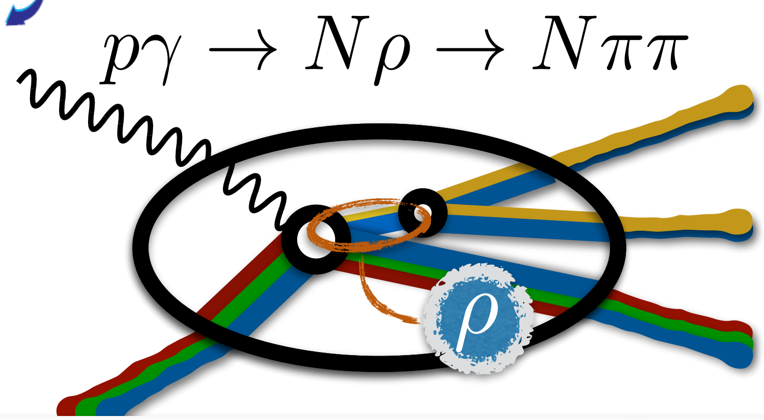
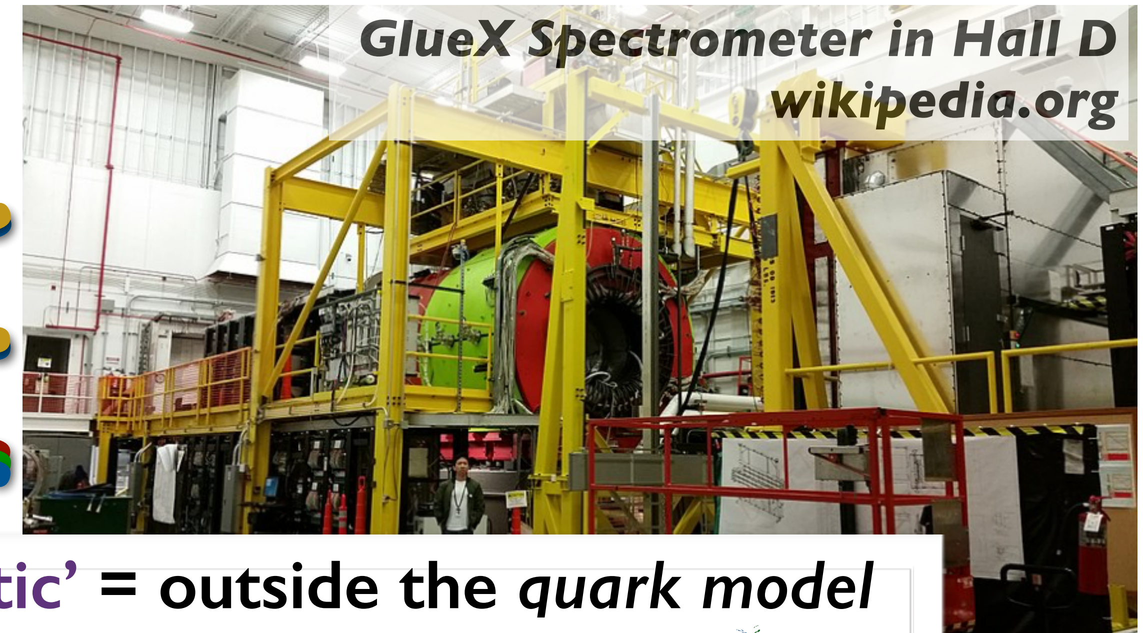
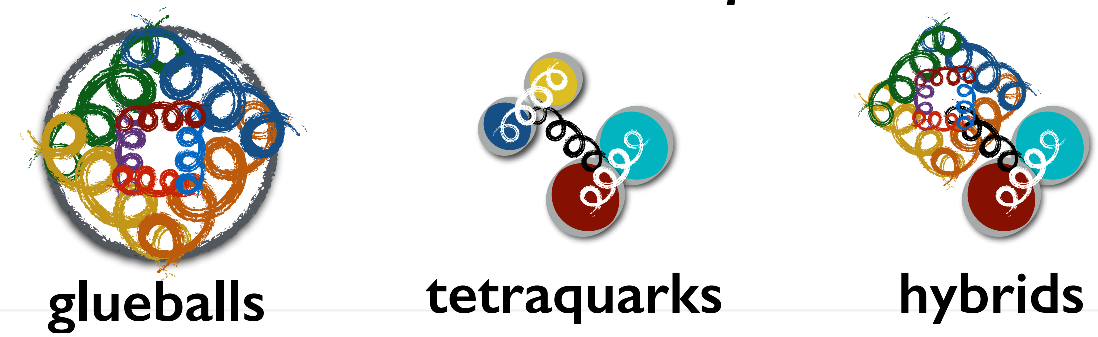






9 GeV photons, ideal for probing **exotic resonance spectrum** of the **strong force**

'exotic' = outside the *quark model*



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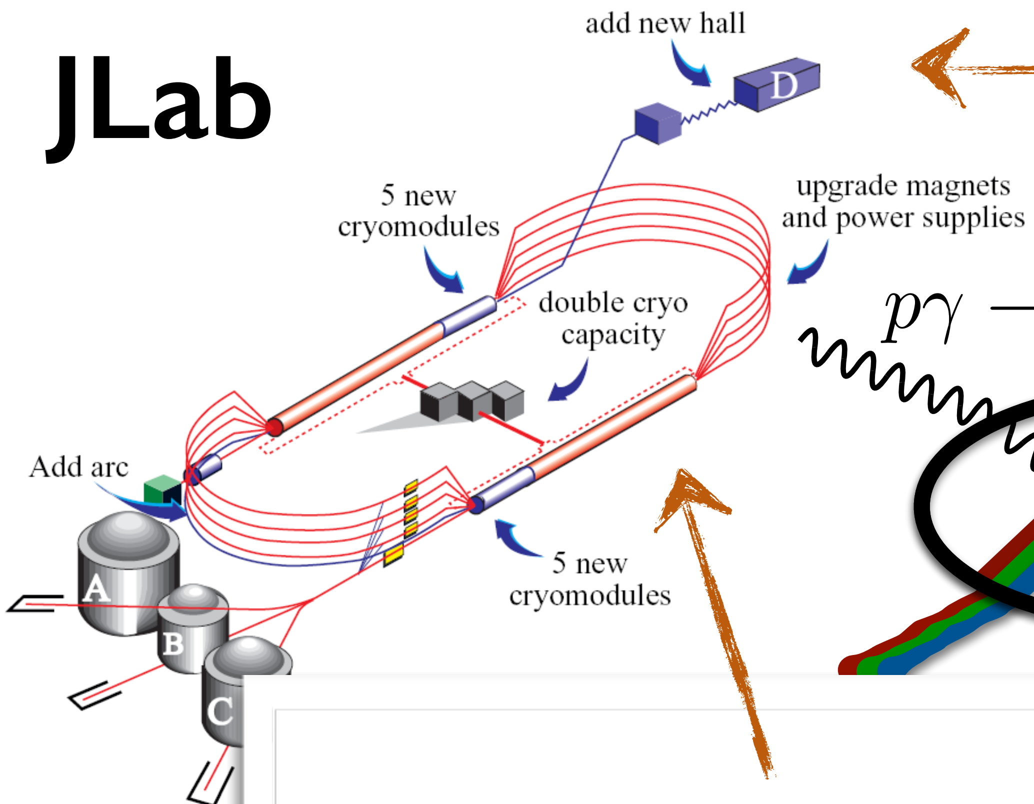

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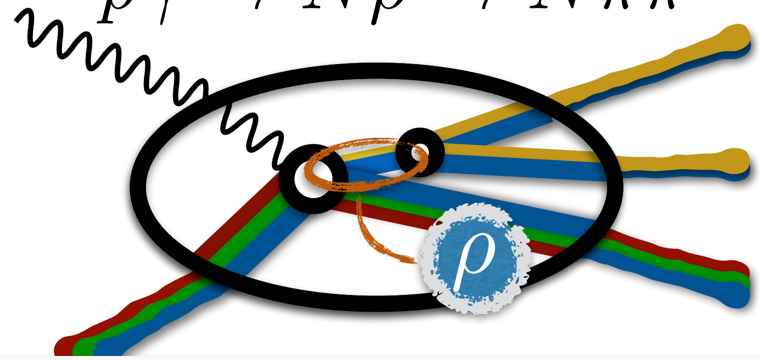

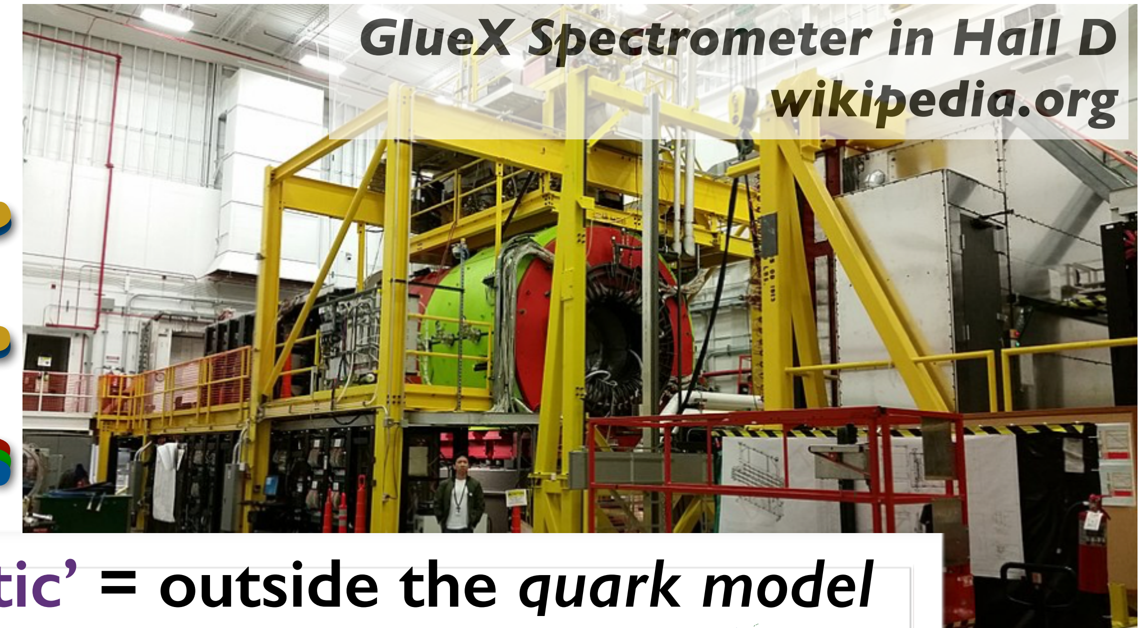

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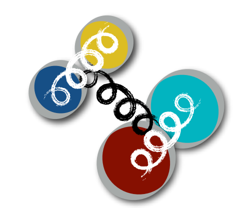
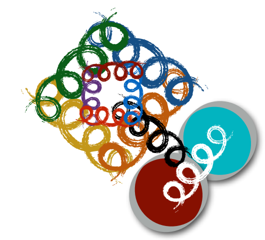
glueballs

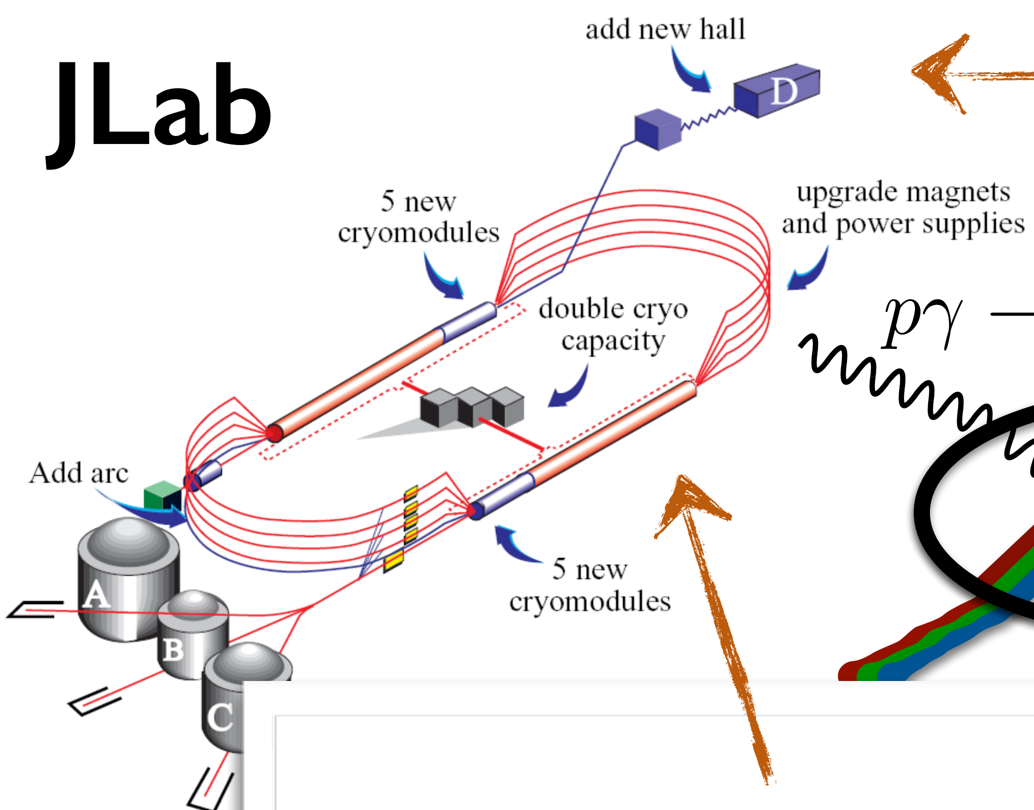

tetraquarks


hybrids

But we know the quark model is not the fundamental theory of the strong force…
Instead, the detailed behavior should be given by an elegant mathematical structure called

Quantum Chromodynamics (QCD)

JLab



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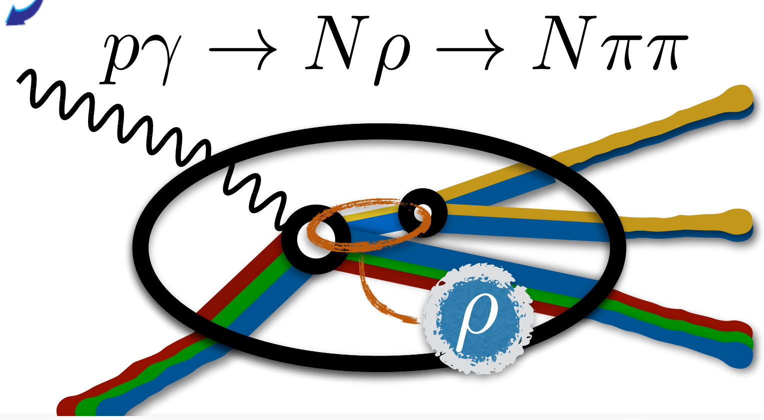

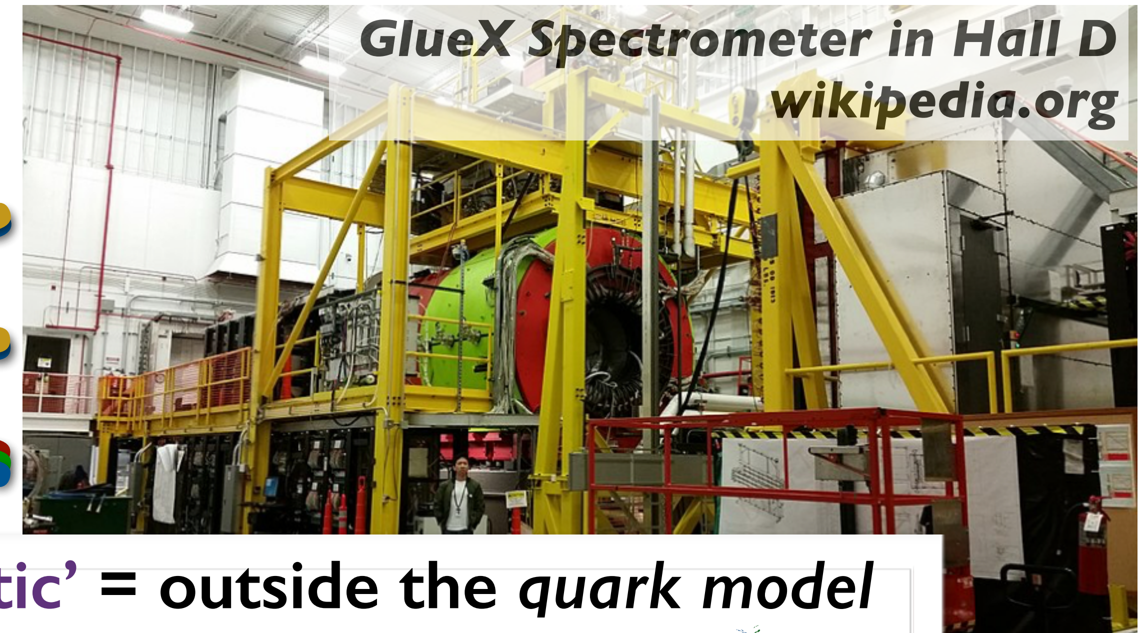

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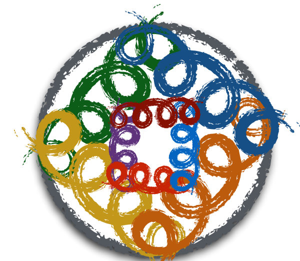

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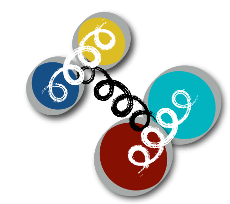

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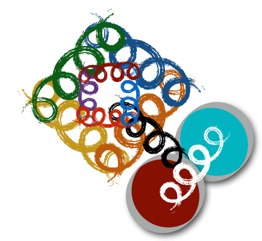

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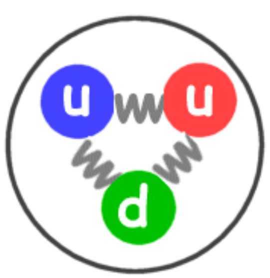
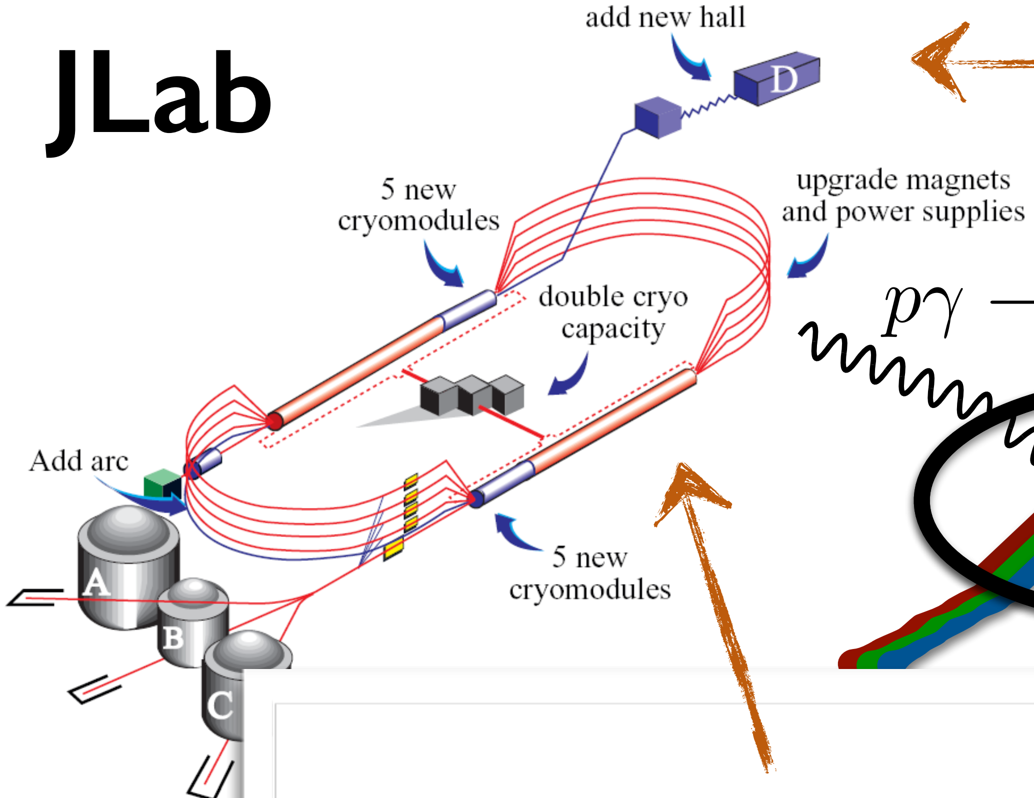


Theory of quarks & gluons with interactions highly constrained by symmetries

JLab



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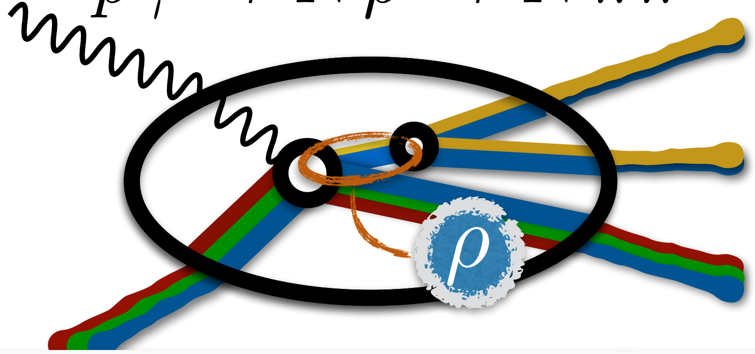

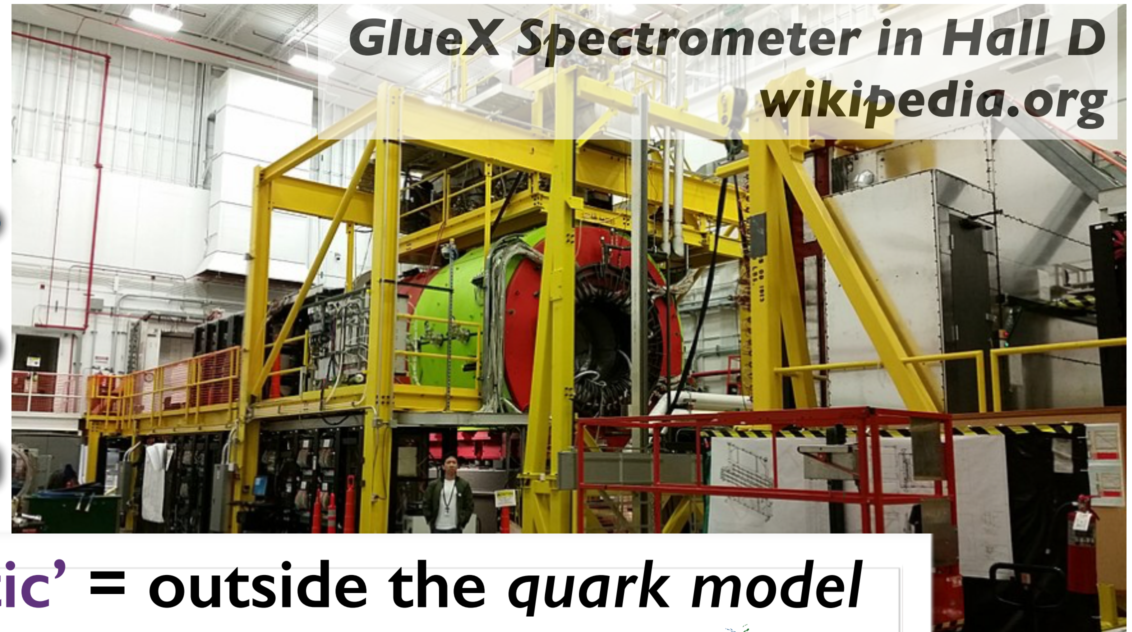

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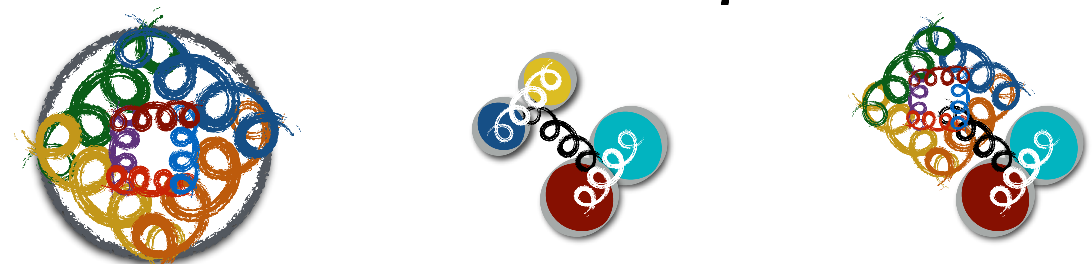

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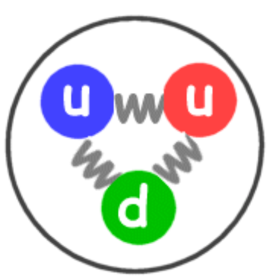

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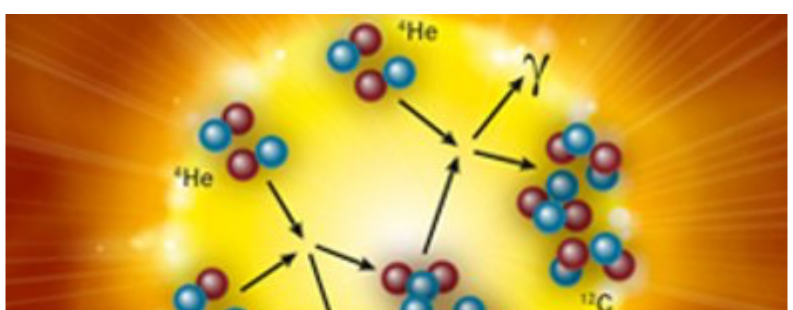
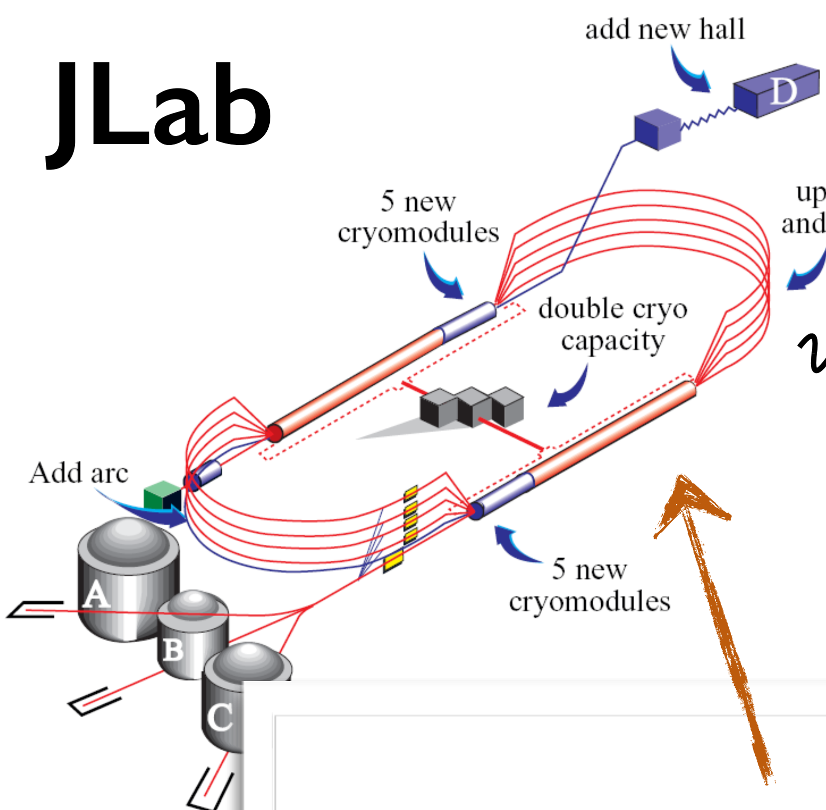


Jülich Forschungszentrum

Simple underlying structure leads to a rich variety of phenomena

JLab



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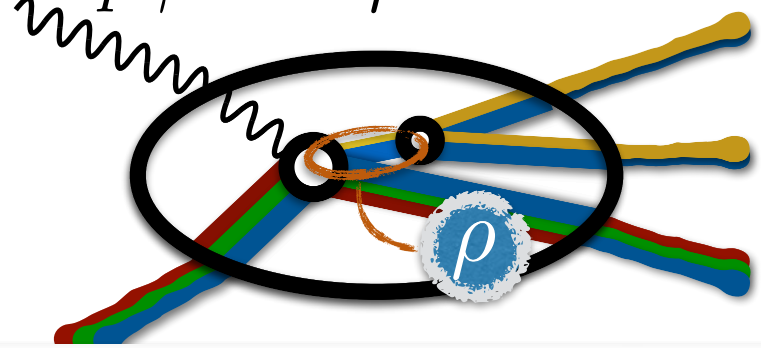

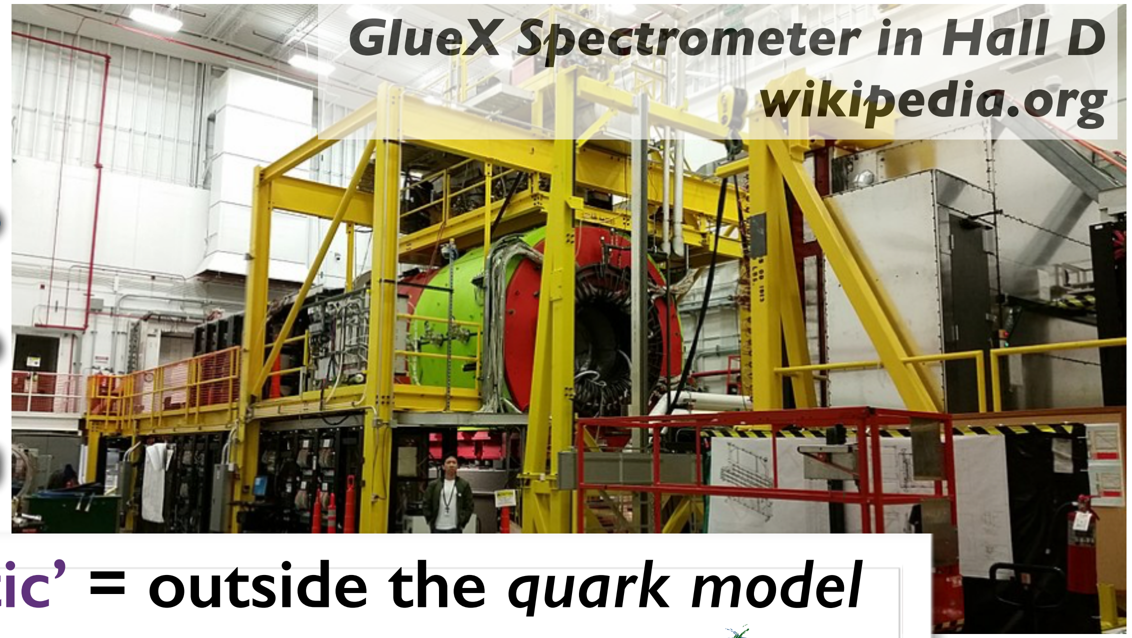

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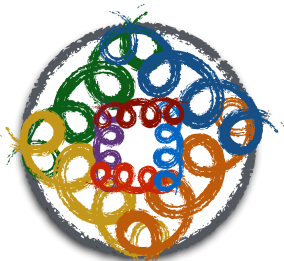

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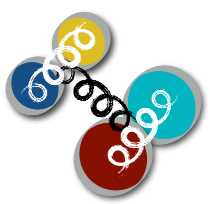

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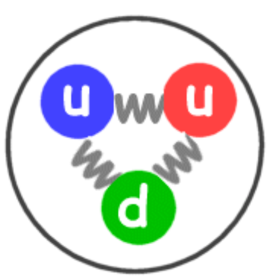

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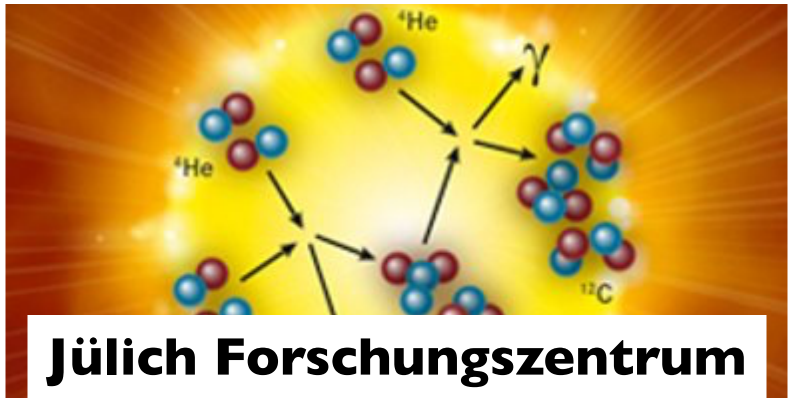


Jülich Forschungszentrum

Simple underlying structure leads to a rich variety of phenomena

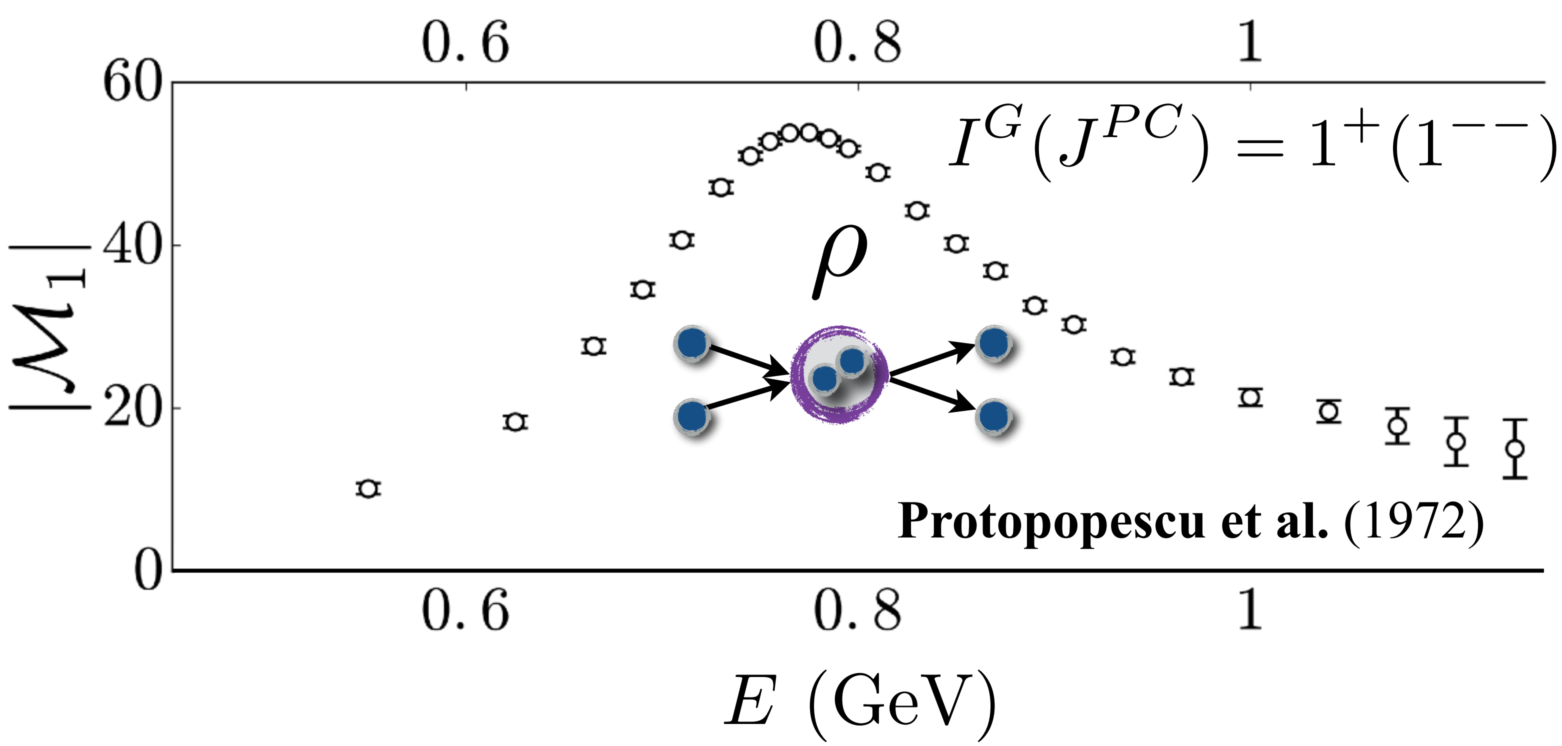
JLab

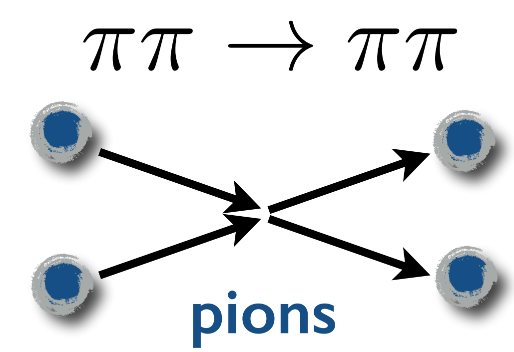
Difficult to extract predictions from the underlying theory

Resonances vs. bound states

A resonance is a bump in $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

scattering rate — unitarity relation





(lightest particle that feels the strong force)

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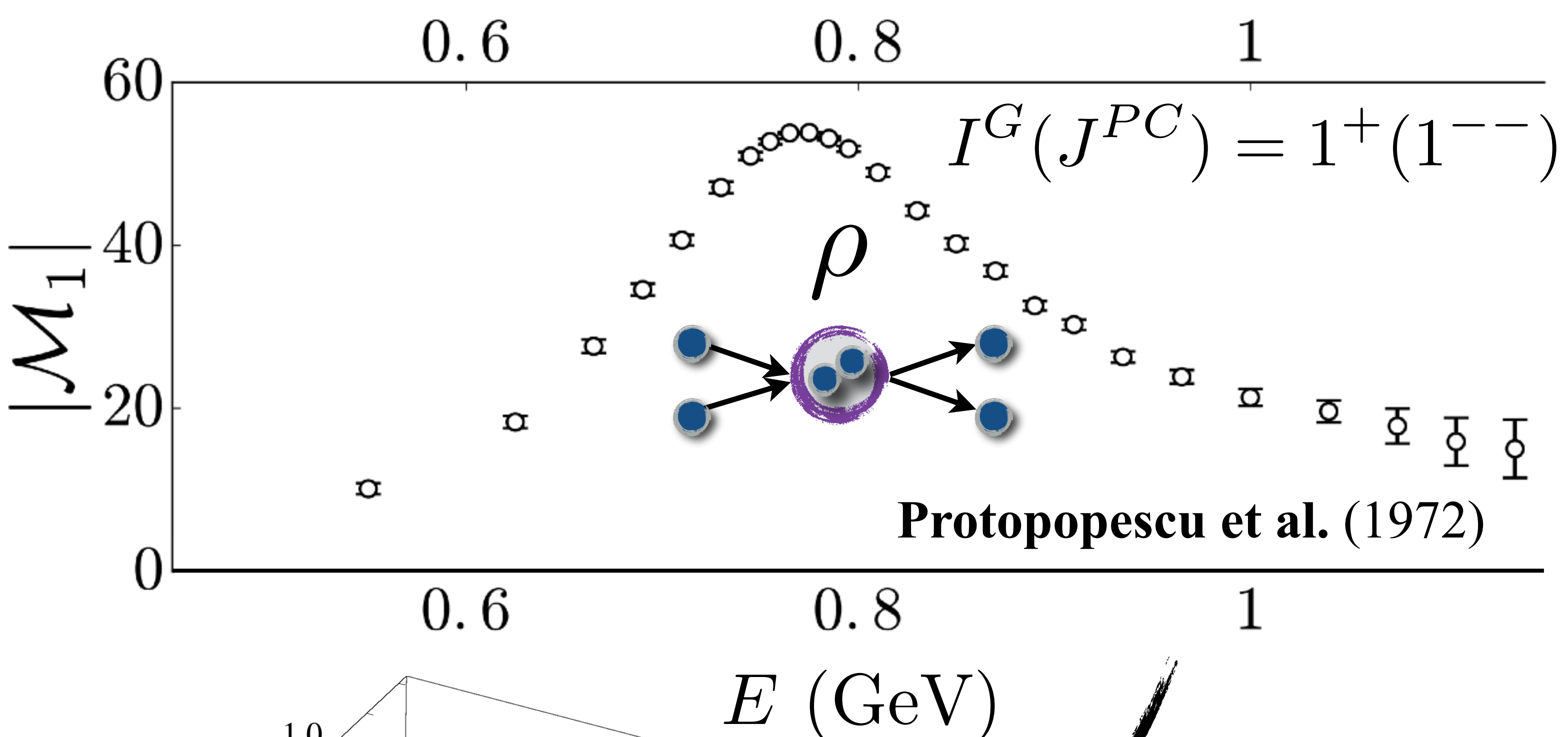

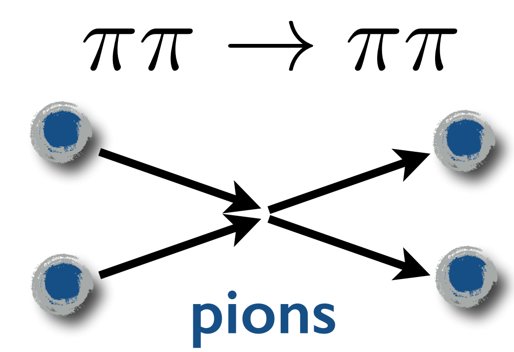

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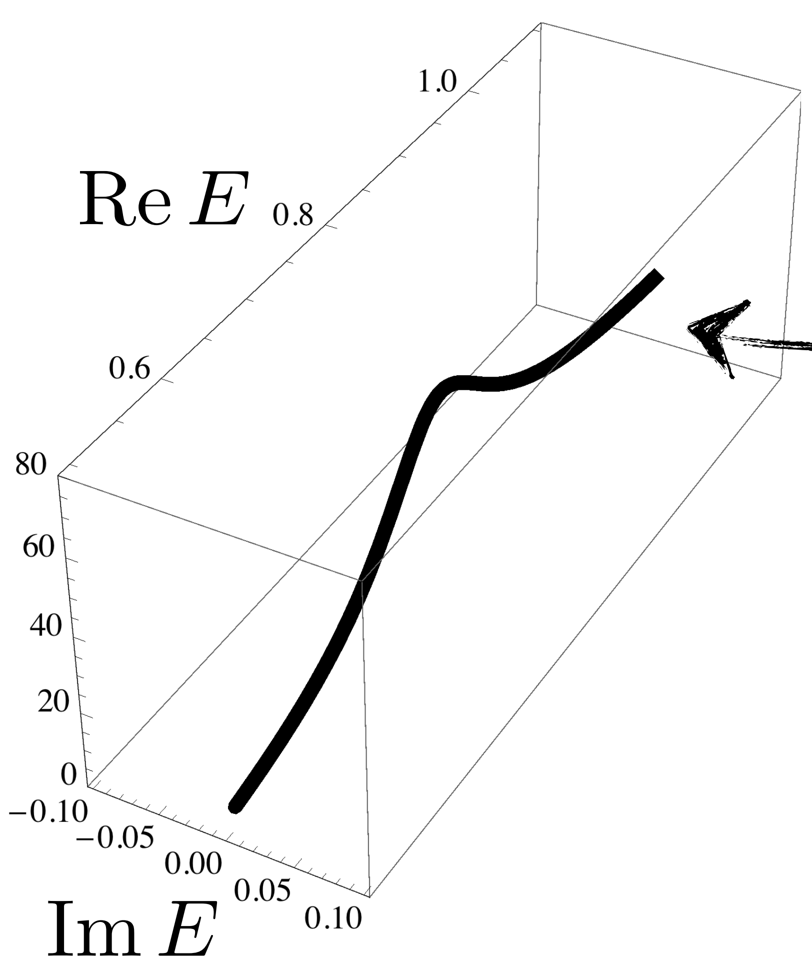

Consider this curve in the complex plane

Resonances vs. bound states

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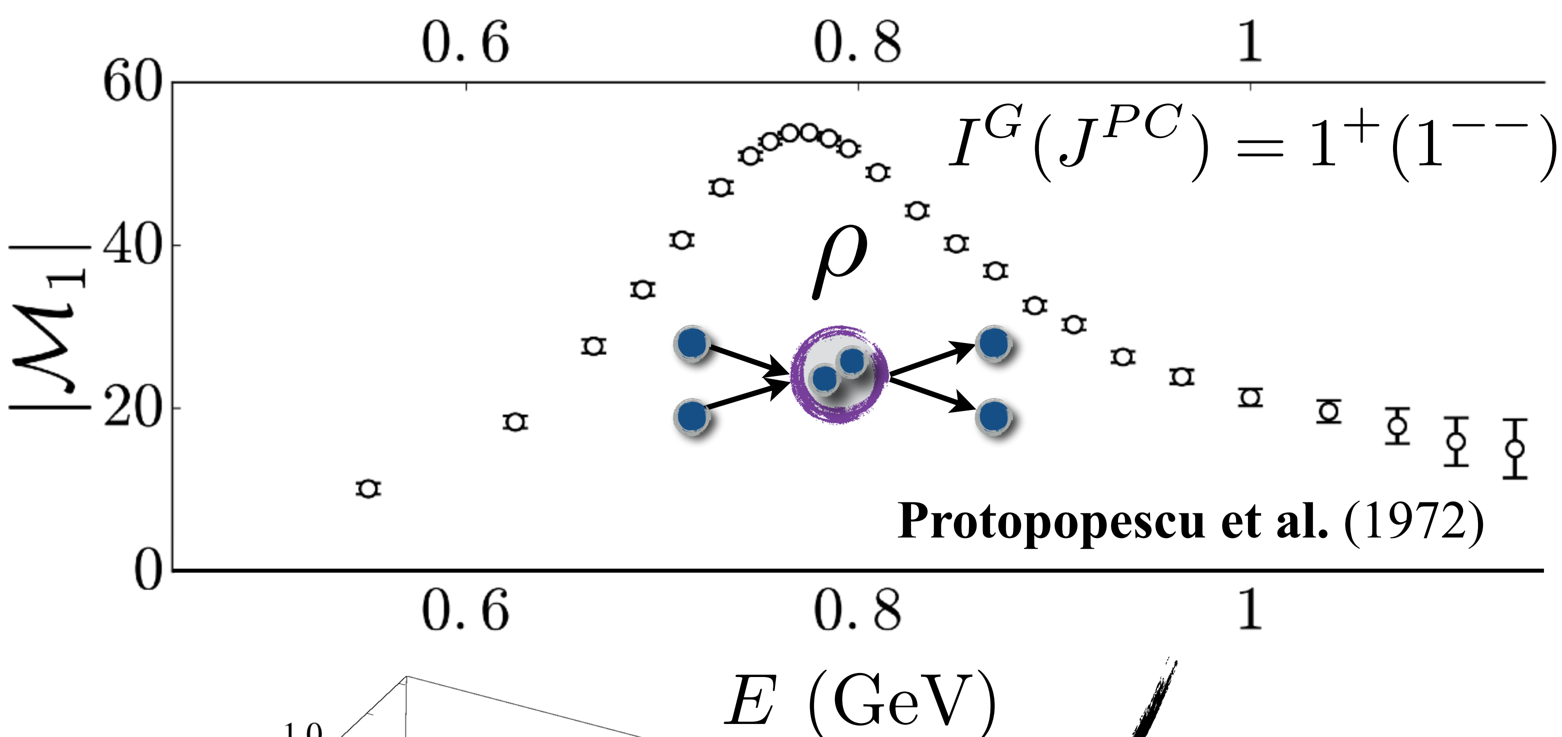

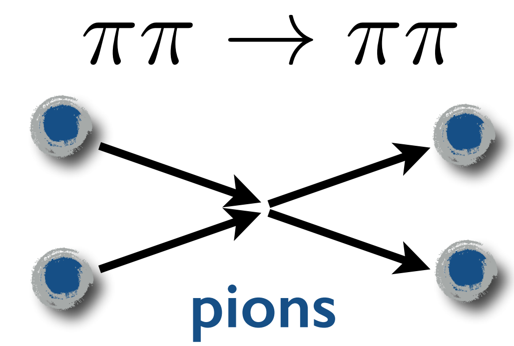

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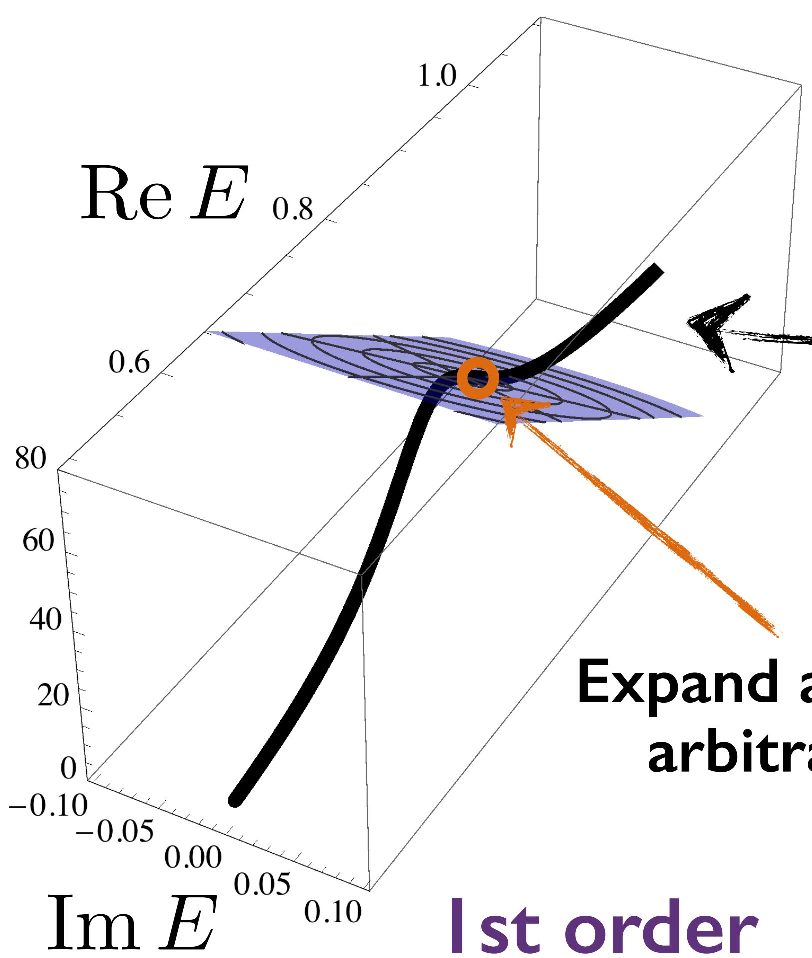

Consider this curve in the complex plane

Expand about some arbitrary point

1st order

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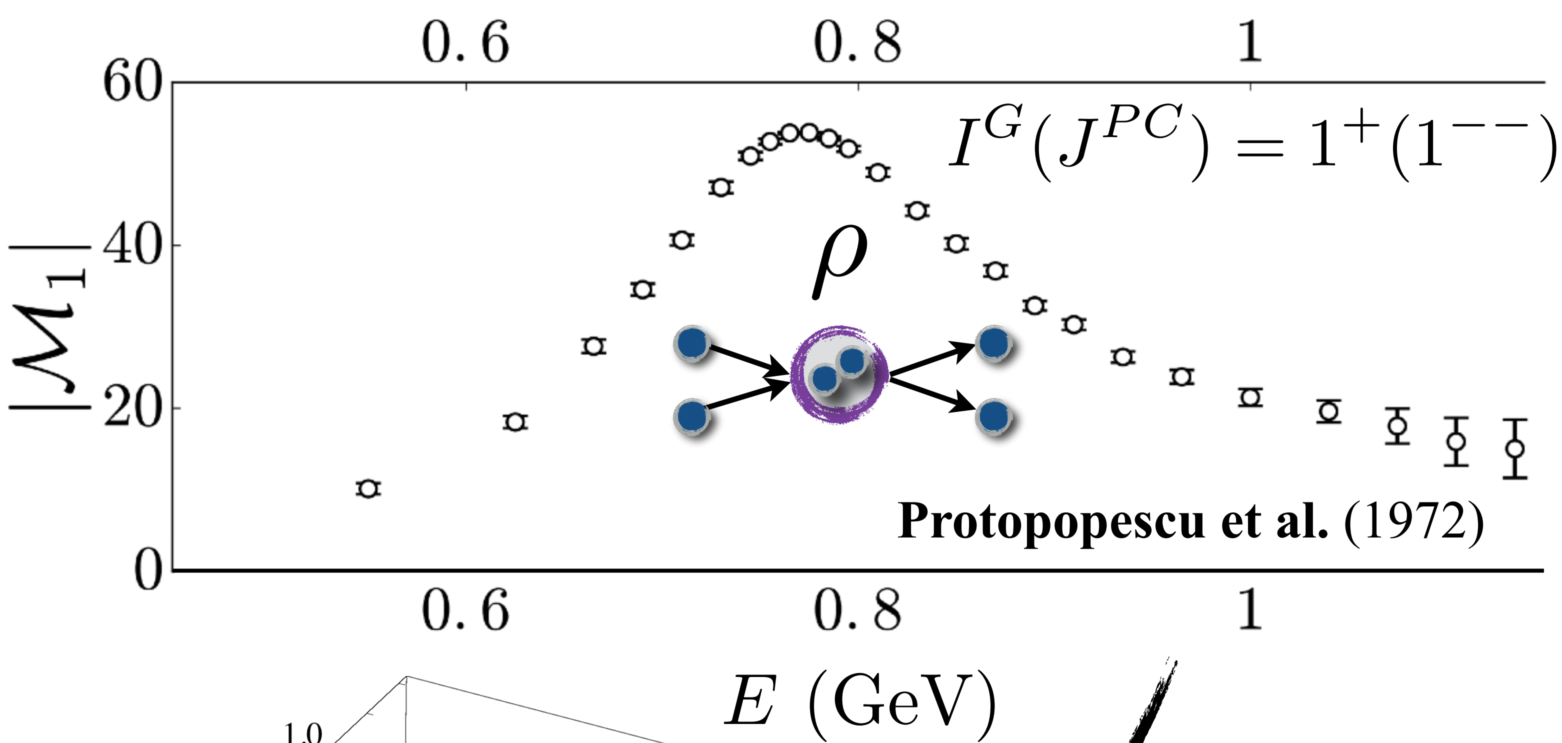
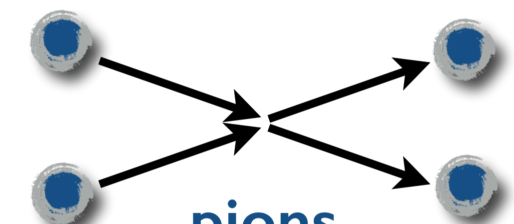


$\pi\pi \to \pi\pi$



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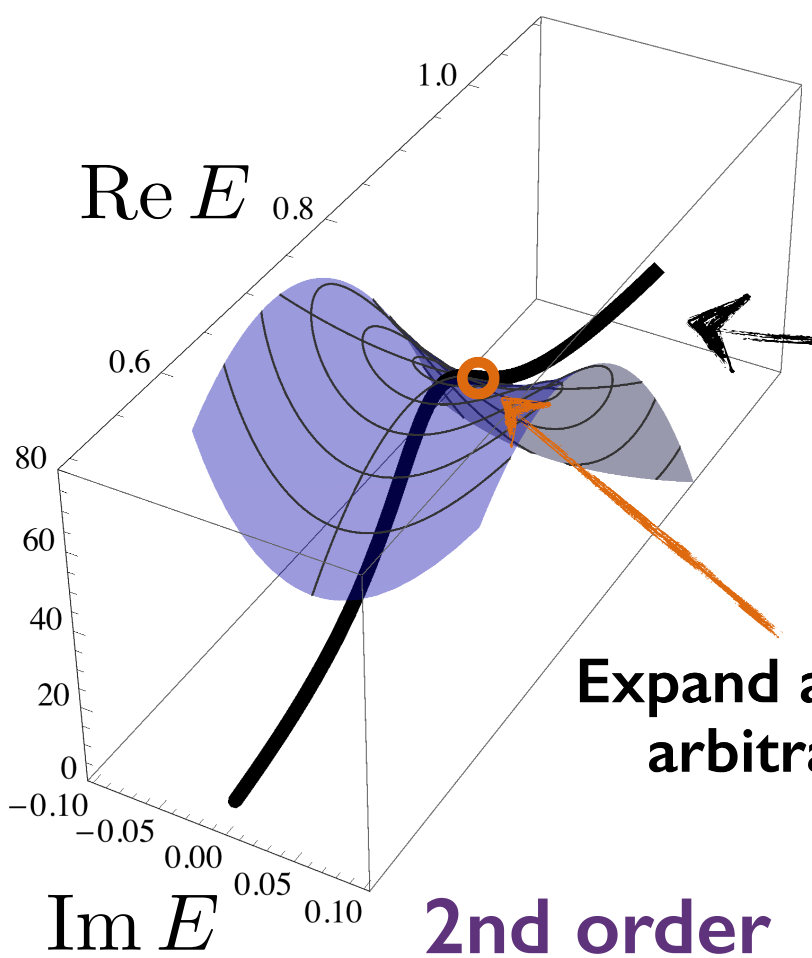

Consider this curve in the complex plane

Expand about some arbitrary point

2nd order

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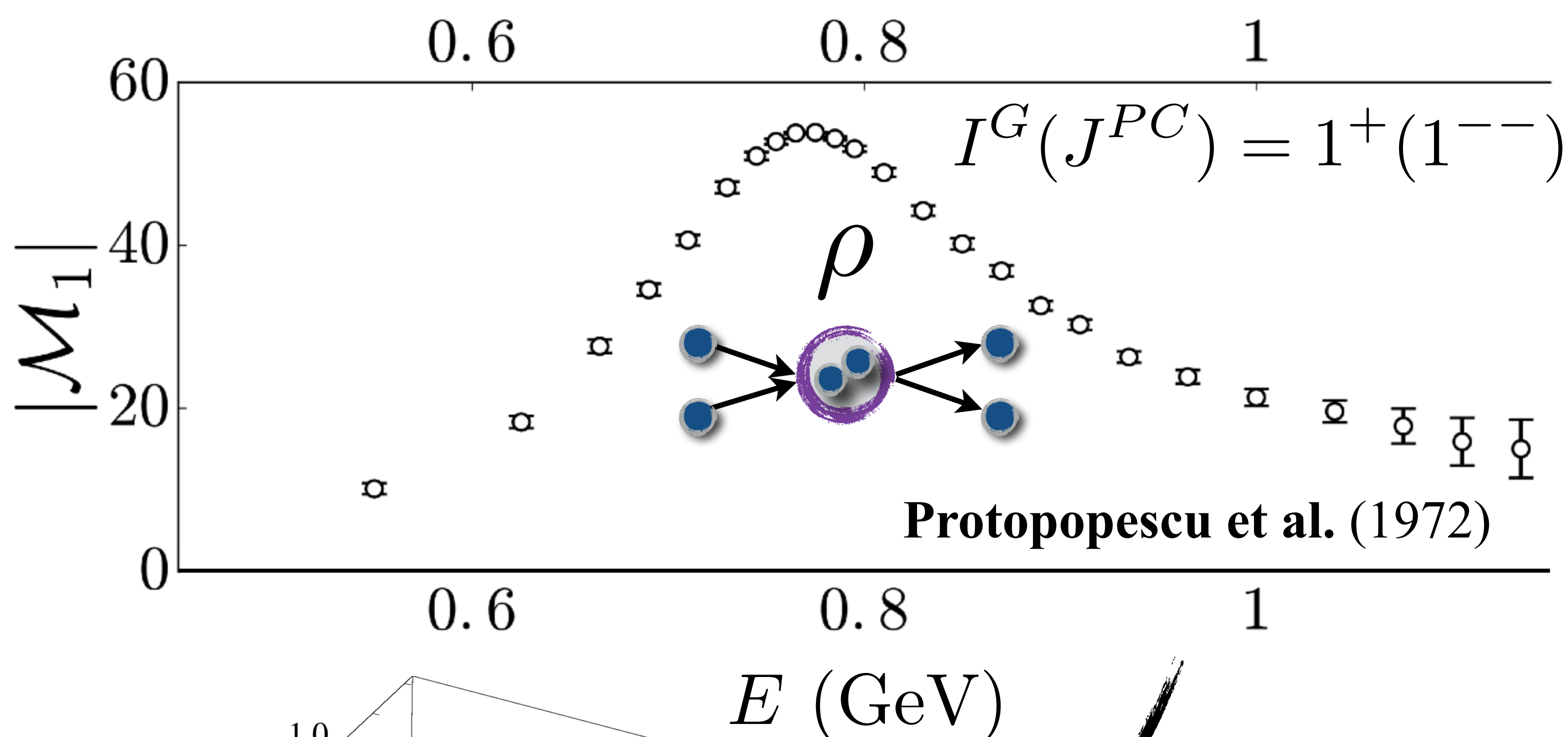
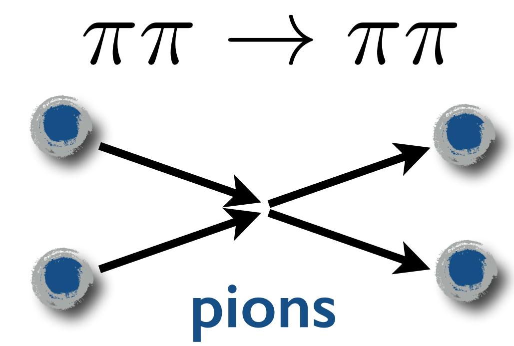
scattering rate   unitarity relation





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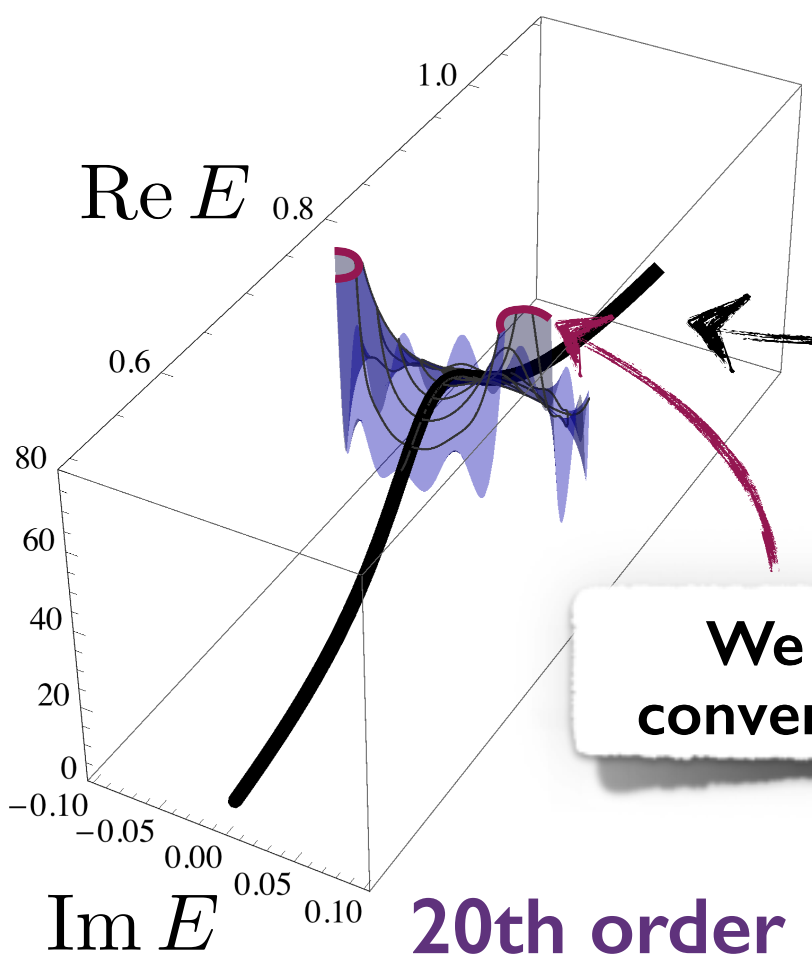

20th order

Consider this curve in the complex plane

We see a feature that limits the convergence of the Taylor expansion

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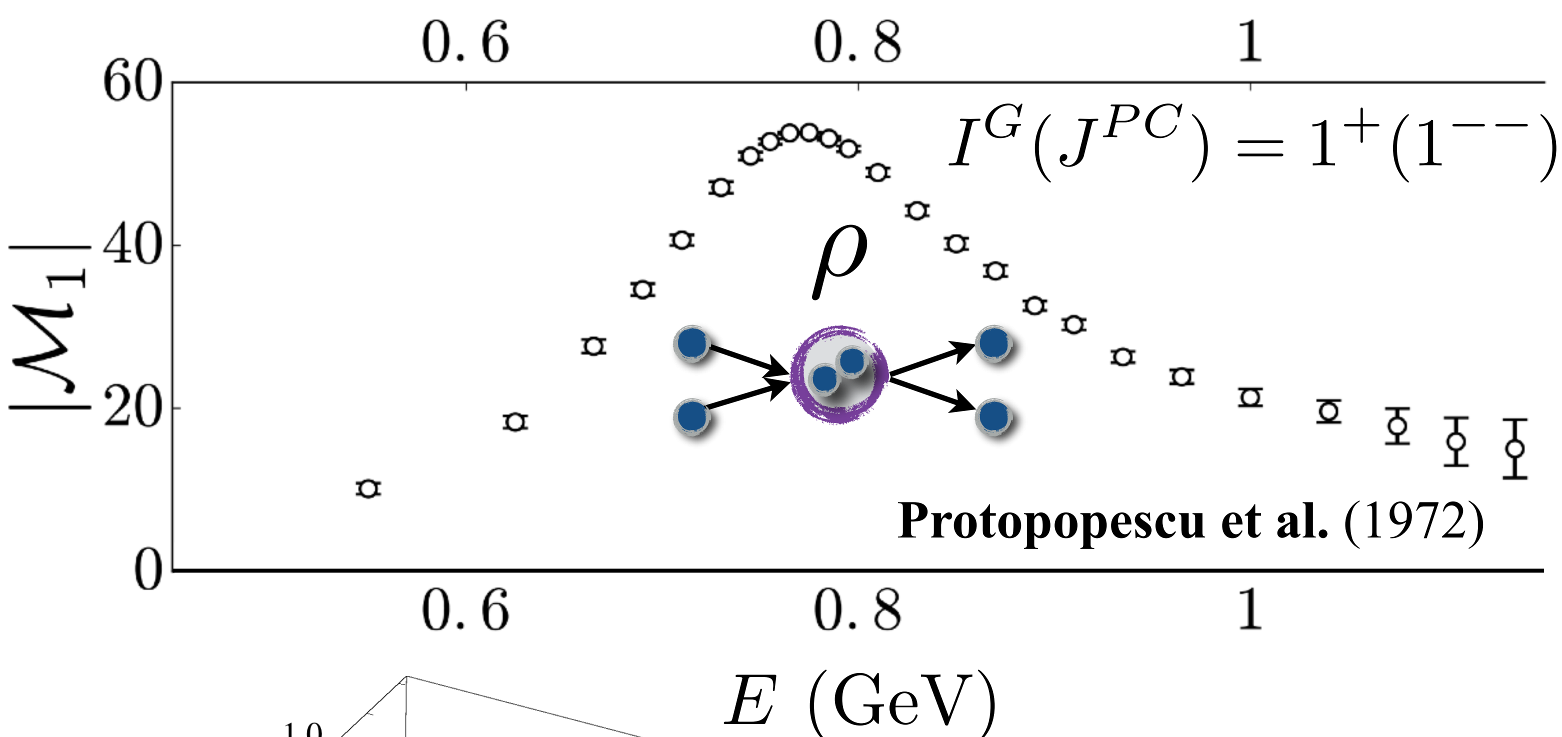

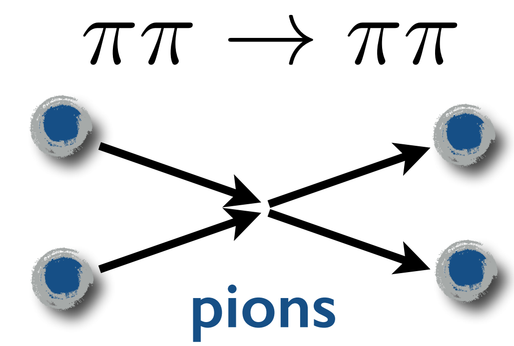

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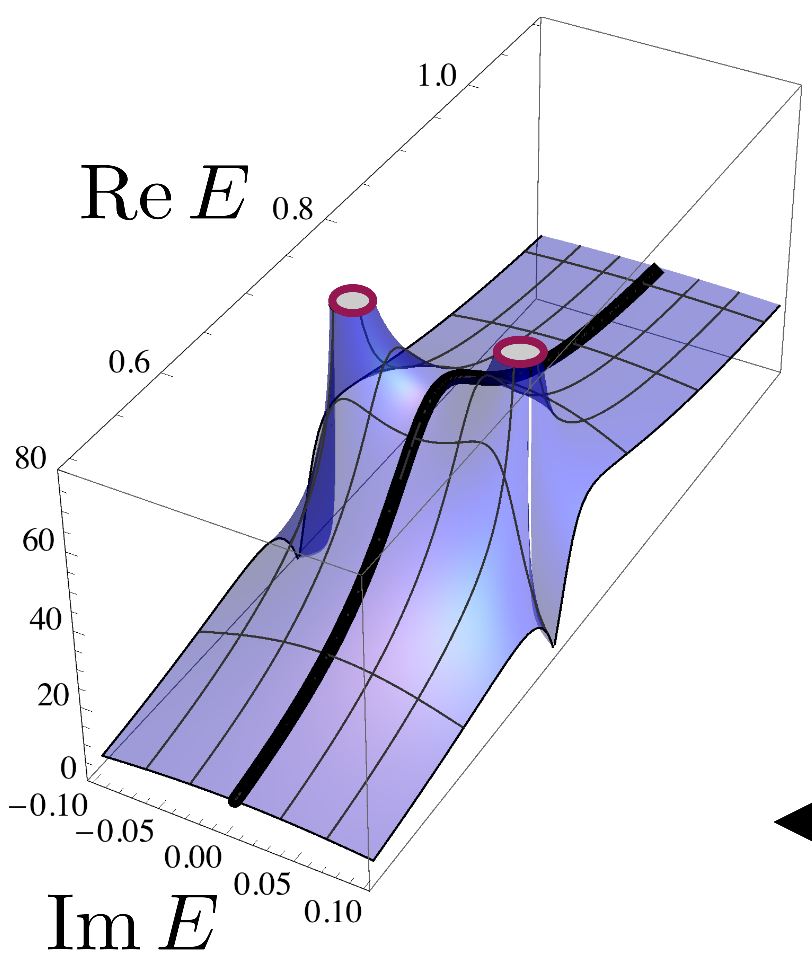
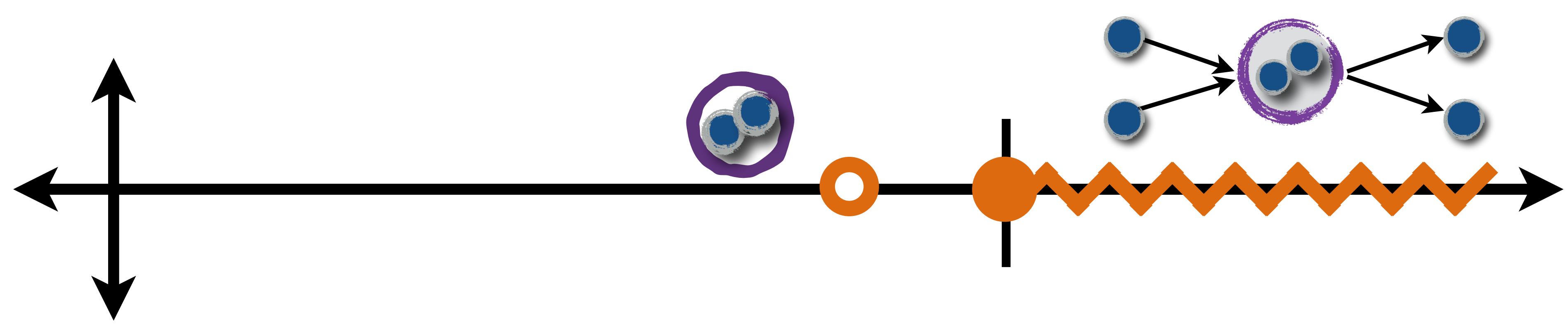


Analytic continuation reveals that the bump corresponds to a pole in the complex plane

This bridges the gap between bound states and resonances

$E_R = M_R + i\Gamma_R/2$



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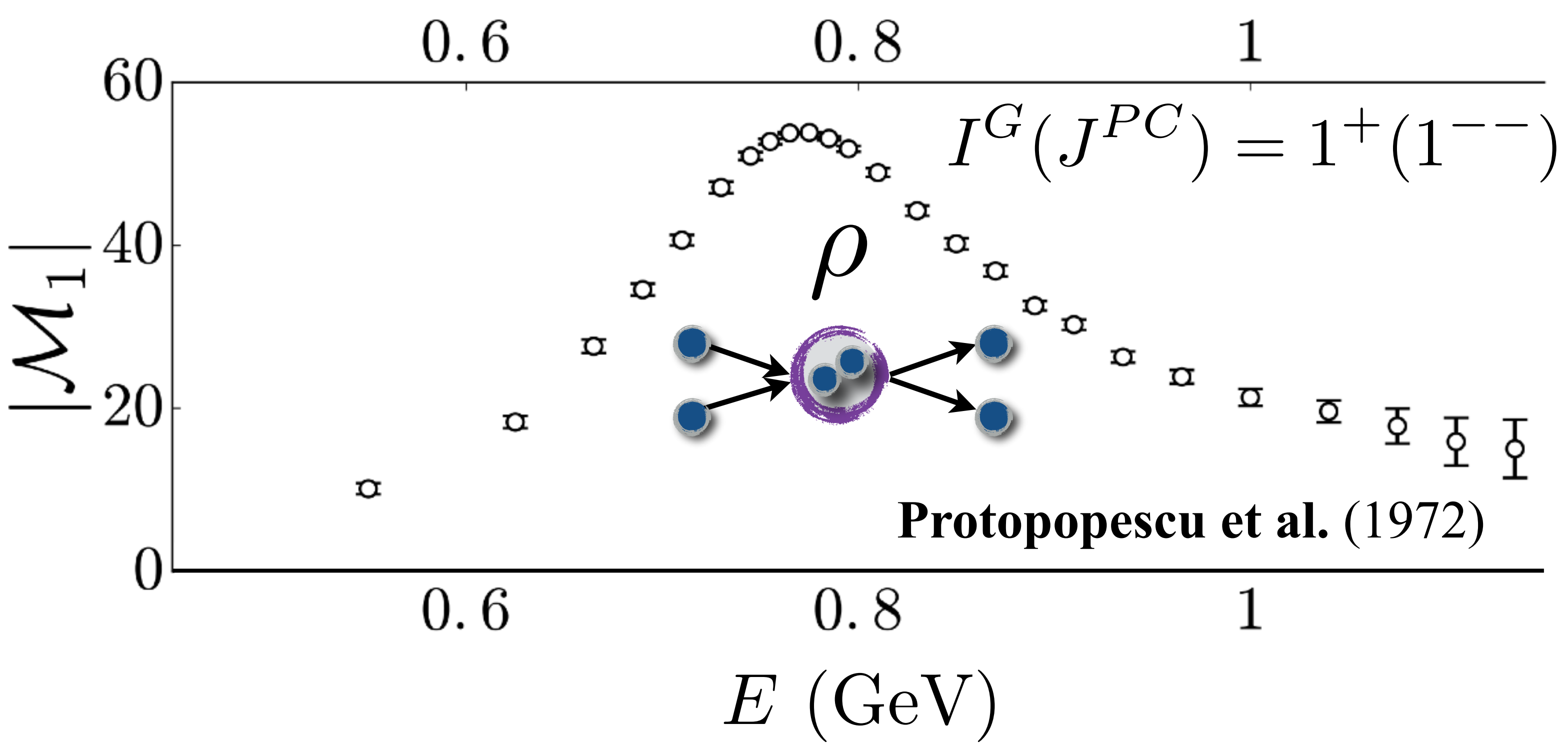

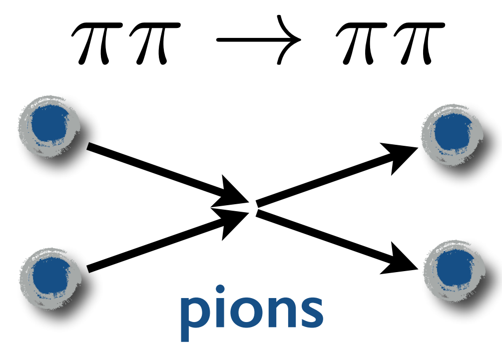

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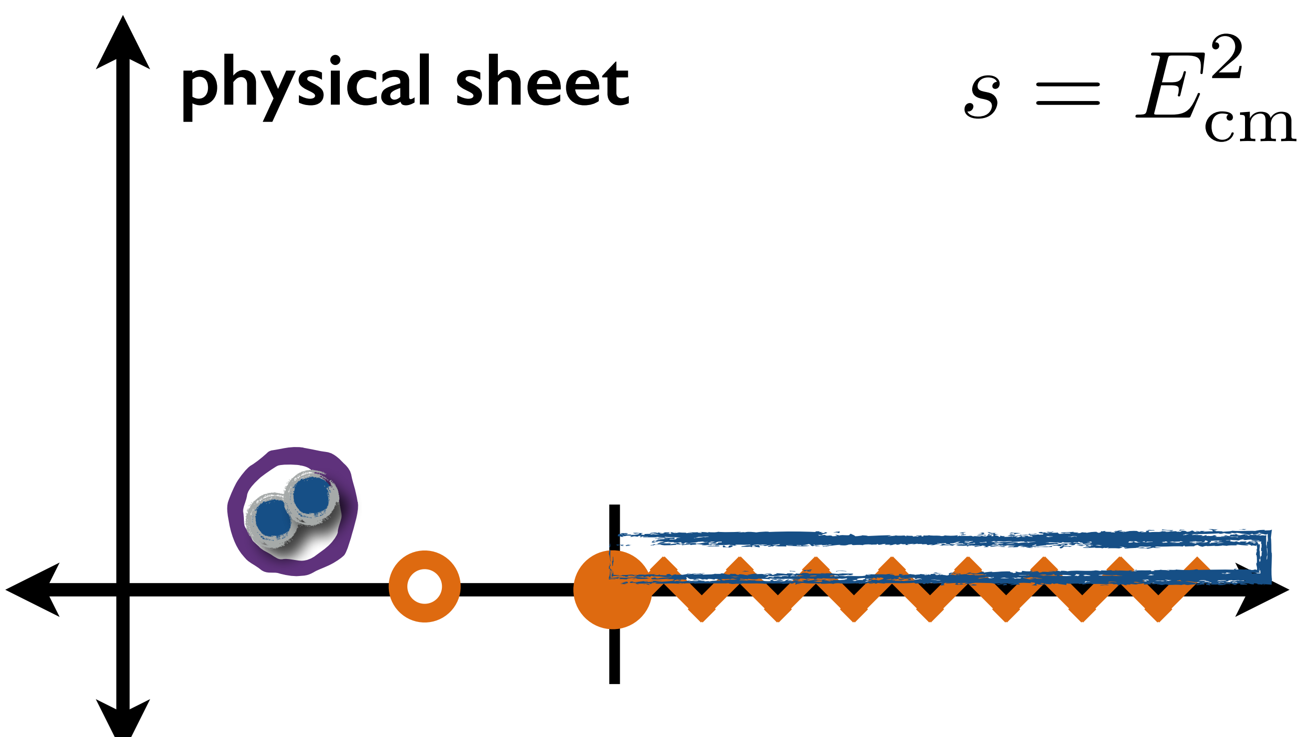
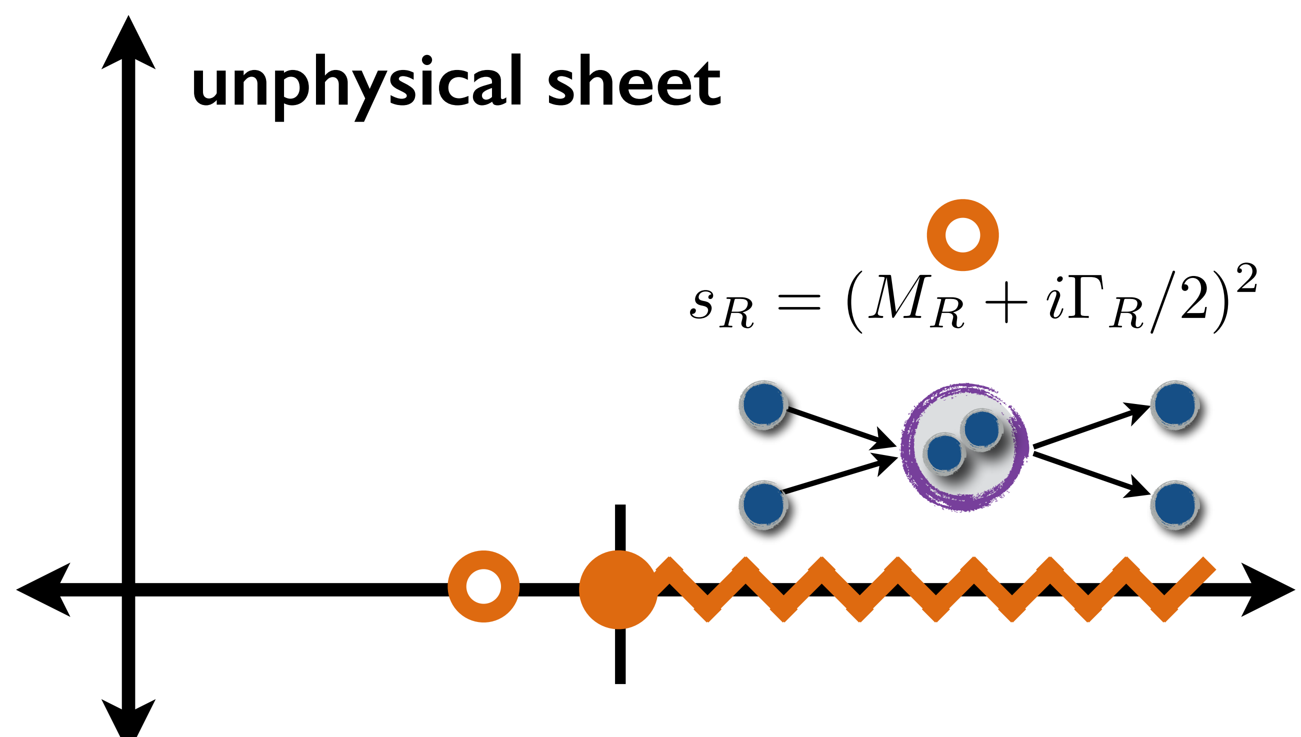
In fact, resonance poles appear on "unphysical sheet"

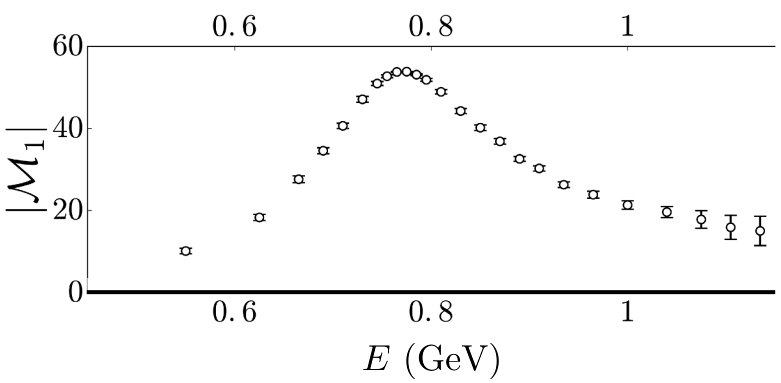


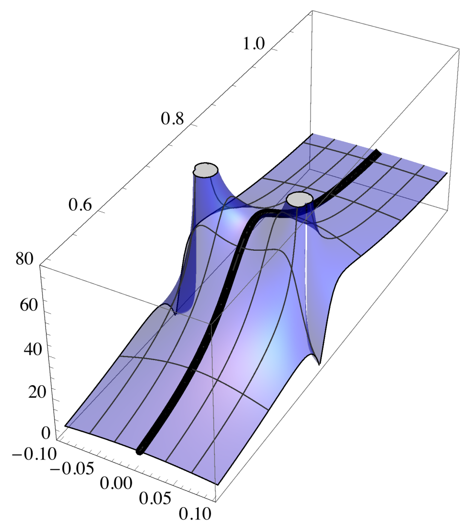


Towards a detailed, first-principles understanding of resonances

Scattering

- ☑ Determine pole positions in S-matrix entries
- ☑ Residues measure the couplings to multi-particle states





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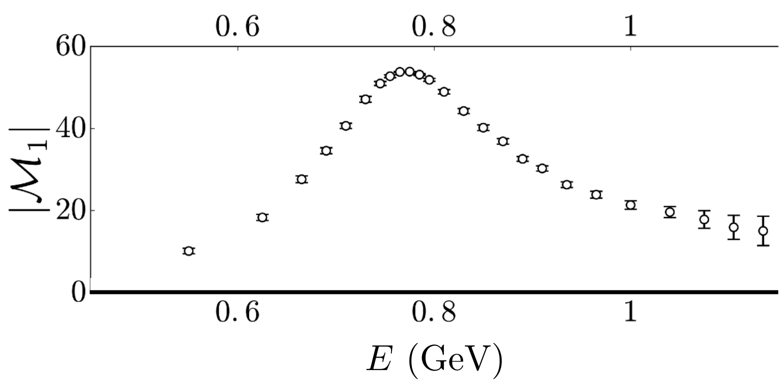
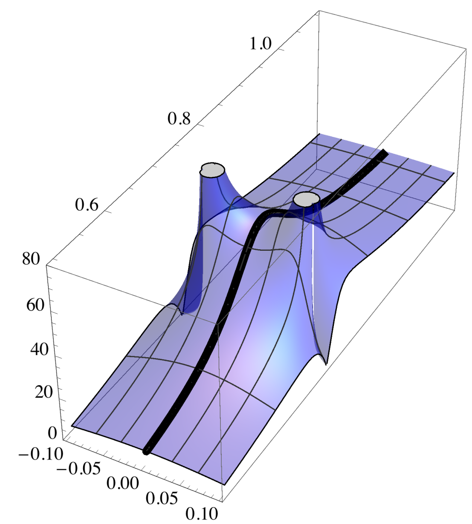
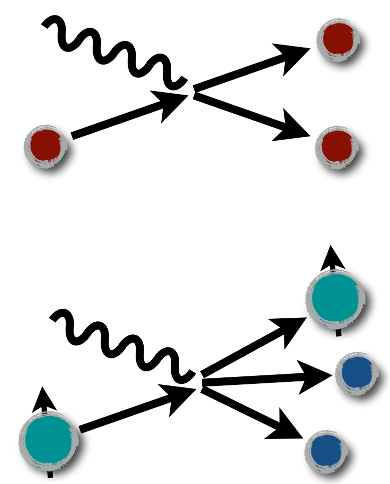




Transition amplitudes

- ☑ Measure how photons and other currents mediate exotic resonance production



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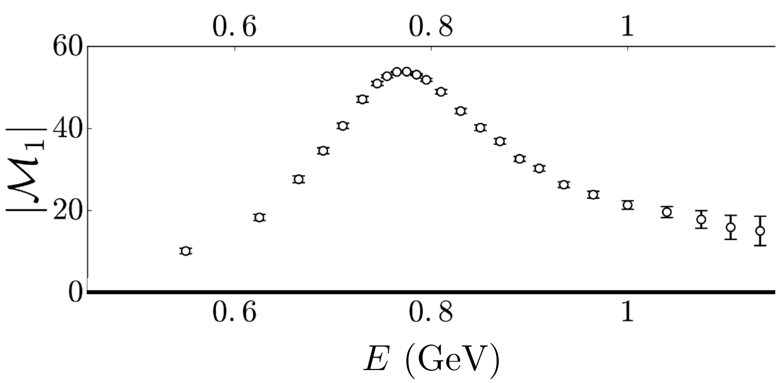

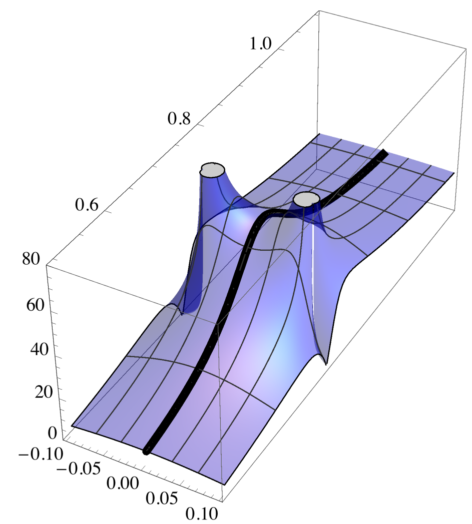

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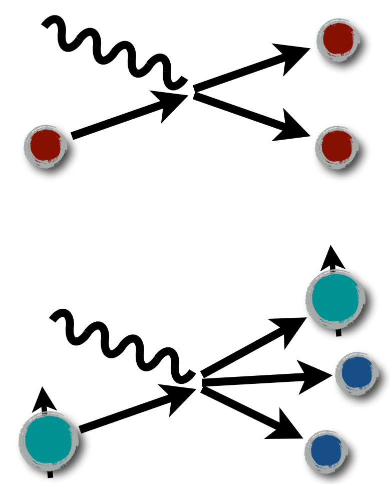
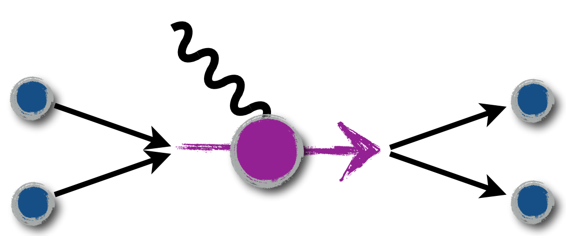
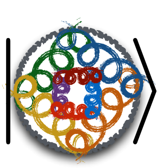
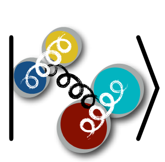
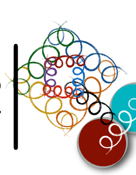


Resonant form factors

- ☑ Predict how the currents couple to the resonance

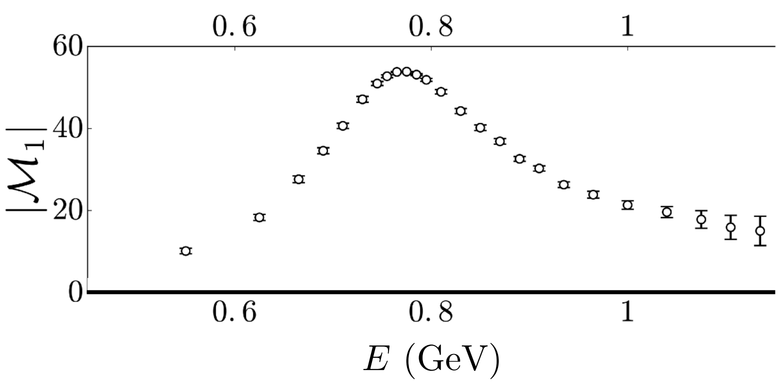


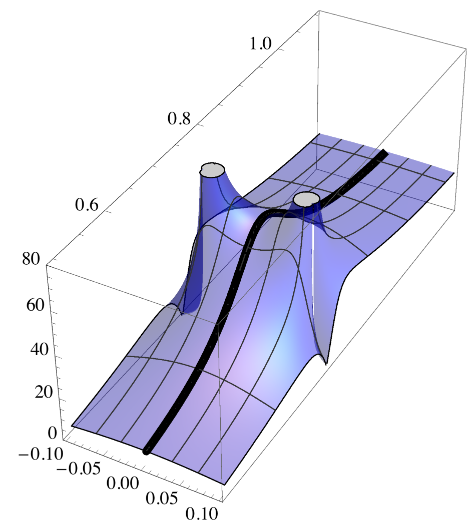
$|\mathrm{Res}\rangle = a|$  $\rangle + b|$  $\rangle + c|$  $\rangle$

Towards a detailed, first-principles understanding of resonances

Scattering

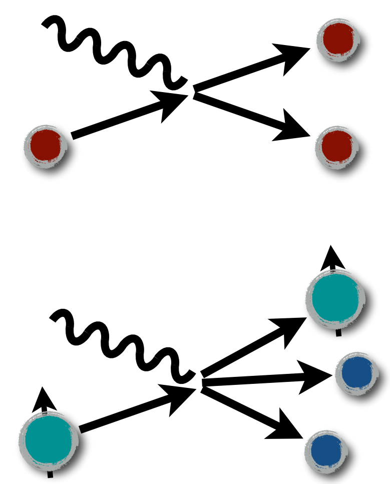
- ☑ Determine pole positions in S-matrix entries
- ☑ Residues measure the couplings to multi-particle states





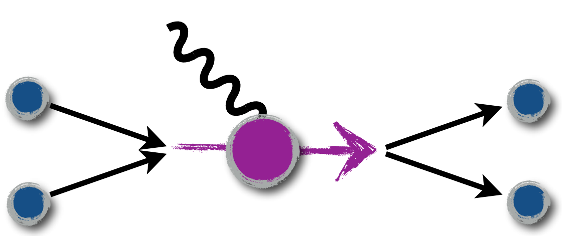
Transition amplitudes

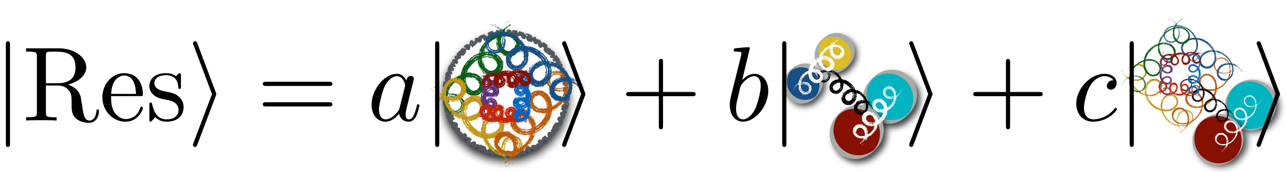
- ☑ Measure how photons and other currents mediate exotic resonance production



Resonant form factors

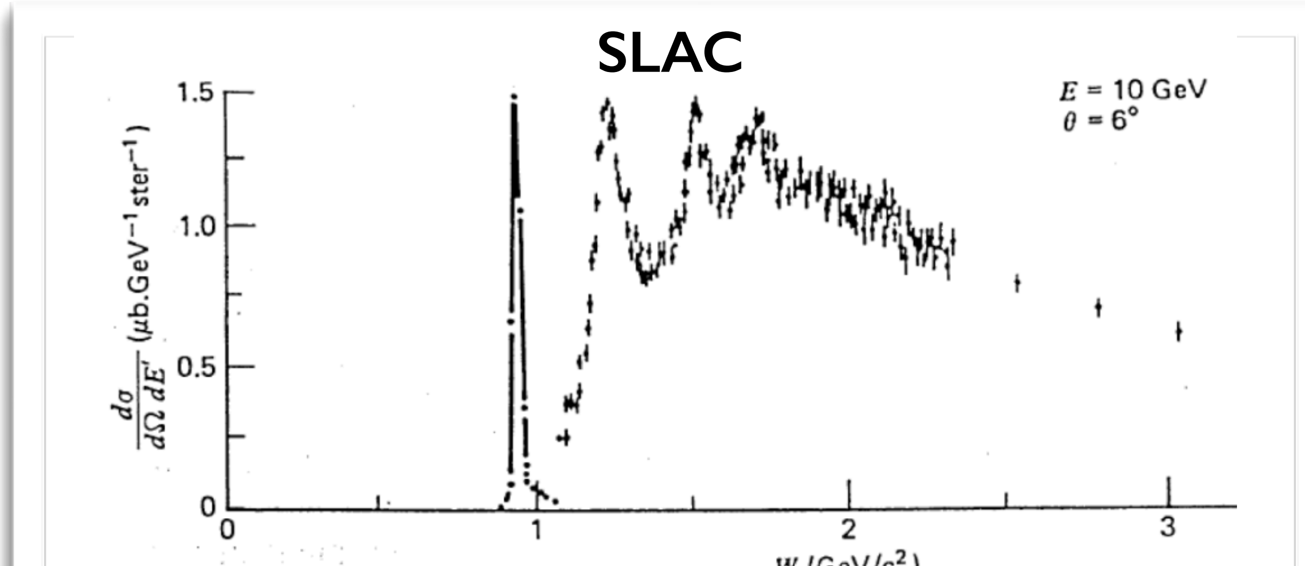
- ☑ Predict how the currents couple to the resonance





From resonances to high-energy scaling

- ☑ Explore the transition from resonance to high energy regimes



taken from Halzen and Martin

Towards a detailed, first-principles understanding of resonances

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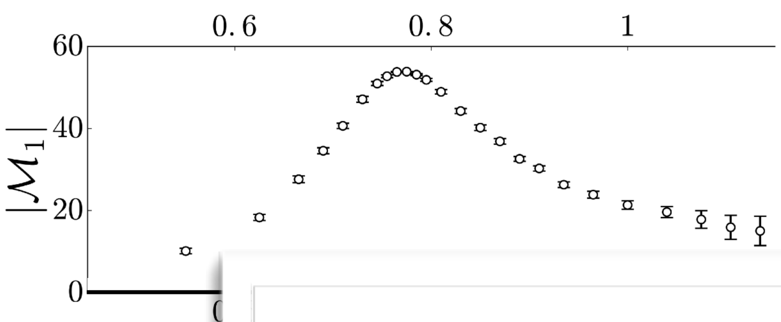

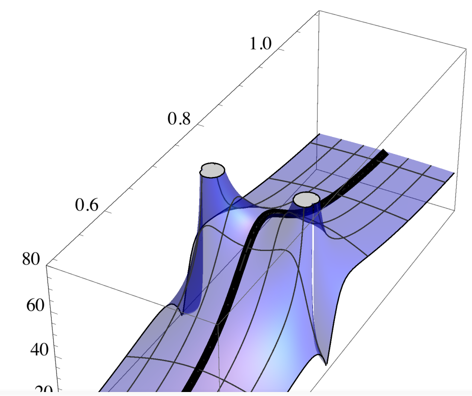

Transition amplitudes

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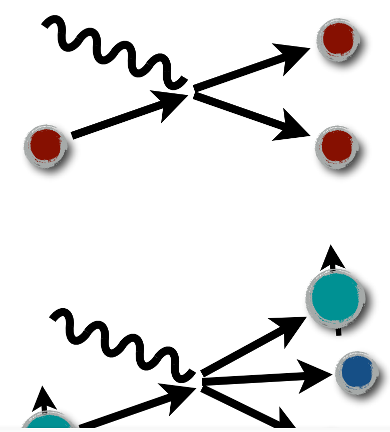
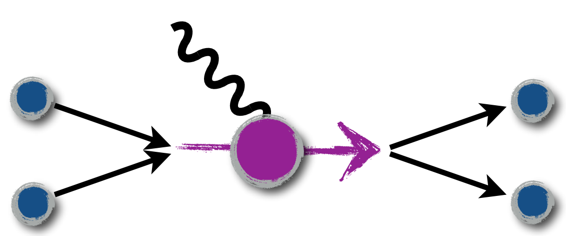
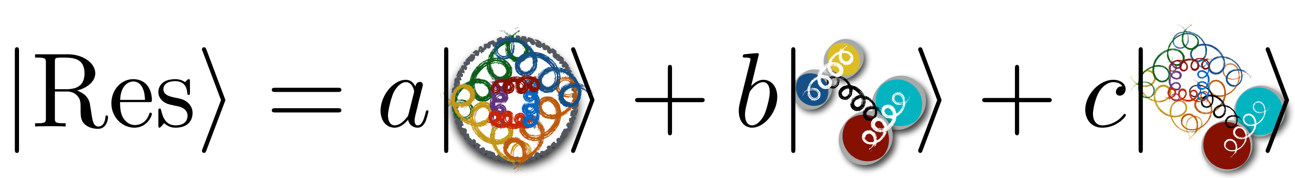


The aim is to determine all such observables in a rigorous, systematically-improvable manner, from first principles QCD

- ☑ Pred… resonance

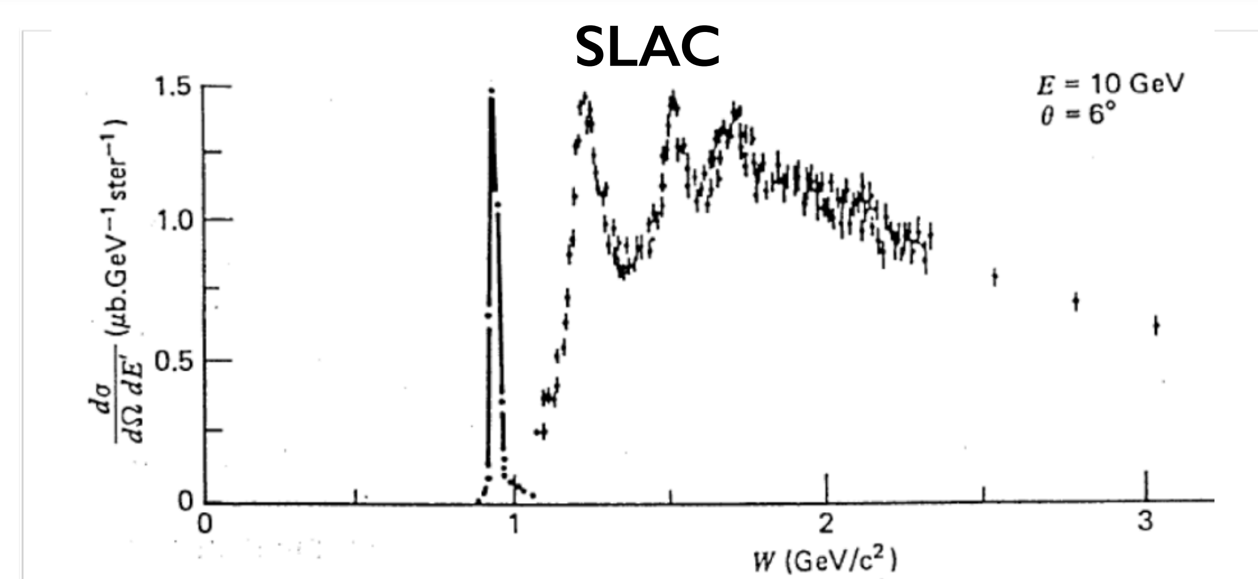




$$|\text{Res}\rangle = a|\ \rangle + b|\ \rangle + c|\ \rangle$$

…caling

- … to high energy regimes



taken from Halzen and Martin

Lattice QCD is a powerful tool for extracting QCD predictions

LQCD = evaluating a difficult integral numerically

$$\textbf{observable} = \int \mathcal{D}\phi \; e^{iS} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

Lattice QCD is a powerful tool for extracting QCD predictions

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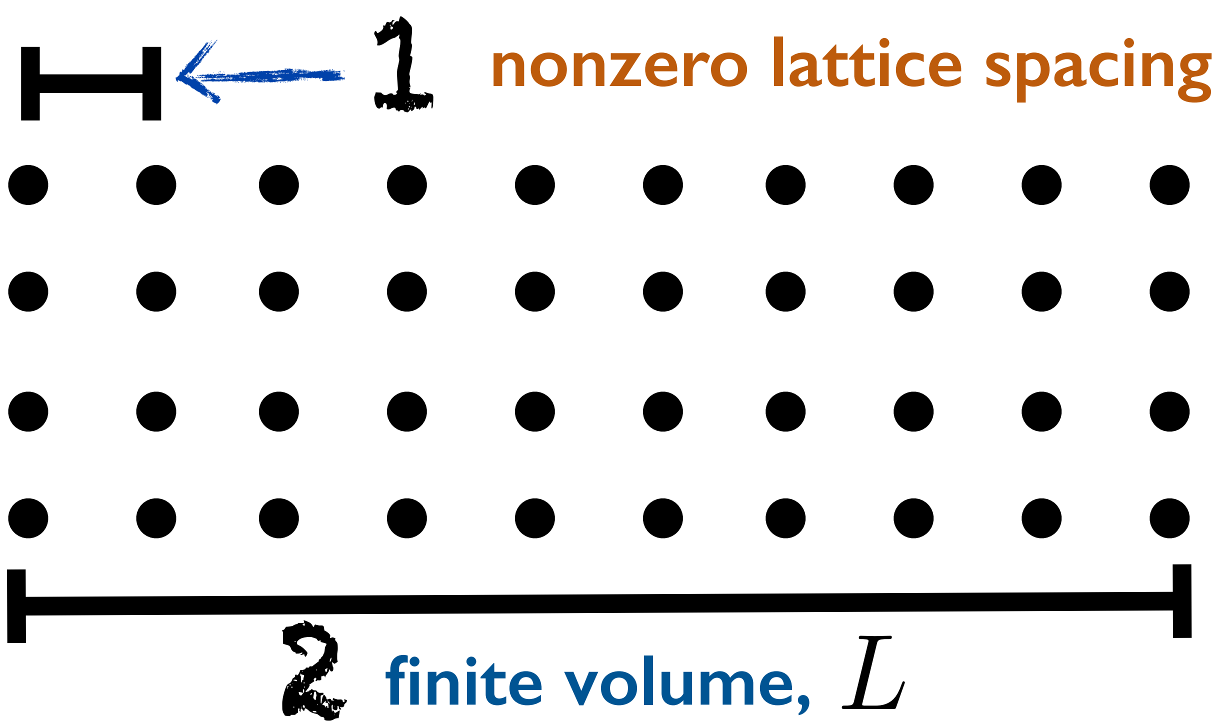
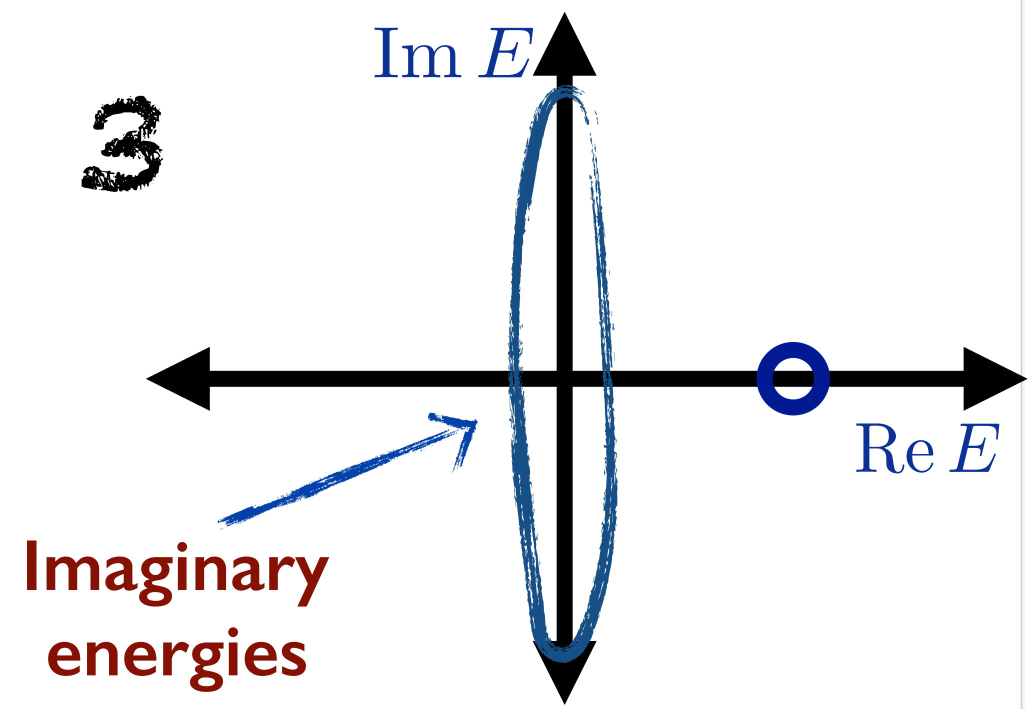
To do so we have to make three compromises





Also... Unphysical pion masses $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

But calculations at the physical pion mass do now exist

Lattice QCD is a powerful tool for extracting QCD predictions

LQCD = evaluating a difficult integral numerically

$$\left(\begin{array}{c} \textbf{observable?} \\ \text{discretized, finite volume,} \\ \text{Euclidean, heavy pions} \end{array} \right) = \int \prod_i^N d\phi_i \; e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

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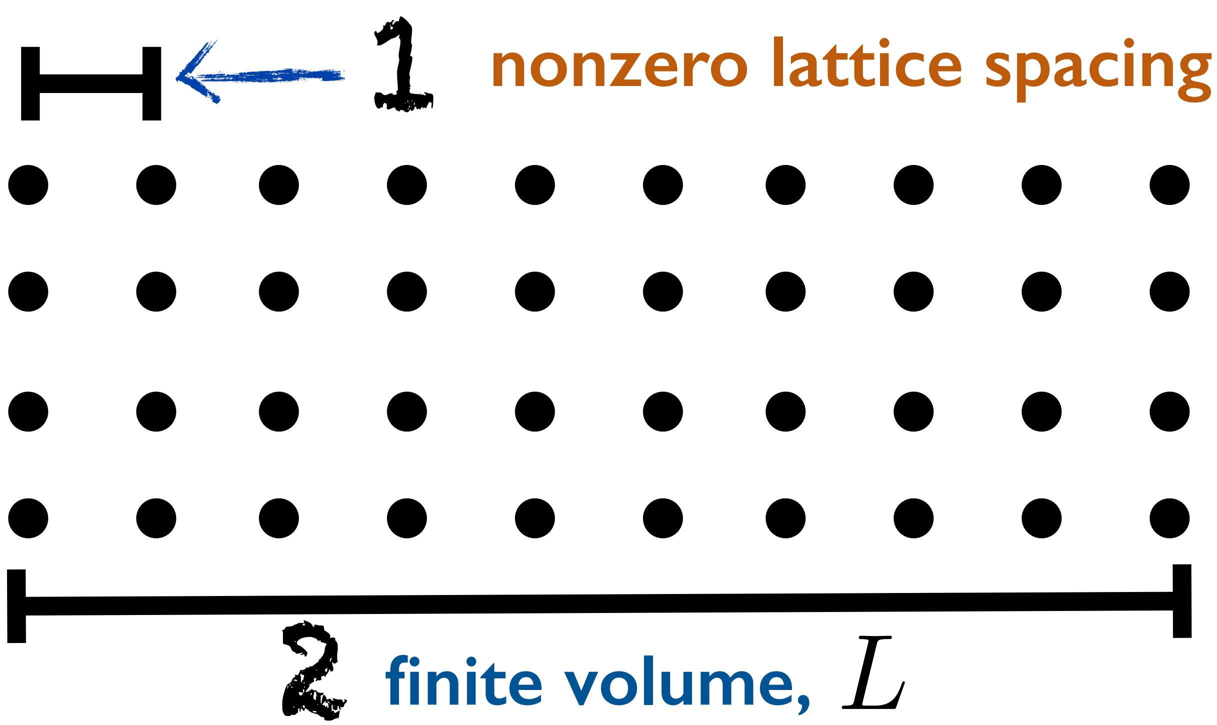

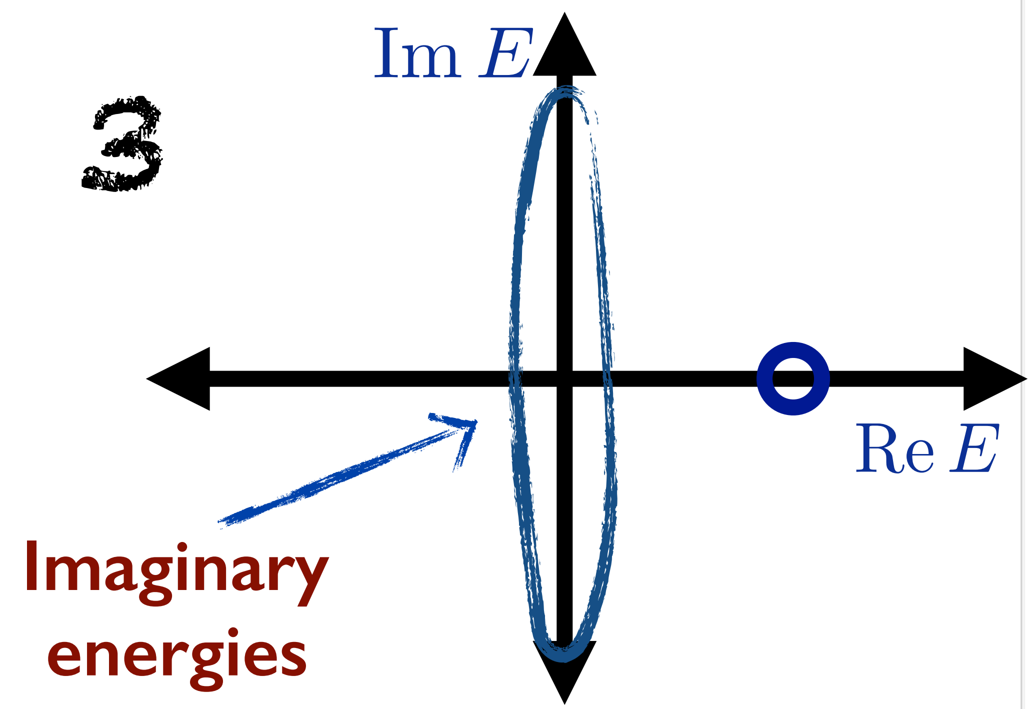

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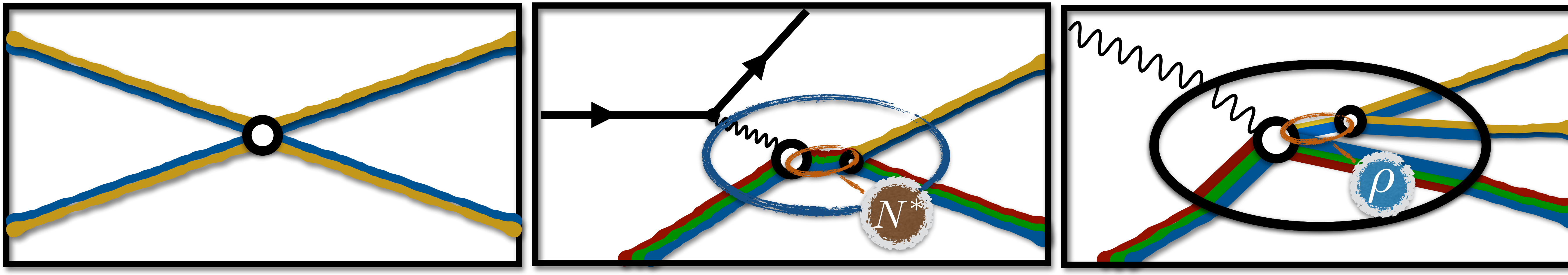
The difficulty with resonances

Not possible to directly calculate



$\langle \underline{\pi\pi} | \pi\pi \rangle$ $\quad\quad$ $\langle \underline{N\pi} | \mathcal{J}_\mu | N \rangle$ $\quad\quad$ $\langle \underline{N\pi\pi} | \mathcal{J}_\mu | N \rangle$

multi-particle in- and outstates

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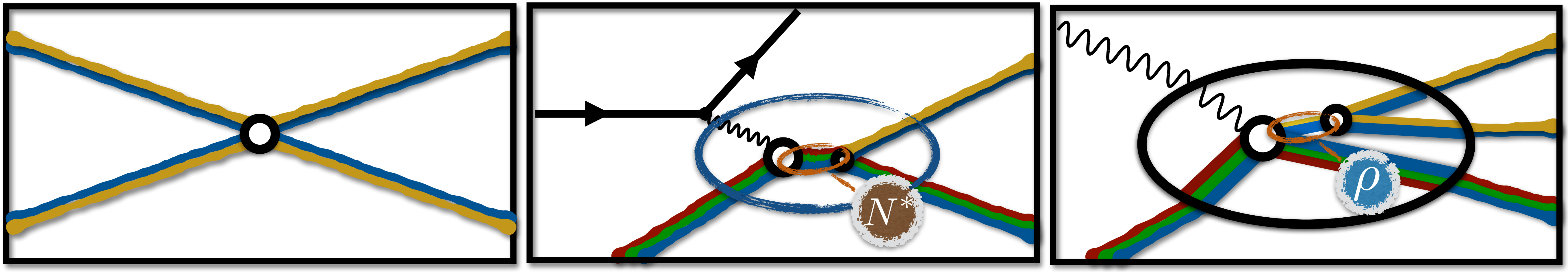

$\langle \pi\pi | \pi\pi \rangle$ $\quad$ $\langle N\pi | \mathcal{J}_\mu | N \rangle$ $\quad$ $\langle N\pi\pi | \mathcal{J}_\mu | N \rangle$

multi-particle in- and outstates

---

amputate and go on-shell

$$\langle N\pi\pi, \mathrm{out} | \mathcal{J}_\mu(x) | N \rangle = \langle 0 | \tilde{N}(p_1') \tilde{\pi}(p_2') \tilde{\pi}(p_3') \ \mathcal{J}_\mu(x) \ \tilde{N}(P) | 0 \rangle$$

Requires real, physical energies and infinite volume

The difficulty with resonances

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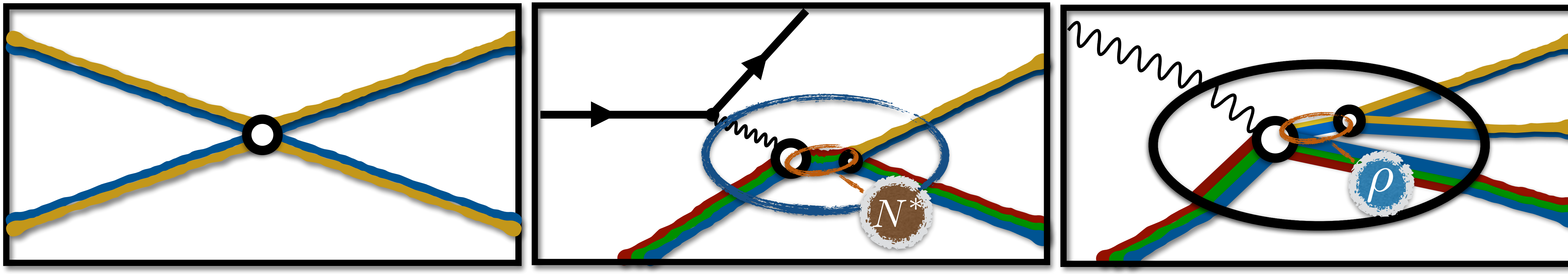

$\langle \pi\pi | \pi\pi \rangle$ $\qquad$ $\langle N\pi | \mathcal{J}_\mu | N \rangle$ $\qquad$ $\langle N\pi\pi | \mathcal{J}_\mu | N \rangle$

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$$\langle N\pi\pi, \mathrm{out} | \mathcal{J}_\mu(x) | N \rangle = \langle 0 | \tilde{N}(p_1') \tilde{\pi}(p_2') \tilde{\pi}(p_3) \; \mathcal{J}_\mu(x) \; \tilde{N}(P) | 0 \rangle$$

~~**Requires real, physical energies and infinite volume**~~

Instead we can only access

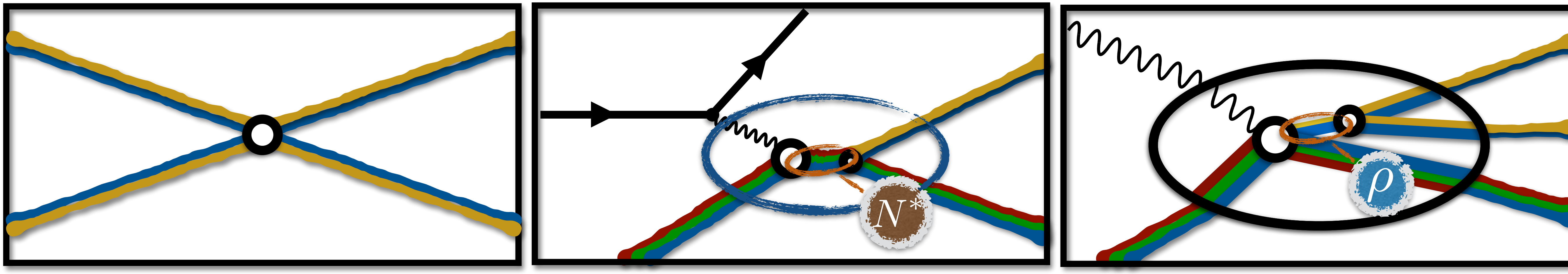
$$H_{\mathrm{QCD}} | n, L \rangle = | n, L \rangle E_n(L) \qquad \langle n, L, \text{“}N\pi\pi\text{”} | \mathcal{J}_\mu(x) | \text{“}N\text{”}, L \rangle$$

finite-volume energies and matrix elements

(labels in quotes indicate quantum numbers)

The difficulty with resonances

Not possible to directly calculate



$\langle \underline{\pi\pi} | \pi\pi \rangle$ $\qquad$ $\langle \underline{N\pi} | \mathcal{J}_\mu | N \rangle$ $\qquad$ $\langle \underline{N\pi\pi} | \mathcal{J}_\mu | N \rangle$

multi-particle in- and outstates

How can we determine

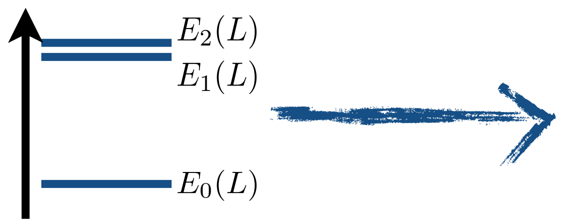
$$\underline{\langle \pi\pi, \mathrm{out} | \pi\pi, \mathrm{in} \rangle \ \ \textbf{and} \ \ \langle N\pi\pi, \mathrm{out} | \mathcal{J}_\mu(x) | N \rangle}$$

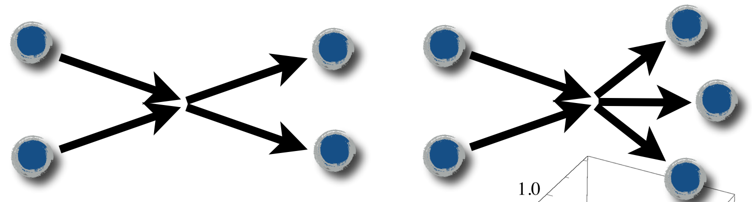
from

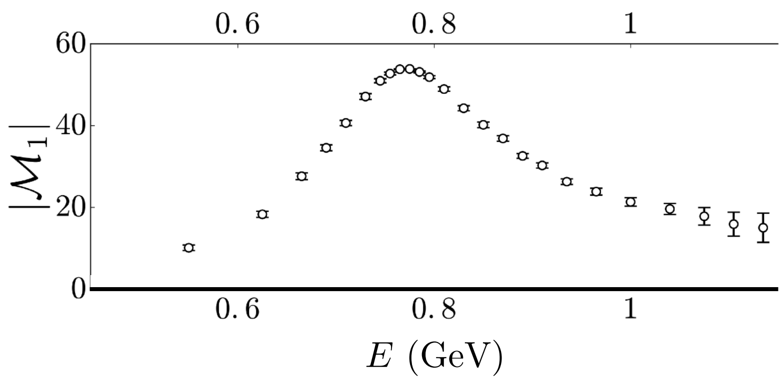
$$\underline{E_n(L) \ \ \textbf{and} \ \ \langle n, L, \text{“}N\pi\pi\text{”} | \mathcal{J}_\mu(x) | \text{“}N\text{”}, L \rangle} \ \textbf{?}$$

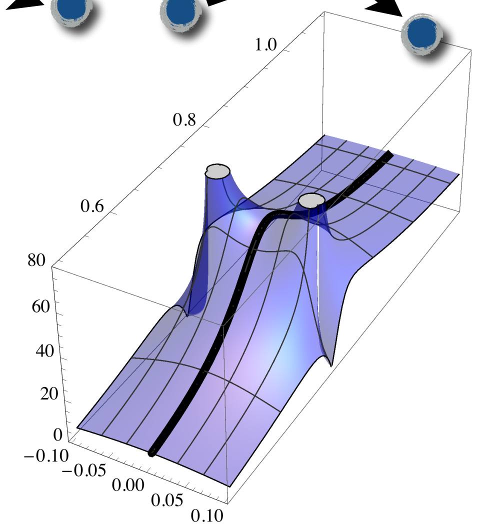
Solution: Relations between finite- and infinite-volume physics

Scattering









Finite-volume energies related to scattering

Solution: Relations between finite- and infinite-volume physics

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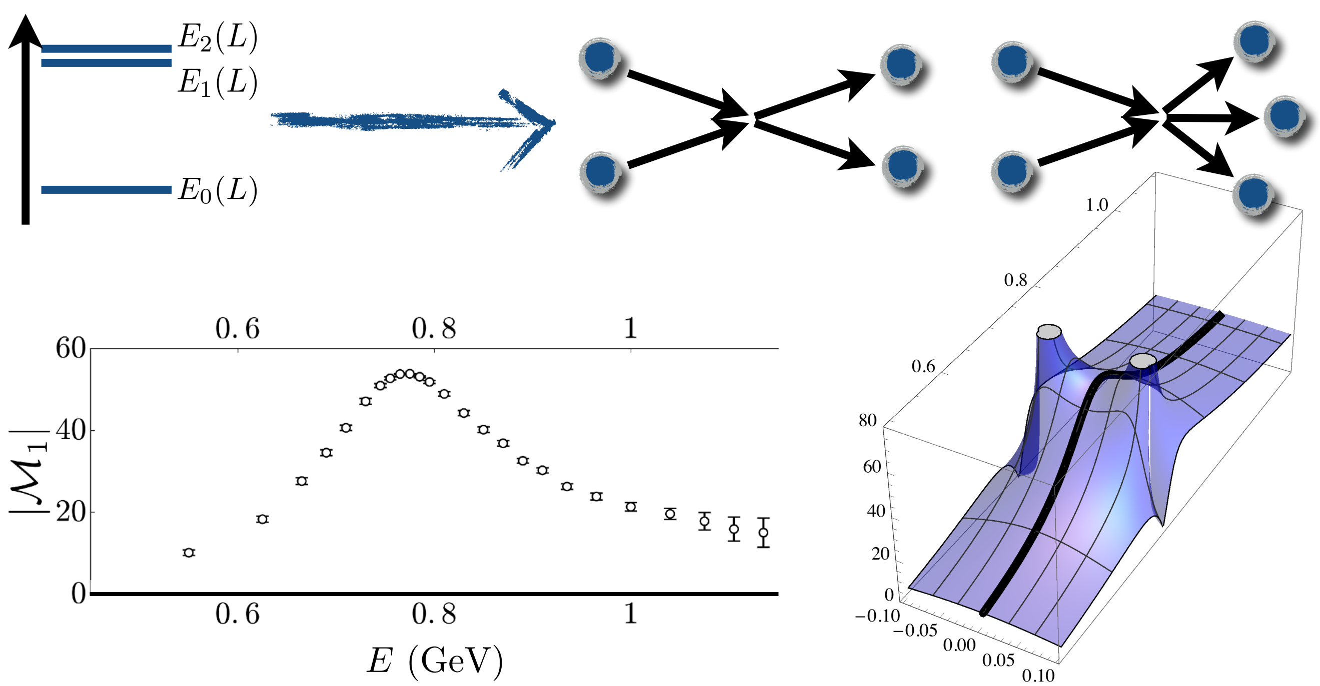

Finite-volume energies related to scattering

Transition amplitudes

Finite-volume matrix elements map to infinite-volume transitions

Solution: Relations between finite- and infinite-volume physics

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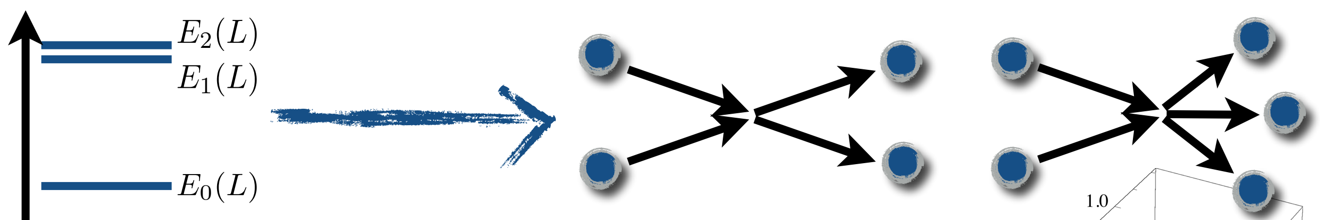

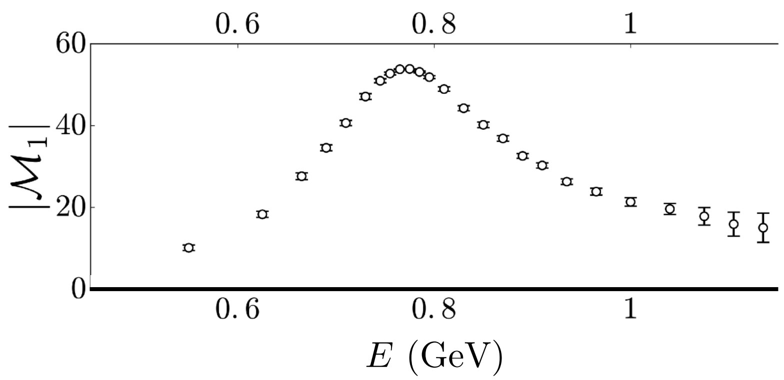

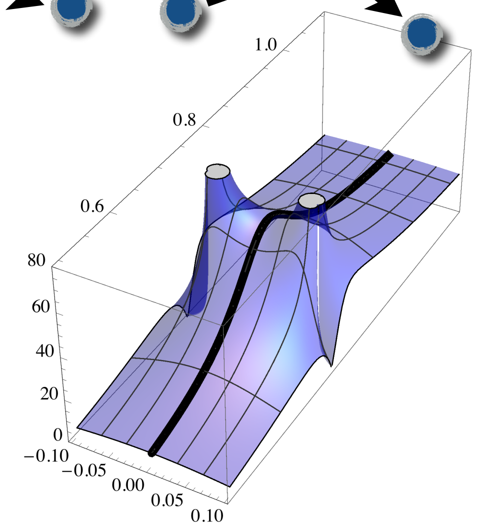

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Resonant form factors

More complicated formalism for two-to-two transitions

Solution: Relations between finite- and infinite-volume physics

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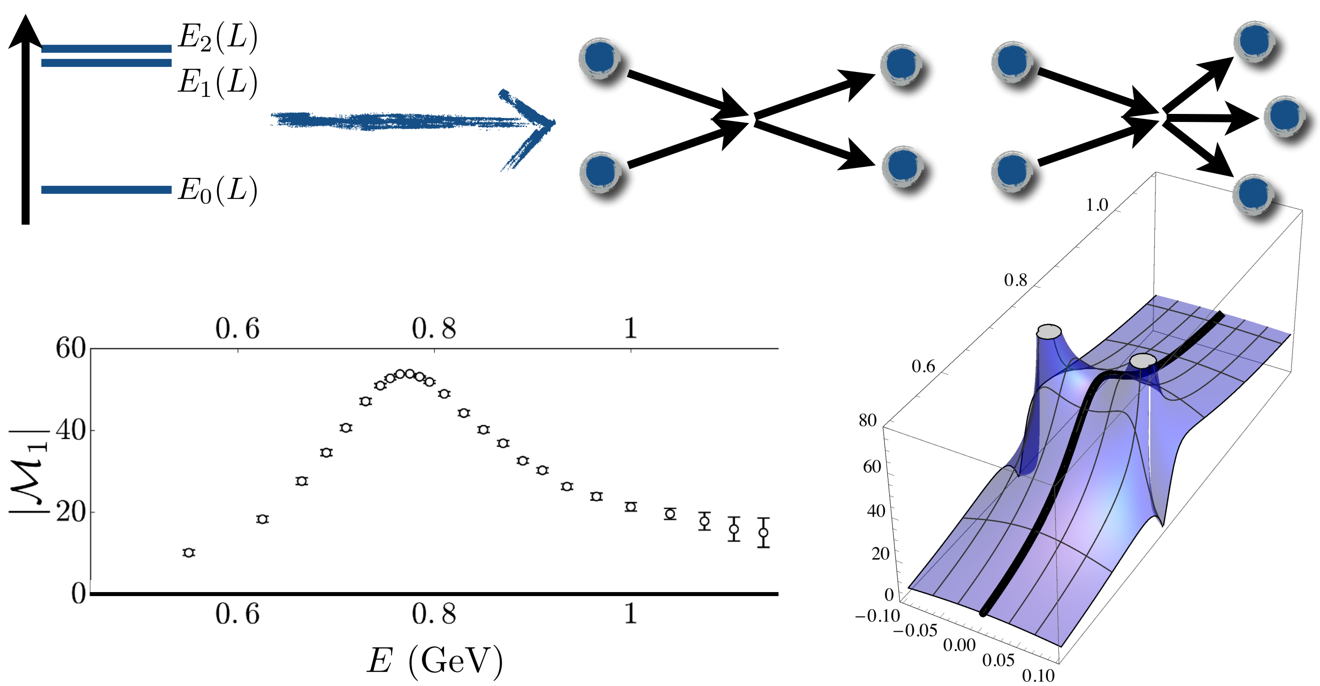

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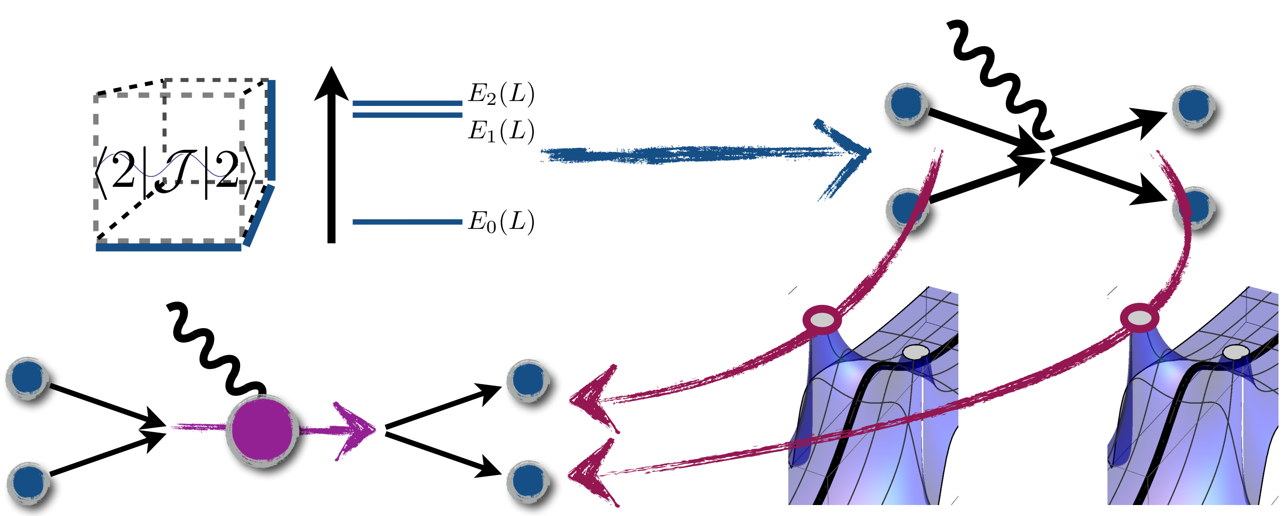

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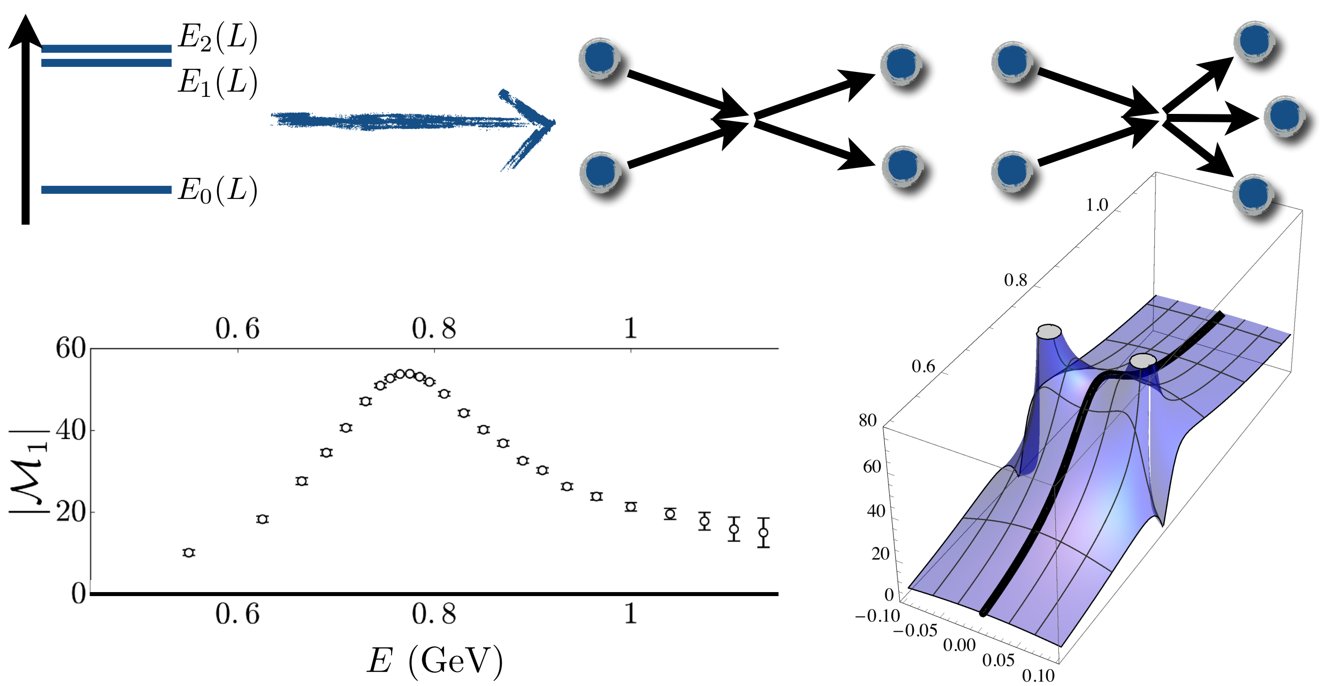

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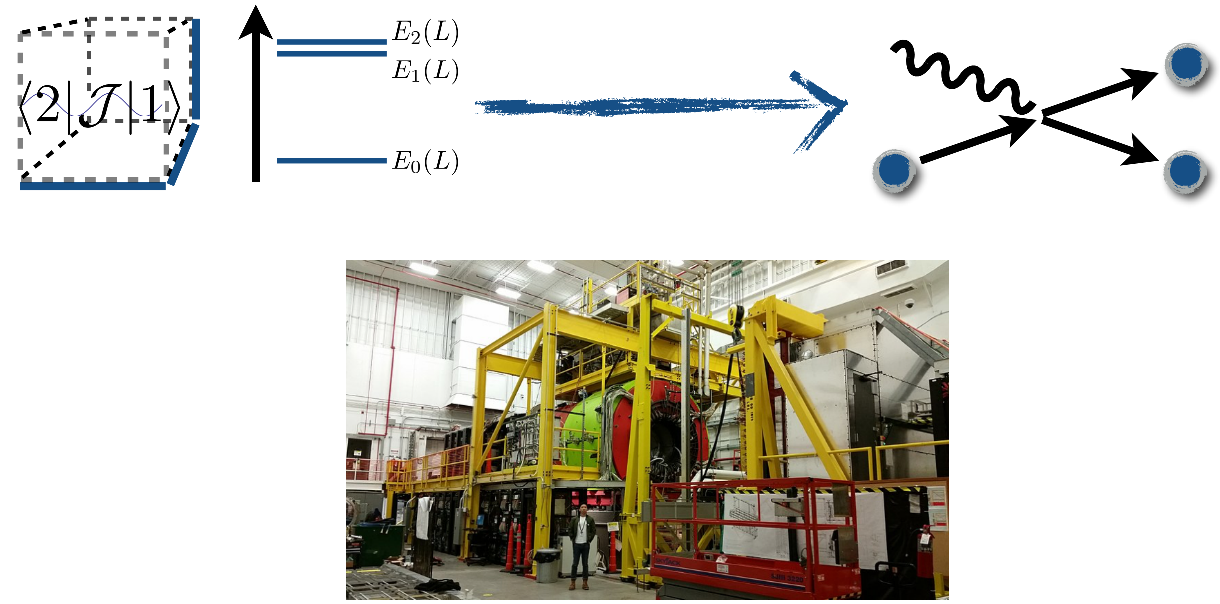

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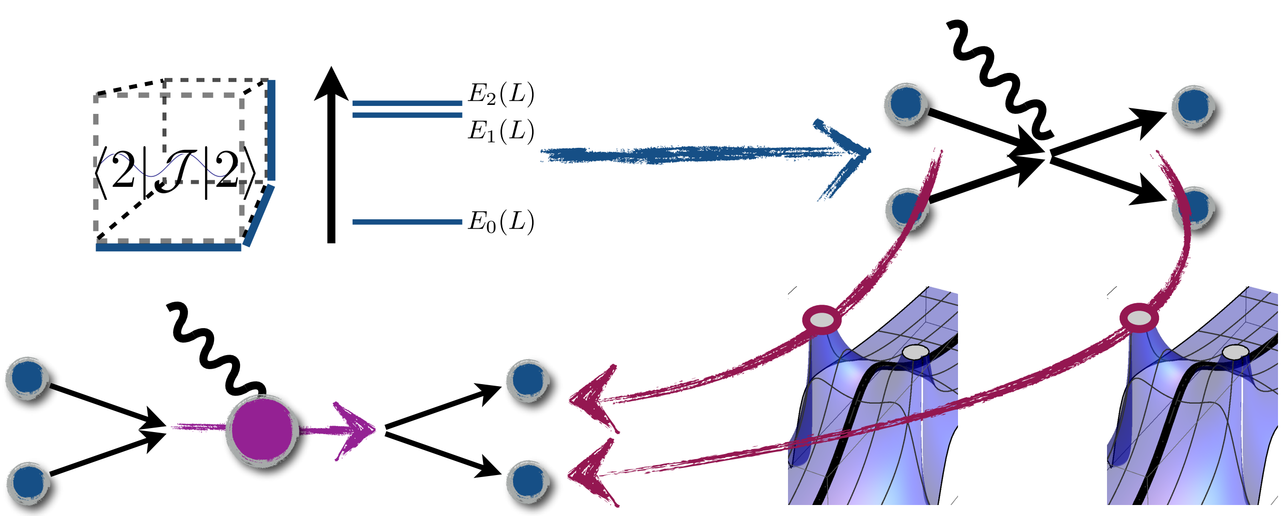
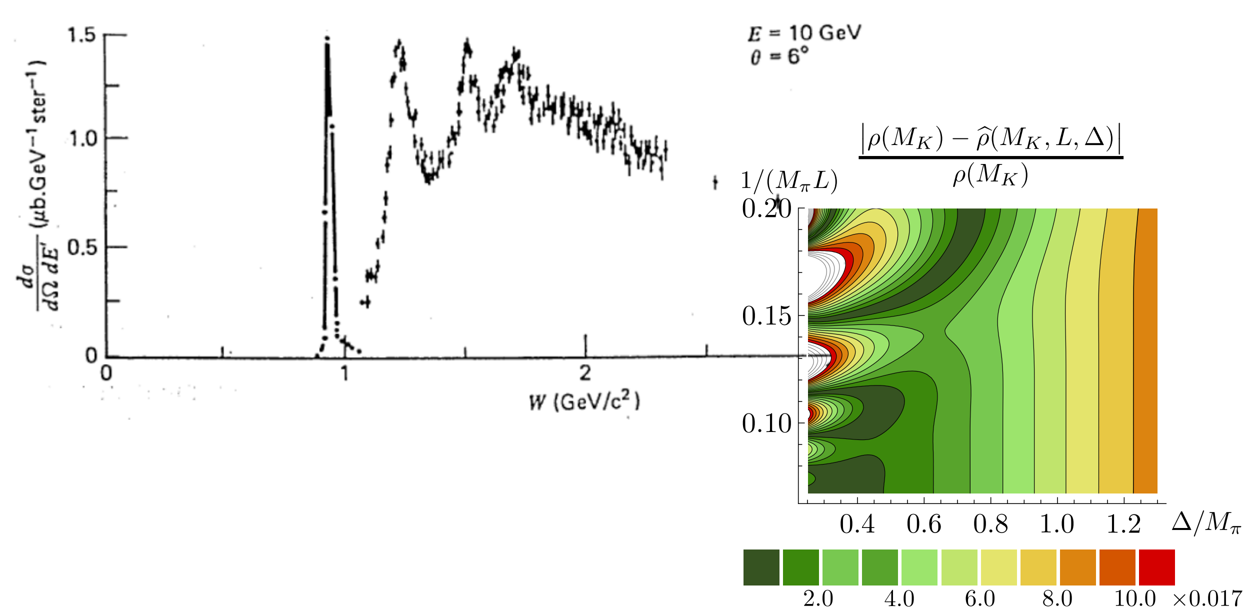


More complicated formalism for two-to-two transitions

From resonances to high-energy scaling



Explore the transition from resonance to high energy regimes

Large body of formal developments spanning many decades

Scattering

Lüscher (1986,1991) ○ **Rummukainen, Gottlieb** (1995)
Li, Liu (2004) ○ **Feng, Li, Liu** (2004)
Detmold, Savage (2004) ○ **Kim, Sachrajda, Sharpe** (2005)
Christ, Kim, Yamazaki (2005) ○ **Bernard, et. al.** (2008)
Davoudi, Savage (2011) ○ **Polejaeva, Rusetsky** (2012)
Leskovec, Prelovsek (2012) ○ **Göckeler et al.** (2012)
MTH, Sharpe (2012) ○ **Briceño, Davoudi** (2012,2013)
Briceño (2014) ○ **MTH, Sharpe** (2015,2016)
Briceño, MTH, Sharpe (2017)
Hammer, Pang, Rusetsky (2017) ○ **Mai, Döring** (2017)
Döring, et al. (2018)

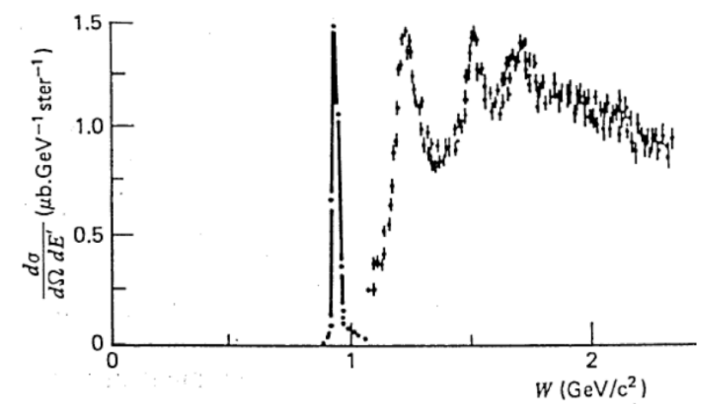
Transition amplitudes

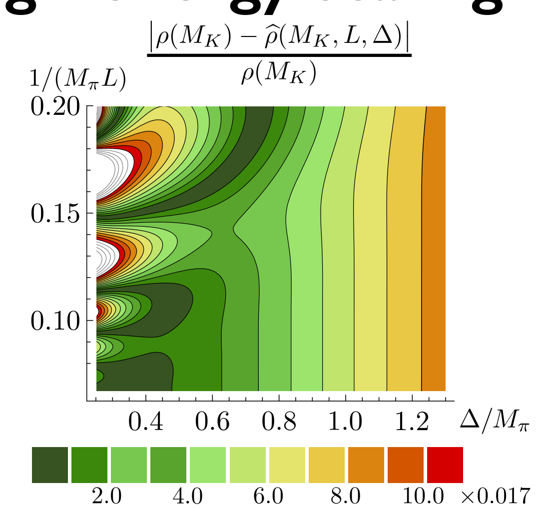
Lellouch, Lüscher (2000)
Kim, Sachrajda, Sharpe (1995)
Christ, Kim, Yamazaki (2005) ○ **Meyer** (2011)
MTH, Sharpe (2012) ○ **Briceño, Davoudi** (2012)
Bernard et al. (2012) ○ **Agadjanov et al.** (2014)
Briceño, MTH, Walker-Loud (2015)
Briceño, MTH (2015)

Resonant form factors

Briceño, Davoudi (2012) ○ **Bernard et al.** (2012)
Detmold, Flynn (2015) ○ **Briceño, MTH** (2016)

From resonances to high-energy scaling

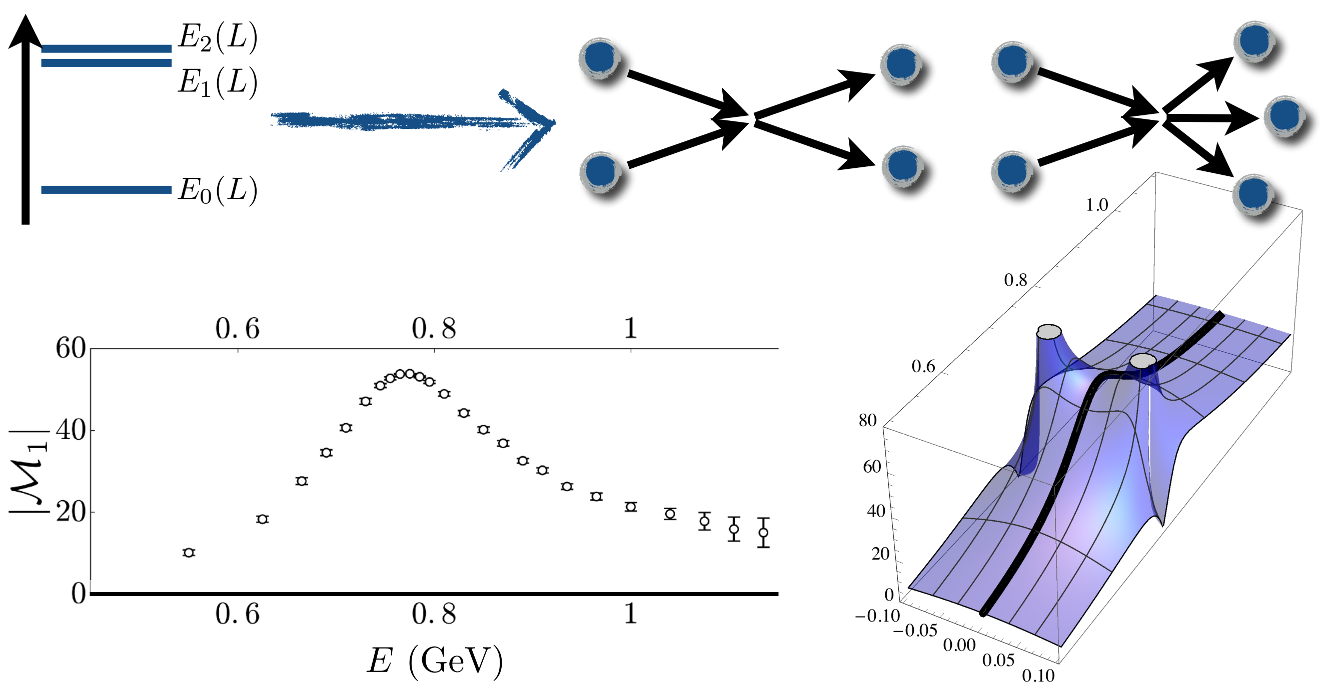




Liu (2017) ○ **Hashimoto** (2017)
Chambers (2017) ○ **MTH, Meyer, Robaina** (2017)

Relations between finite- and infinite-volume physics

Scattering



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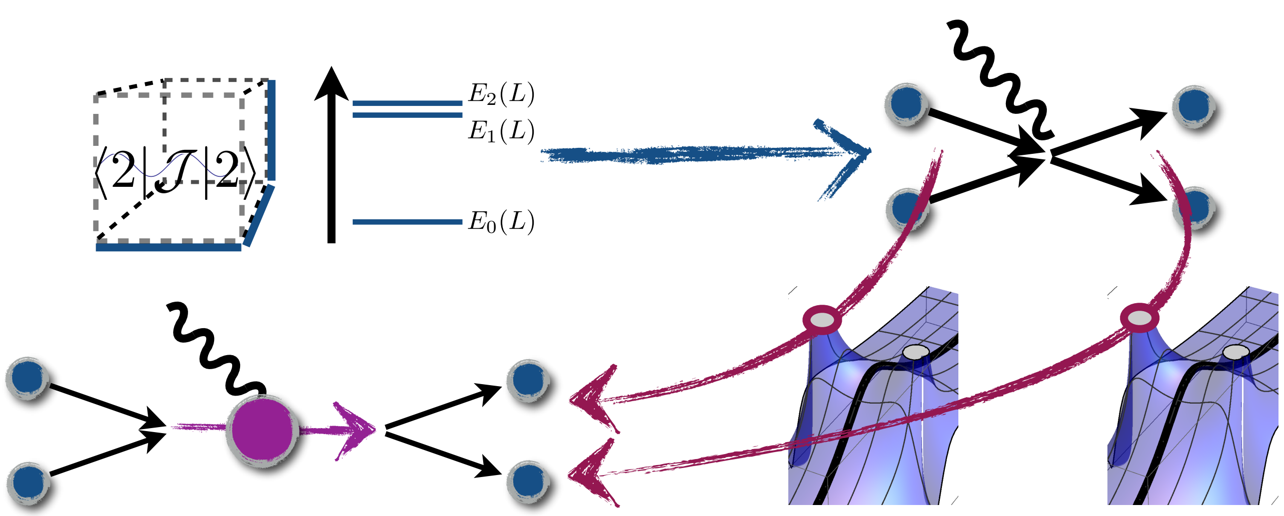
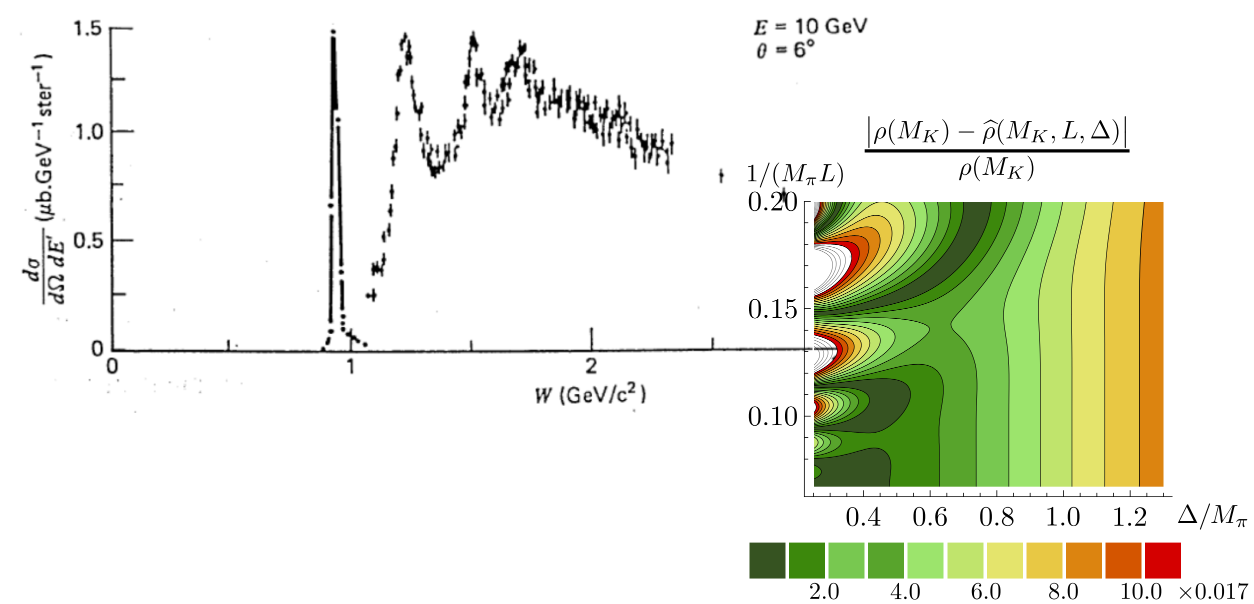


More complicated formalism for two-to-two transitions

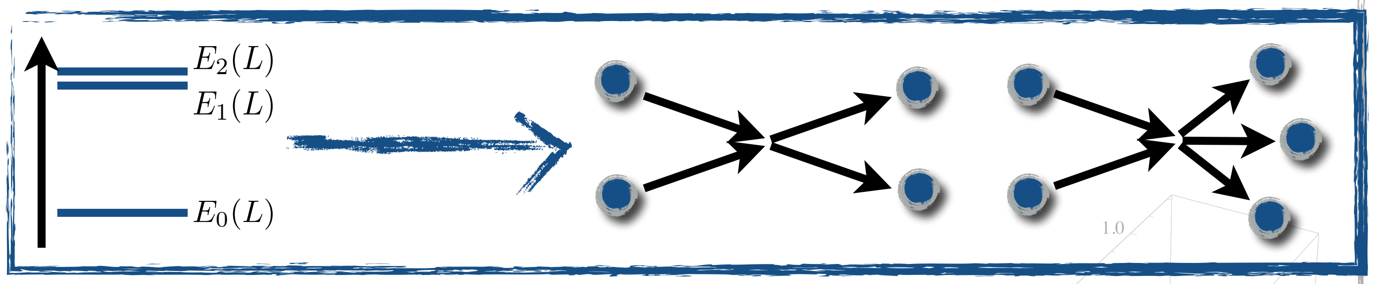
From resonances to high-energy scaling

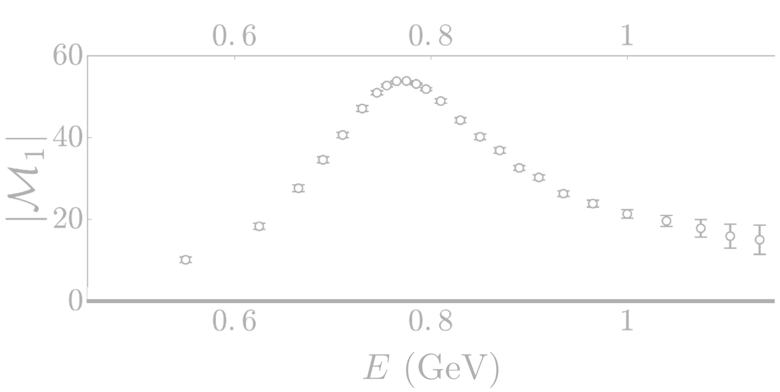


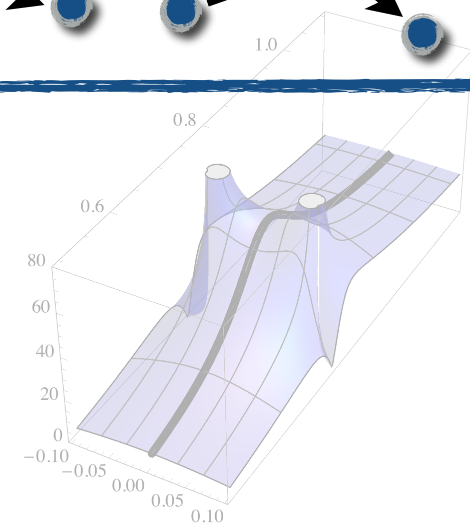
Explore the transition from resonance to high energy regimes

Solution: Relations between finite- and infinite-volume physics

Scattering







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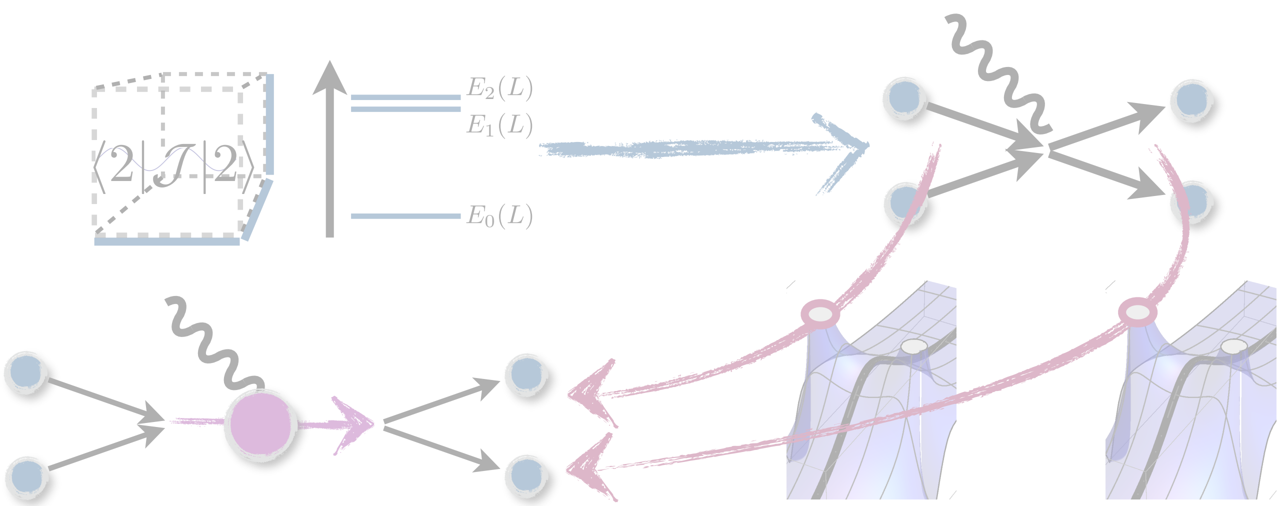

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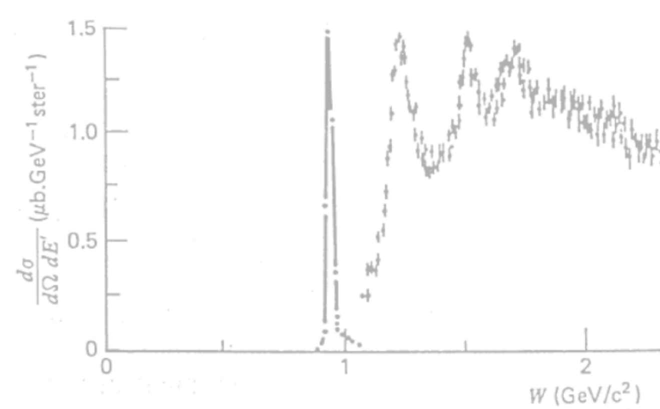
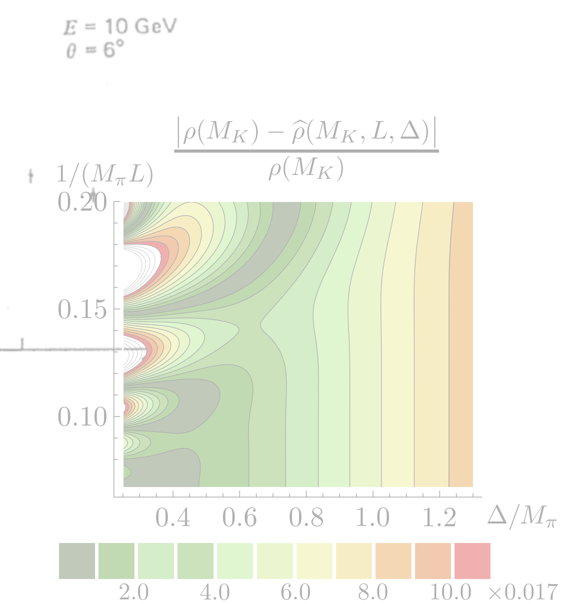
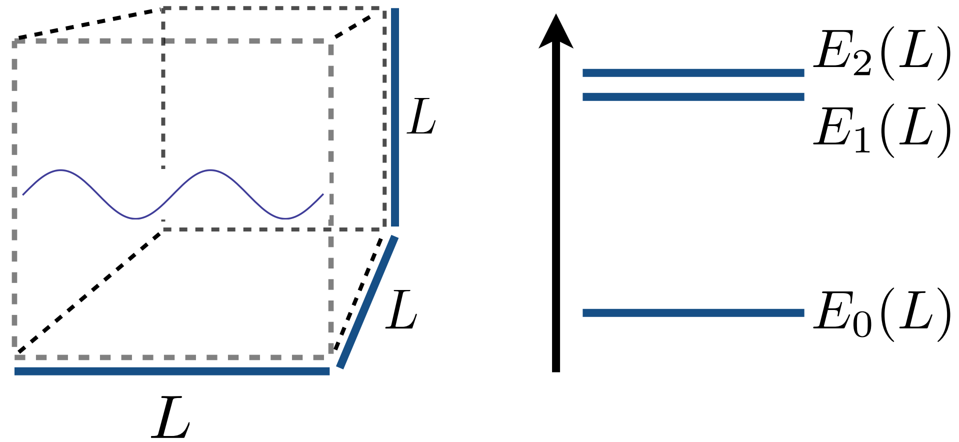




Explore the transition from resonance to high energy regimes

Basic set-up



cubic, spatial volume (extent L)

periodic boundary conditions

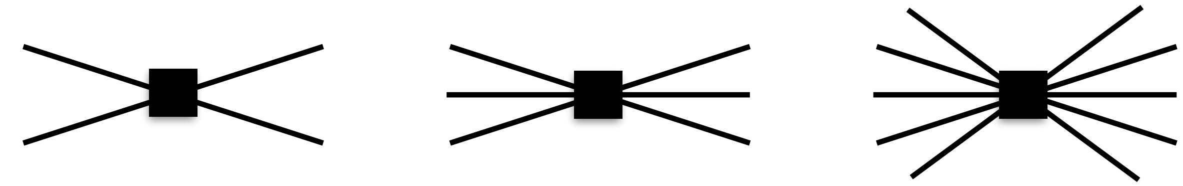
$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite**

L large enough to ignore e^{-mL}

Generic relativistic QFT

1. Include all interactions



2. no power-counting scheme

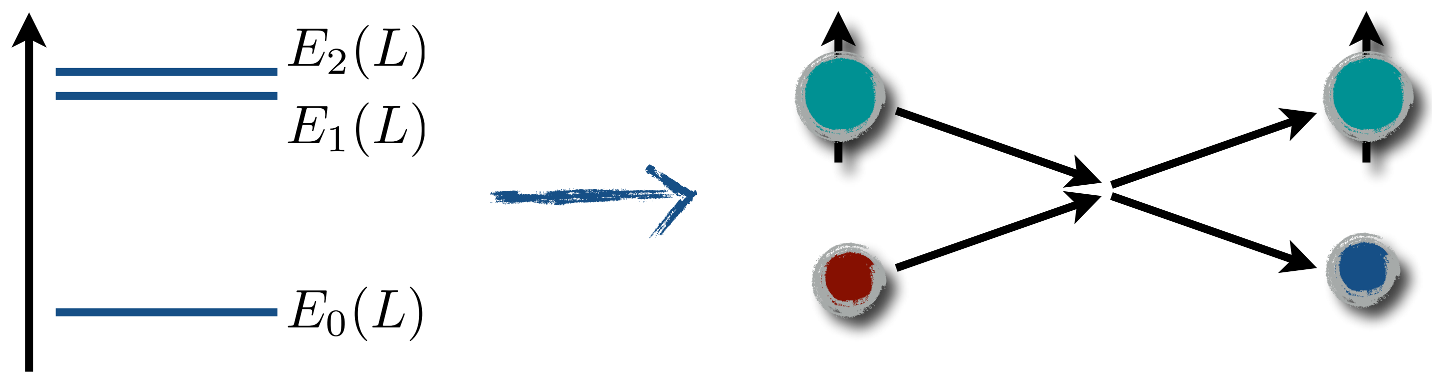
Not possible to directly calculate scattering observables to all orders

But it is possible to derive *general, all-orders relations* to finite-volume quantities

Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

In the case of two-to-two scattering
Lüscher's formalism + extensions give a general mapping



All results contained in a generalized *quantization condition*

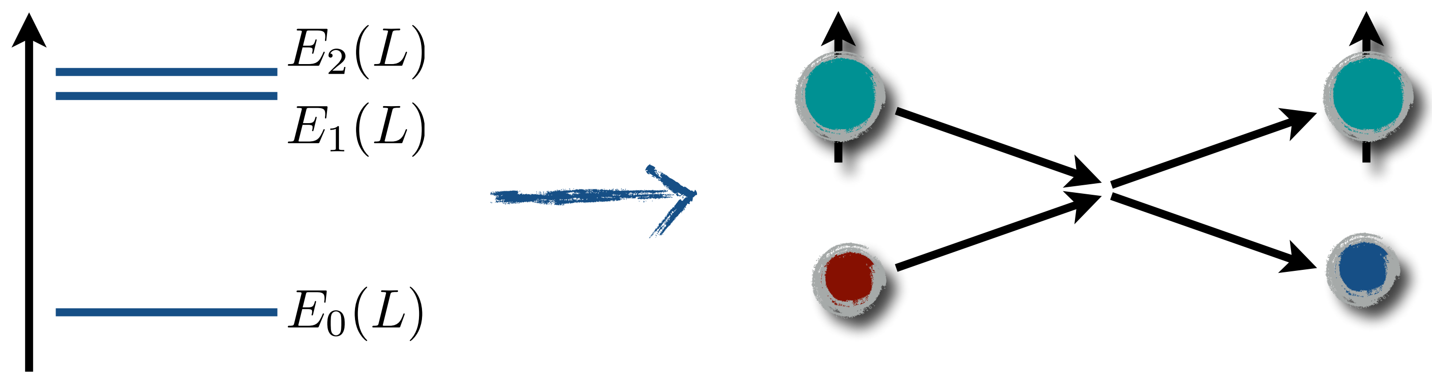
$$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$$

scattering amplitude known geometric function

Matrices in angular momentum, spin and channel space

Lüscher, Rummukainen, Gottlieb, Li, Liu, Feng, Detmold, Savage, Kim, Sachrajda, Sharpe, Christ, Kim, Yamazaki, Bernard, Döring, Lage, Meißner, Rusetsky, Davoudi, Savage, Polejaeva, Leskovec, Prelovsek, Göckeler, Horsley, Rakow, Schierholz, Zanotti, MTH, Sharpe, Briceño, Davoudi

In the case of two-to-two scattering
Lüscher's formalism + extensions give a general mapping



All results contained in a generalized *quantization condition*

$$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$$

scattering amplitude | known geometric function

Matrices in angular momentum, spin and channel space

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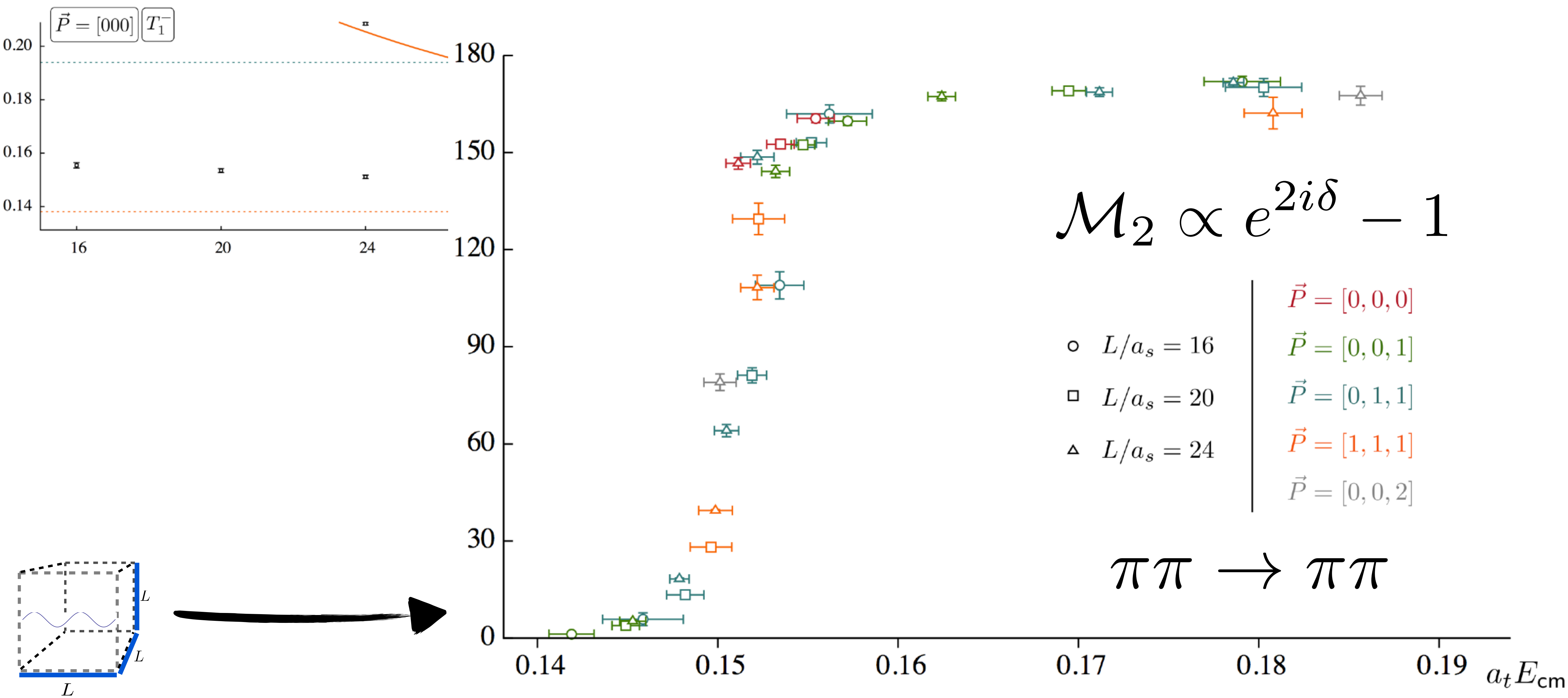
- ☑ Varying $E, \vec{P}$ gives more constraints on functions of $E^{*2} = E^2 - \vec{P}^2$
- ☑ Requires that energy is below lowest three-particle threshold
- ☑ Derivation ignores (drops) suppressed volume effects ($e^{-M_\pi L}$)
- ☑ Only useful if one truncates angular momentum space

Using the result: simplest case is a single-channel

e.g. for pions in a p-wave the relation reduces to

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$



from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

Talks here from John Bulava, Raúl Briceño

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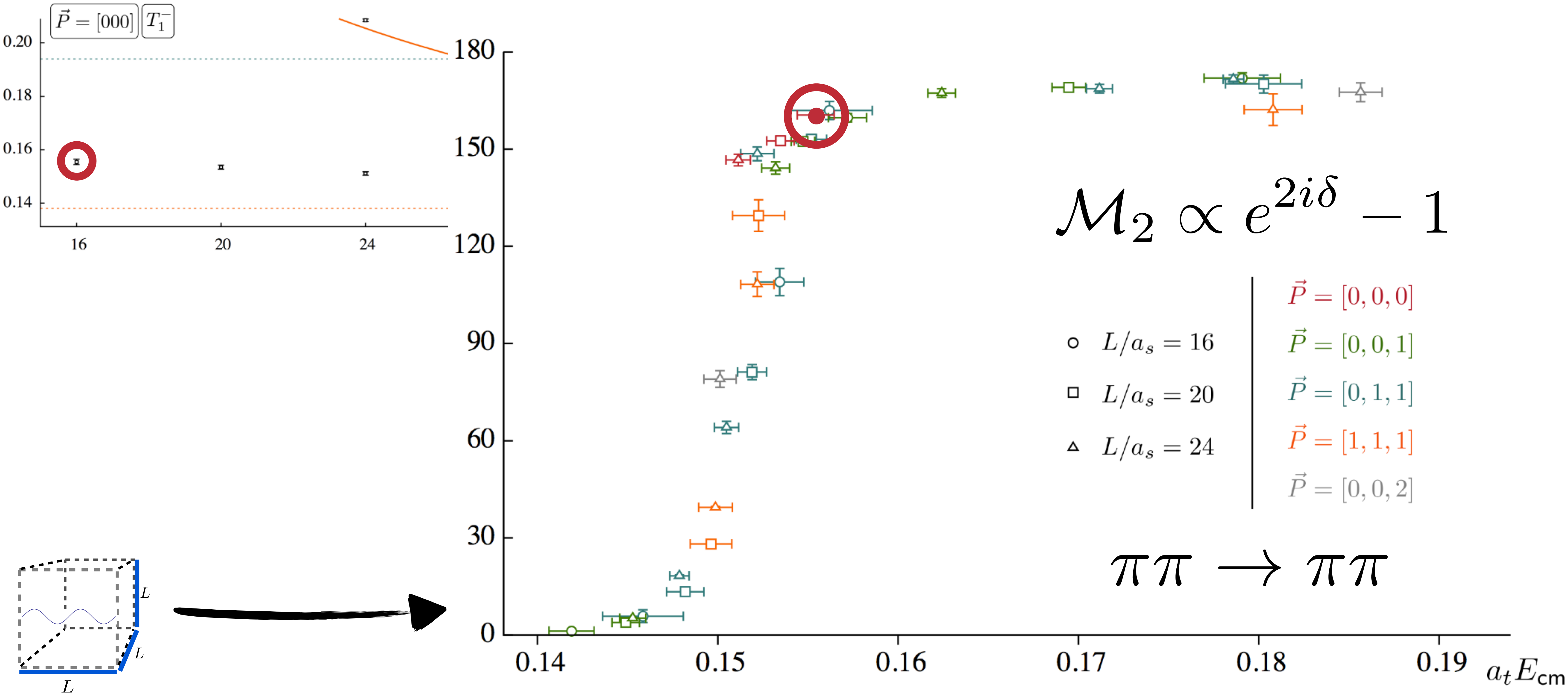

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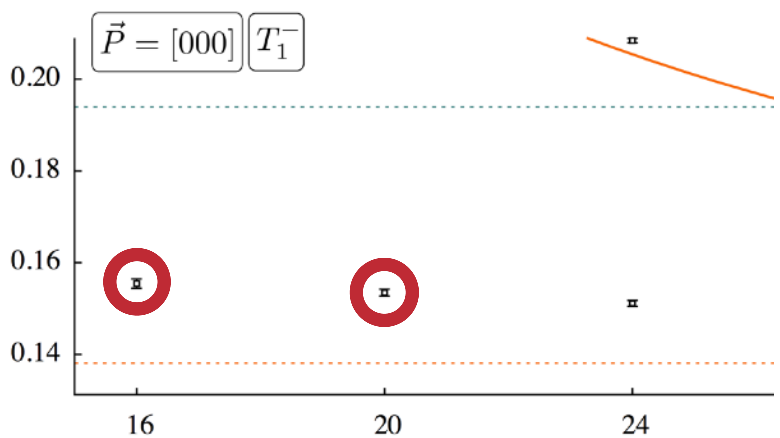

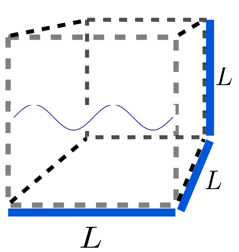

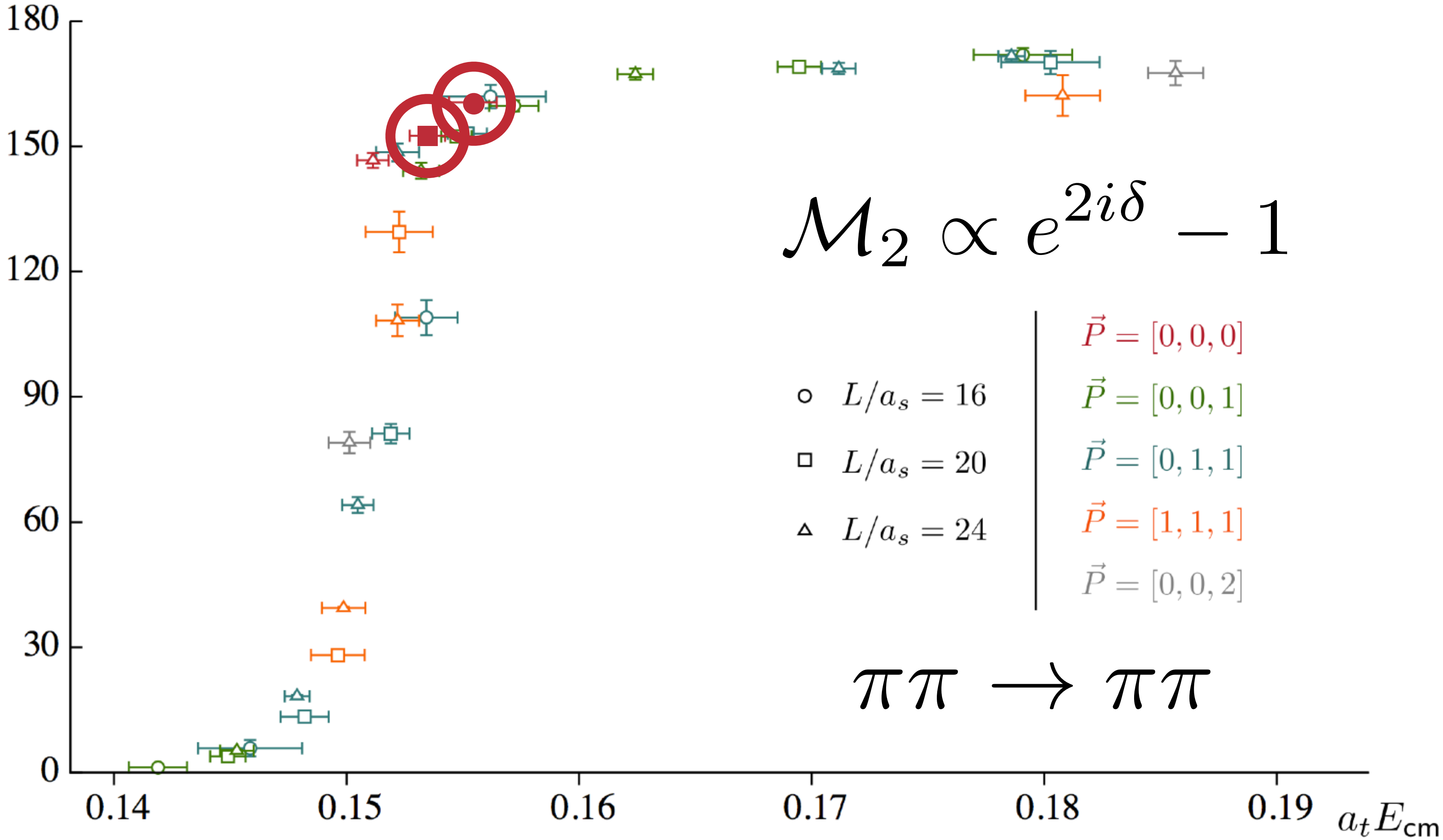

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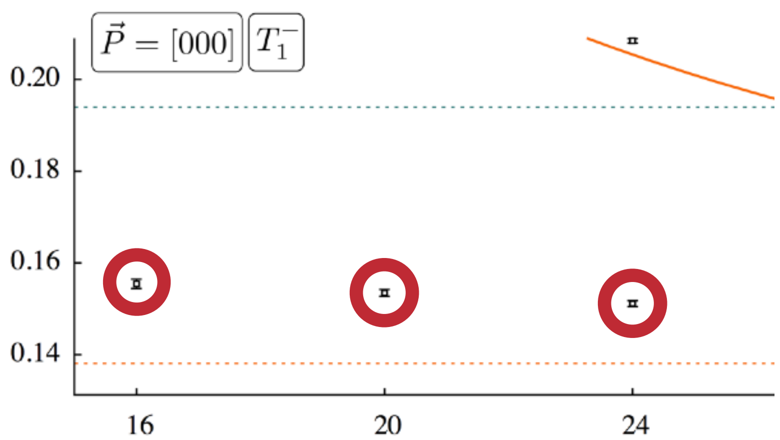

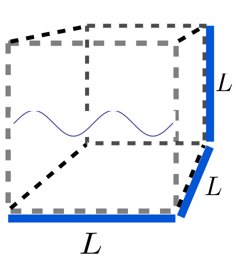

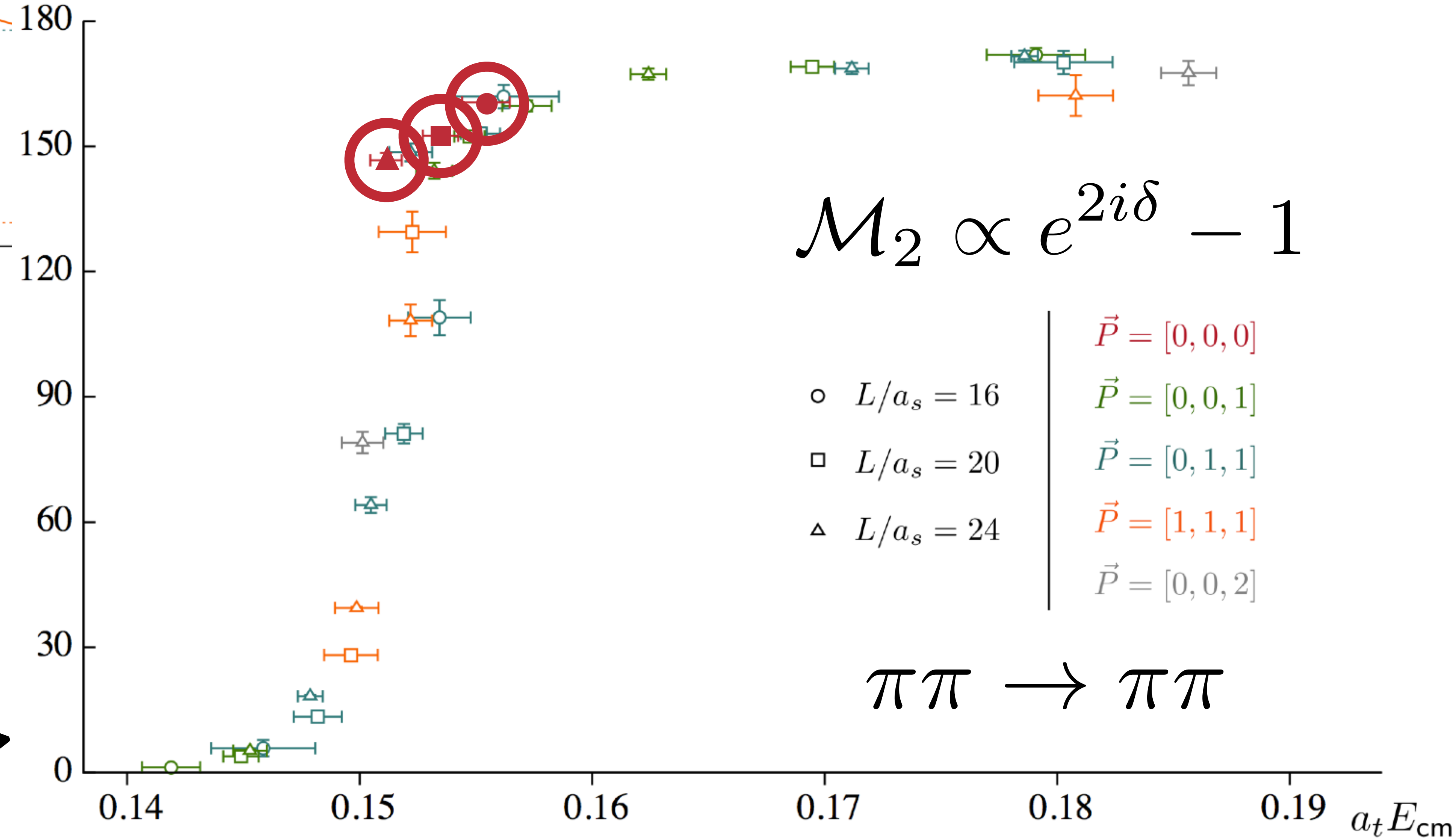

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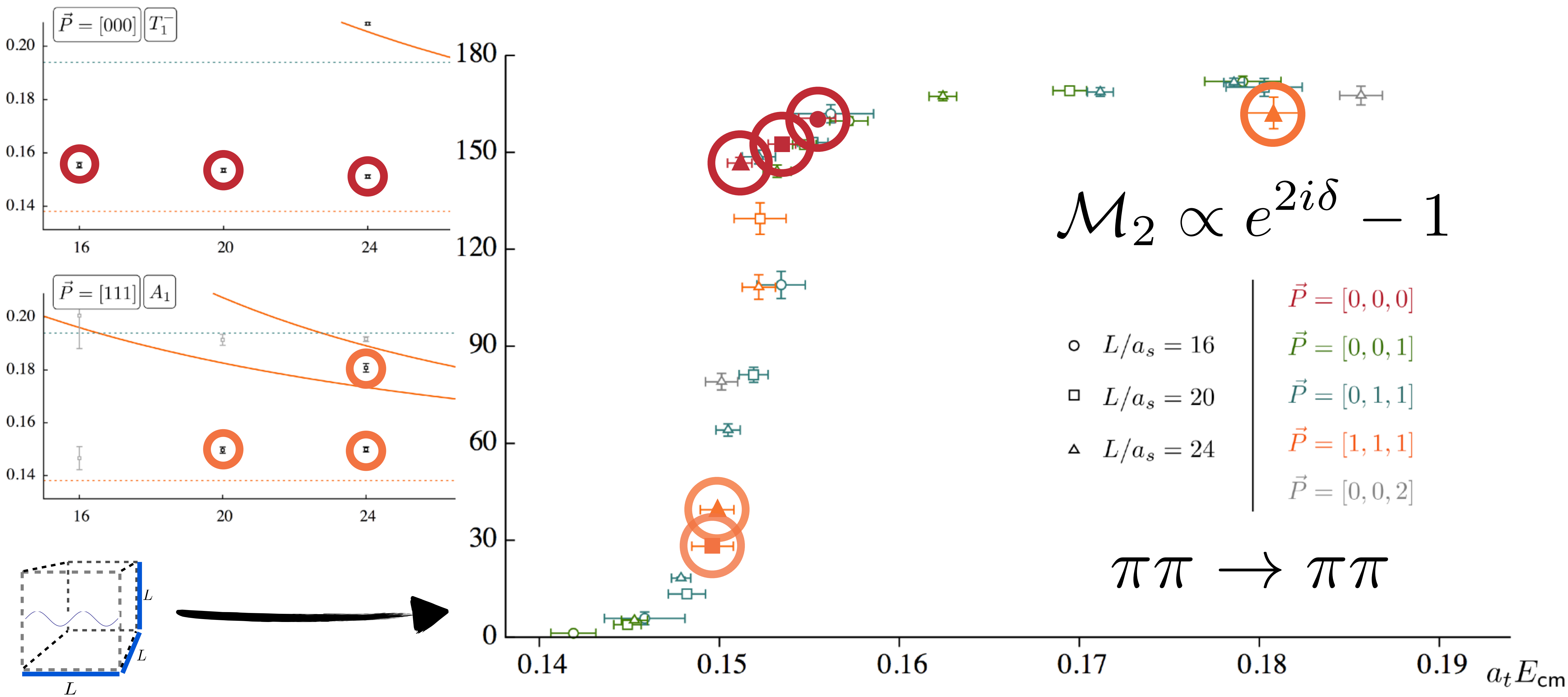

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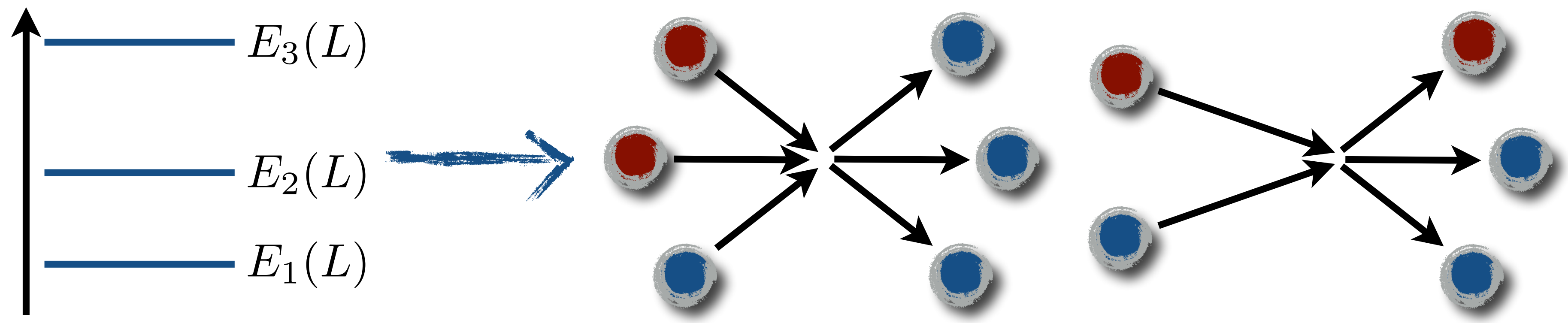
Talks here from John Bulava, Raúl Briceño

Our aim is to extend the derivation for arbitrary

relativistic two- and three-particle systems



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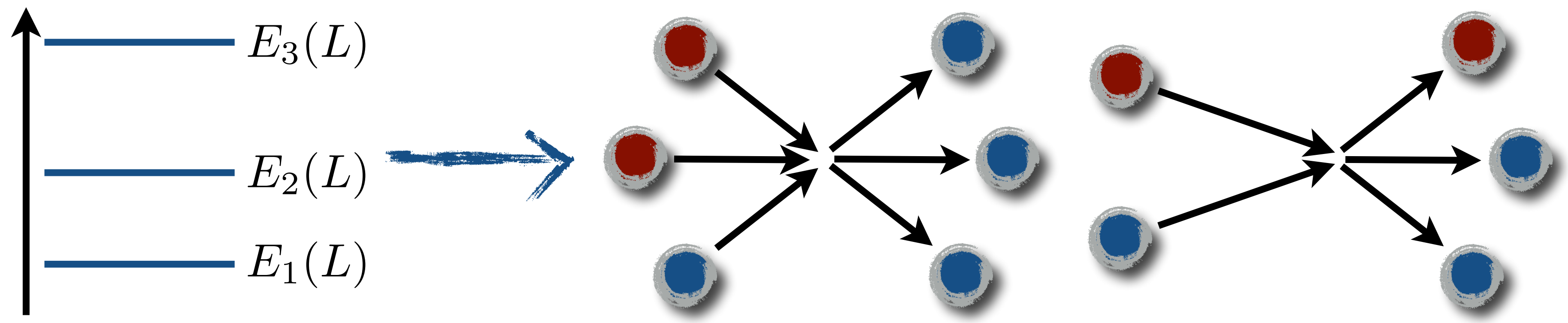
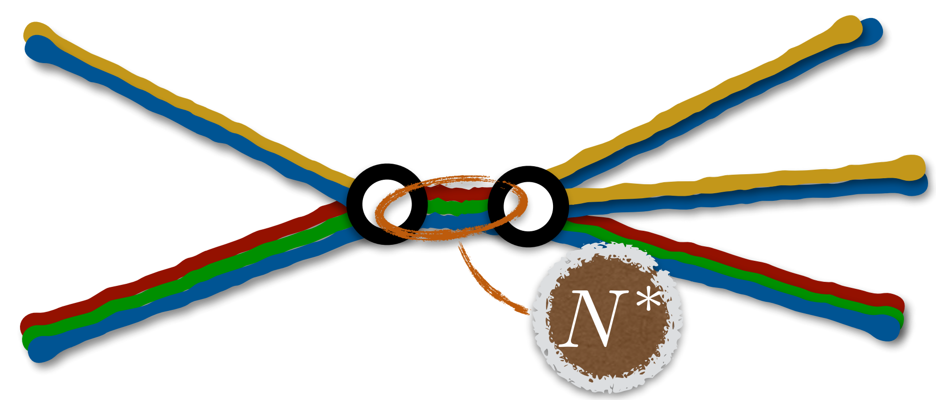


Potential applications…

Studying three-particle resonances

$$\omega(782) \to \pi\pi\pi$$

$$N(1440) \to N\pi, N\pi\pi$$

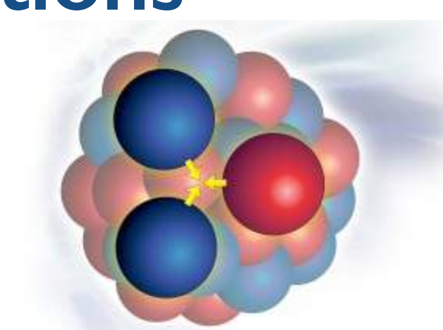


Calculating weak decay amplitudes and form factors

$$K \to \pi\pi\pi$$

Determining three-body interactions

NNN three-body forces needed as EFT input for studying larger nuclei and nuclear matter



Current status

Model- & EFT-independent relation between
finite-volume energies and relativistic two-and-three particle scattering

- ☑ Requires energy is below four-particle production threshold
- ☑ Derivation ignores (drops) suppressed volume effects ($e^{-M_\pi L}$)
- ☑ Only useful if one truncates angular momentum space

MTH, Sharpe (2015),(2016) ○ **Briceño, MTH, Sharpe** (2017)

see also… **Hammer, Pang, Rusetsky (2017)** ○ **Mai, Döring (2017)** ○ **Döring, et al. (2018)**

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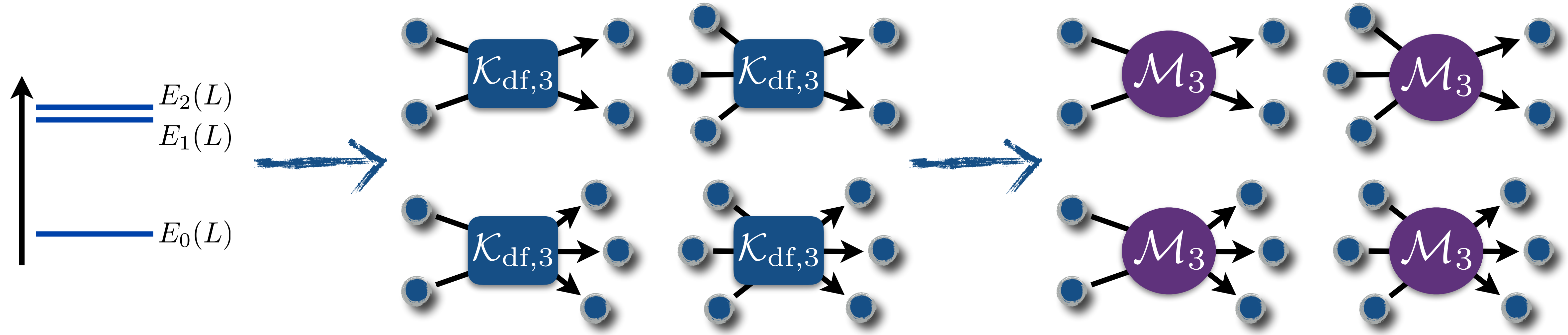
- ☑ **Result relies on an intermediate three-particle quantity**



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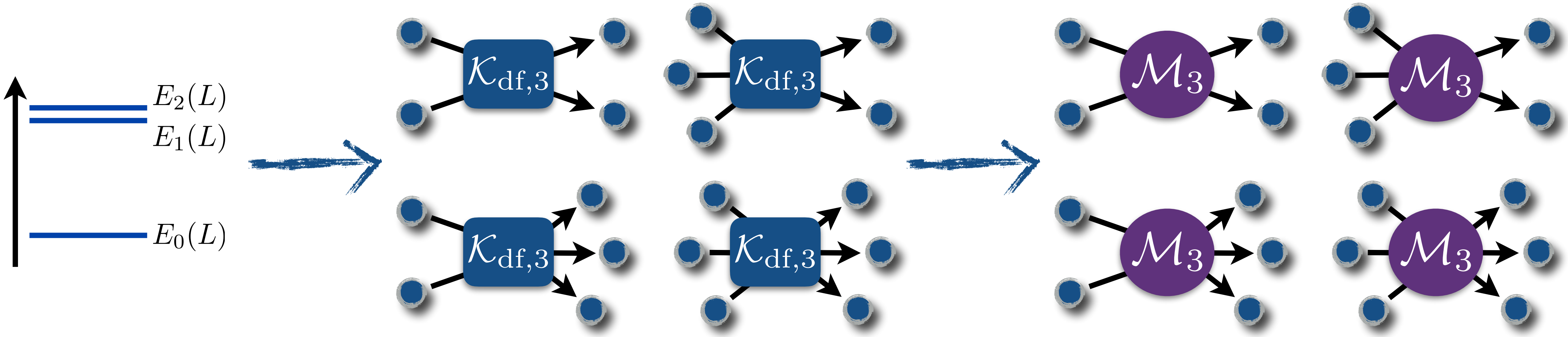
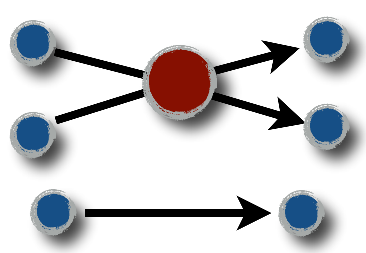


- ☐ Assumes no sub-channel two-particle resonances



Extension to remove this restriction is nearing completion

MTH, Sharpe (2015),(2016) ○ **Briceño, MTH, Sharpe** (2017)

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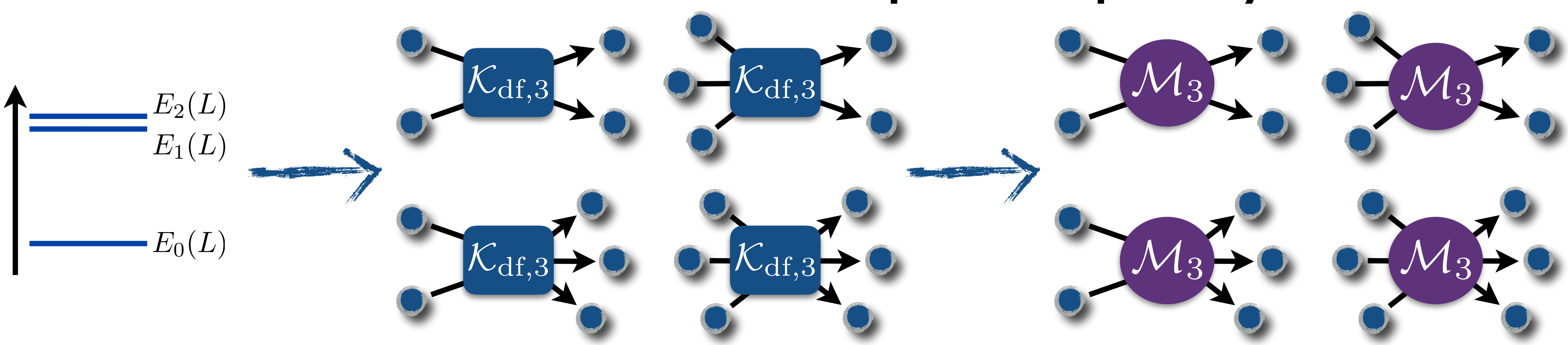

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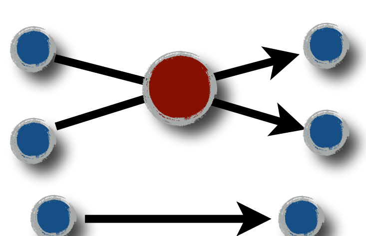


Extension to remove this restriction is nearing completion

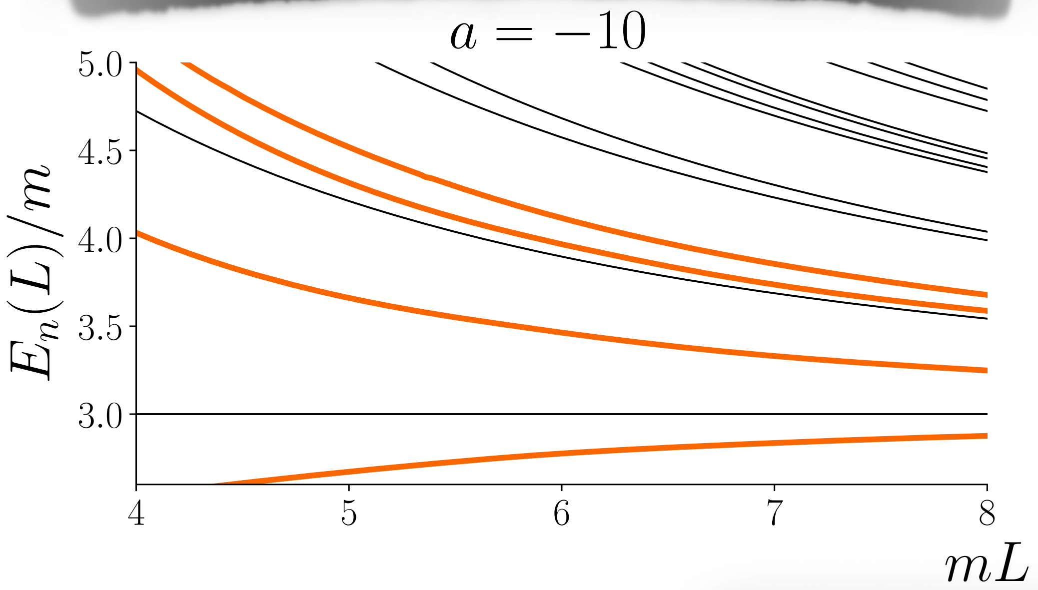
- ☐ Derivation assumes identical scalar particles

MTH, Sharpe (2015),(2016) ○ **Briceño, MTH, Sharpe** (2017)

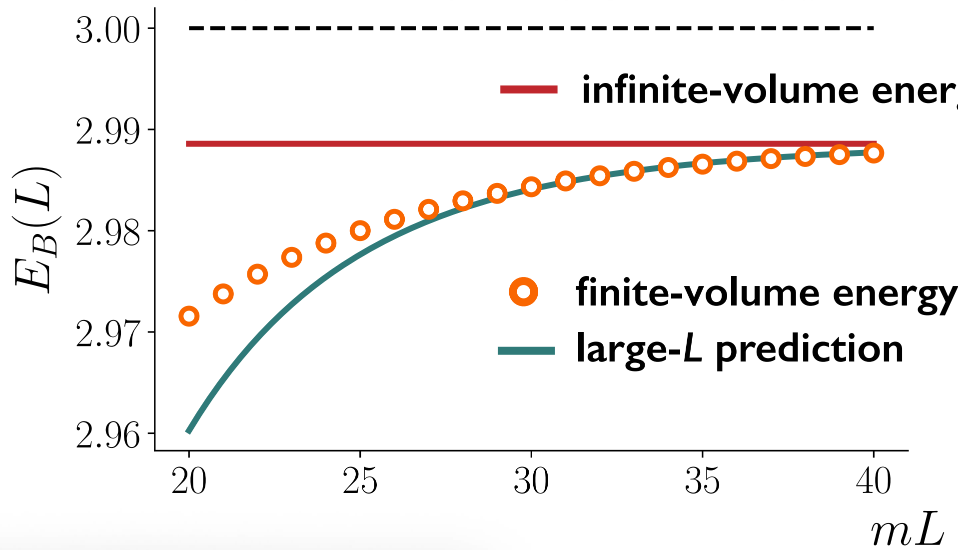
Toy numerics

We are also exploring the result numerically to quantitatively understand the relation between finite- and infinite-volume three-particle physics

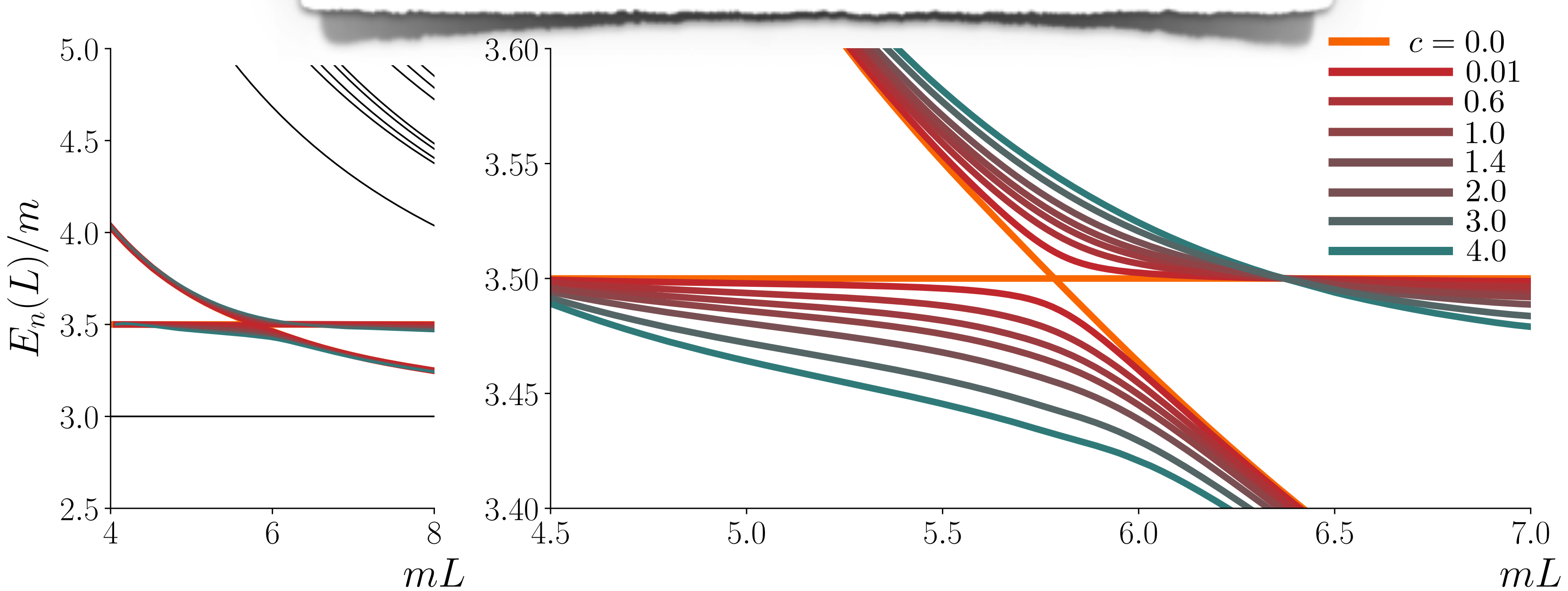
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state

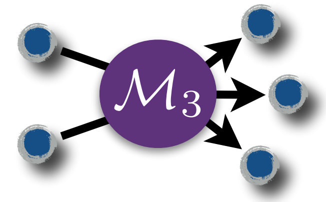


Model of a 3-particle resonance



Briceño, MTH, Sharpe *(tomorrow)*

Simplifications (for numerical investigation)

- Impose a Z_2 symmetry  $= 0$

- Set total momentum to zero $\vec{P} = 0$

- Take two-to-two scattering to be dominated by s-wave scattering length

$$\underbrace{\mathcal{M}_2(E_2^*, \cos\theta^*) \approx \mathcal{M}_2^s(E_2^*)}_{\text{s-wave dominance}} \qquad \underbrace{\mathcal{M}_2^s(E_2^*) \propto \frac{1}{p^* \cot\delta(p^*) - ip^*}}_{\text{unitarity}} \underbrace{\approx \frac{1}{-1/a - ip^*}}_{\text{LO threshold expansion}}$$

- Take three-to-three scattering to be "isotropic"

$$\mathcal{K}_{\mathrm{df},3}(E^2;\ \underbrace{p_1 \cdot p_2, p_1 \cdot p_2', \cdots}_{\text{7 compact degrees of freedom}}) \approx \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E)$$

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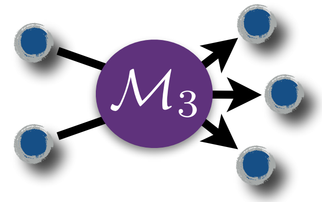
☑ Impose a Z_2 symmetry 

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unitarity

LO threshold expansion

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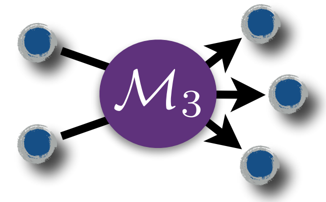
Simplified quantization condition

$$F_3^{\mathrm{iso}}(E_n, L, a) = -1/\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E_n)$$

known function
(high precision straightforward)

Simplifications (for numerical investigation)

☑ Impose a Z₂ symmetry 

$= 0$

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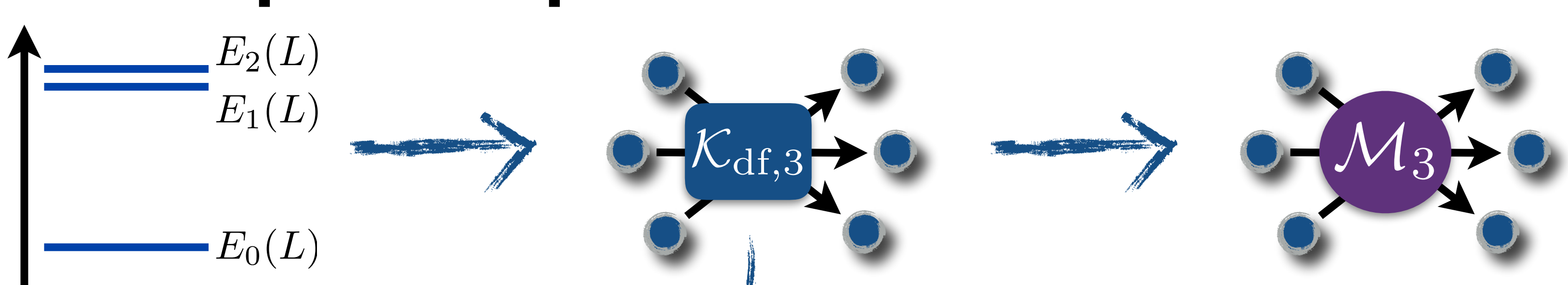


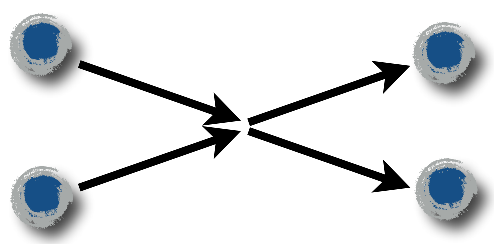
$$F_3^{\mathrm{iso}}(E_n, L, a) = -1/\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E_n)$$

known function
(high precision straightforward)

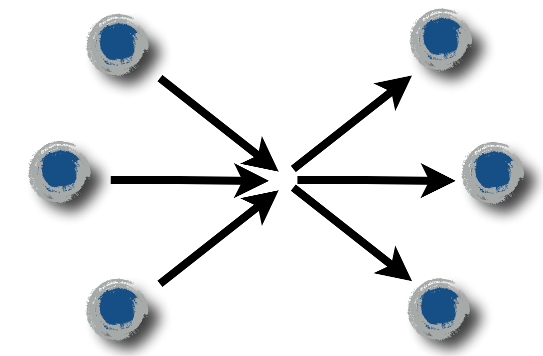
$$\mathcal{M}_3(E, \Omega_3', \Omega_3) = \mathcal{I}\left[\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E), a\right]$$

known function
(requires solving an integral equation, straightforward below threshold)

Relating E and $K_{df,3}$



$$F_2^s(E_{2,n}, L) = -1/\mathcal{K}_2(E_{2,n})$$



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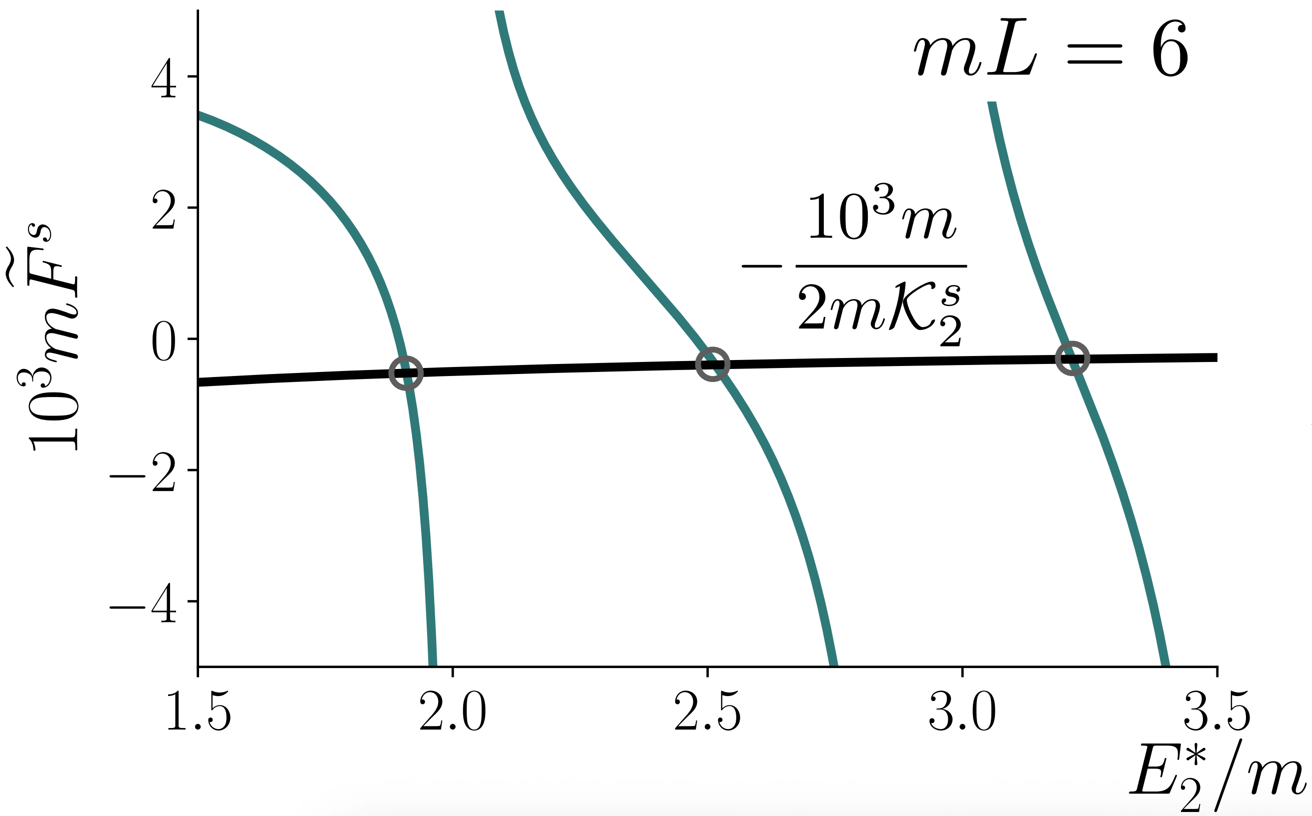
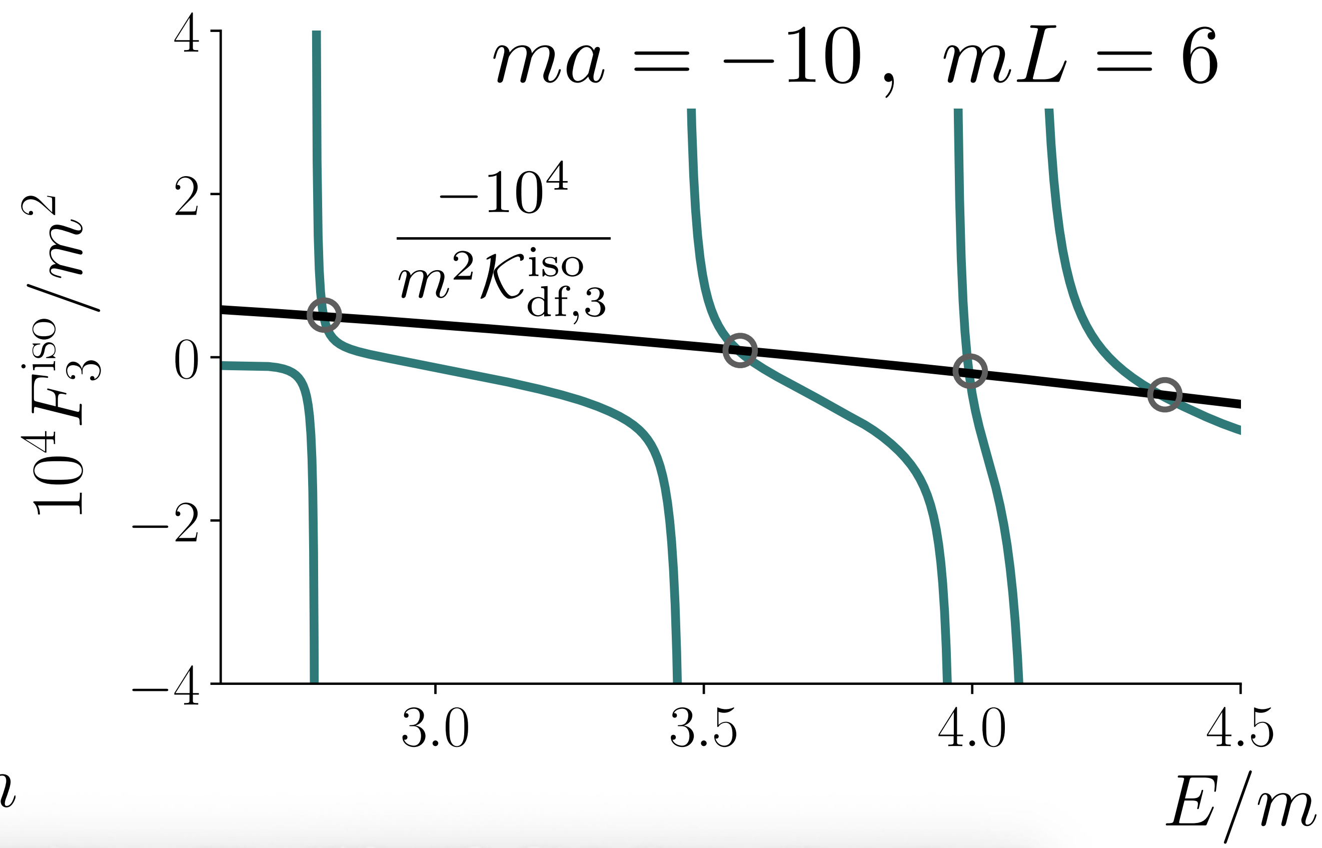




Within this approximation, the two- and three-particle sectors are closely analogous

Unitary bound state

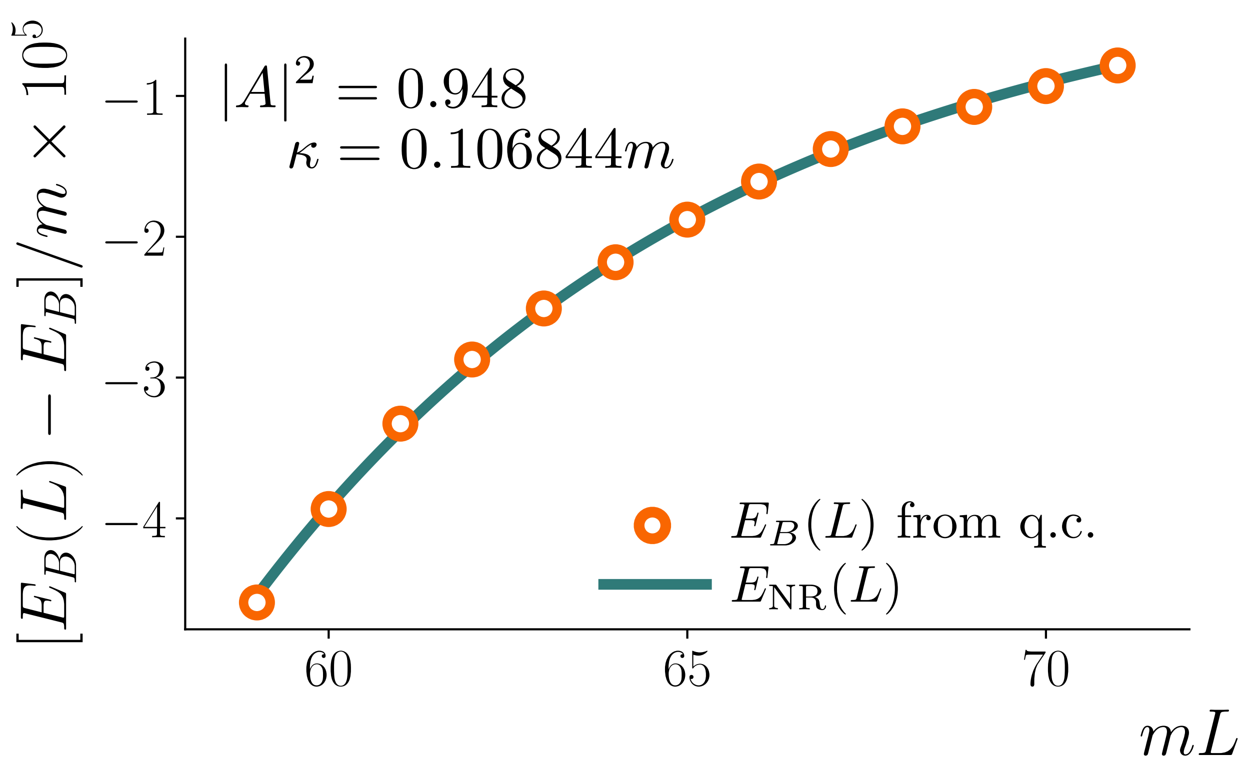
The parameters $a = -10^4$, $\mathcal{K}^{\text{iso}}_{\text{df},3}(E) = 2500$ lead to a shallow bound state

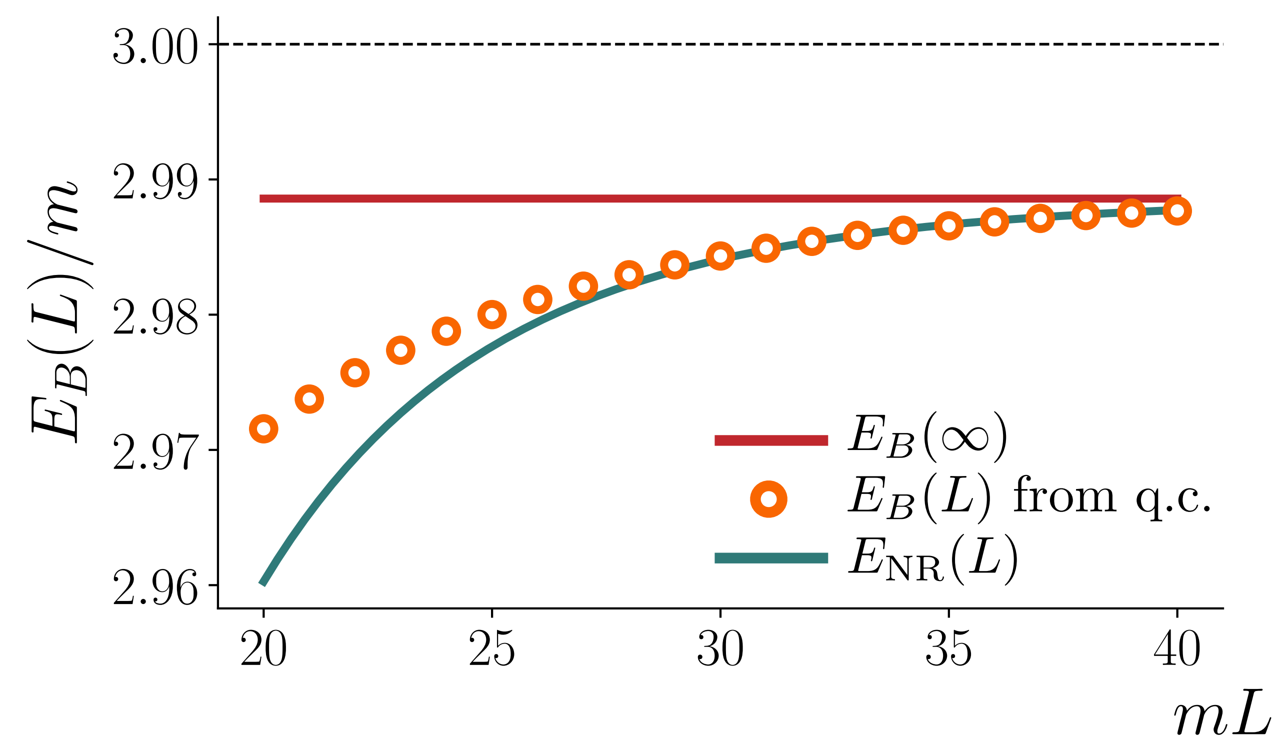
$\kappa \approx 0.1m$ **where** $E_B = 3m - \kappa^2/m$

Finite-volume behavior of this state has a known asymptotic form

Meißner, Rios, Rusetsky (2015)

$$E_B(L) = 3m - \frac{\kappa^2}{m} - (98.35\cdots)|A|^2 \frac{\kappa^2}{m} \frac{e^{-2\kappa L/\sqrt{3}}}{(\kappa L)^{3/2}} \left[1 + \mathcal{O}\left(\frac{1}{\kappa L}, \frac{\kappa^2}{m^2}, e^{-\alpha\kappa L}\right)\right]$$





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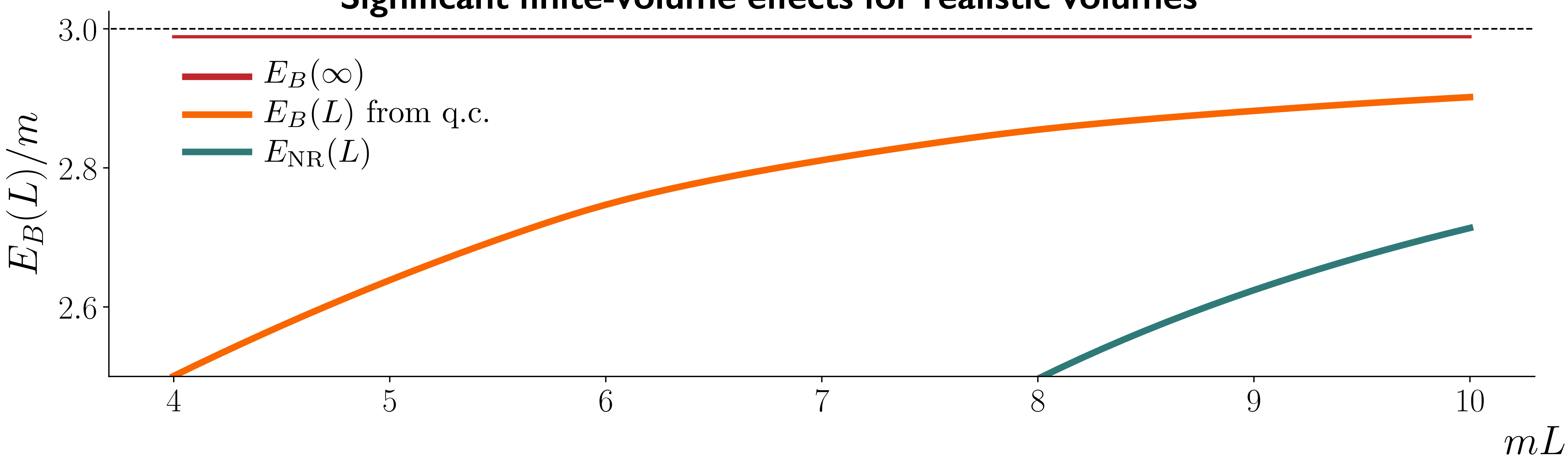
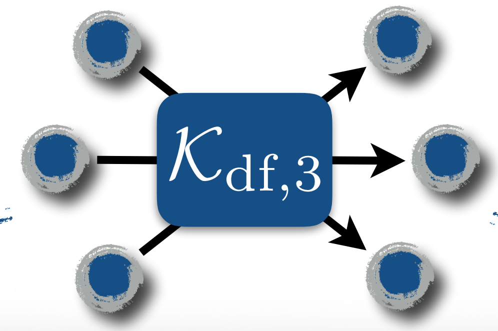
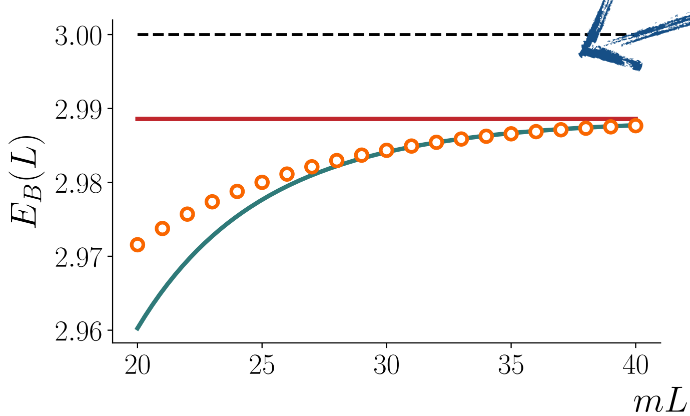
Significant finite-volume effects for realistic volumes



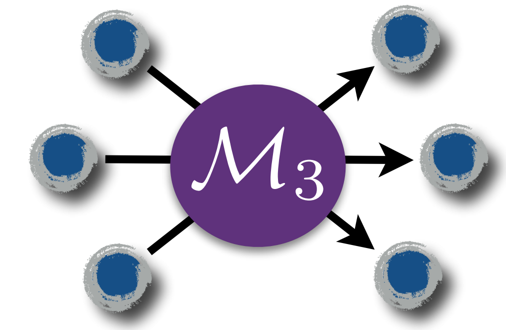
Relation to the scattering amplitude



Discussed so far

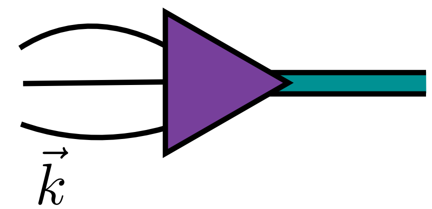


What about this?



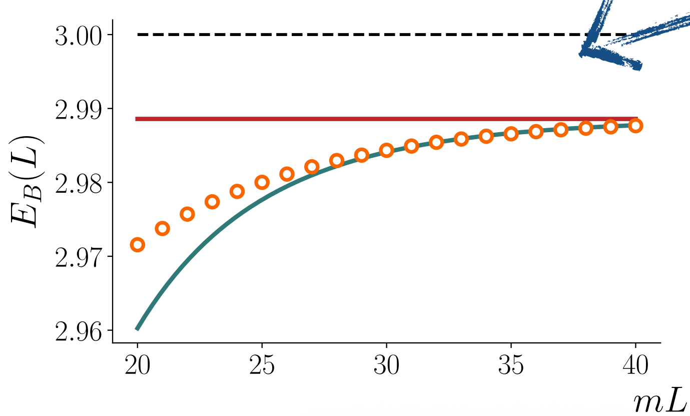
Bound state wave function

$$\Gamma^u(k) \equiv$$



$$\mathcal{M}_3 \sim -\frac{\Gamma(\Omega_3')\overline{\Gamma}(\Omega_3)}{E^2 - M_B^2}$$

Relation to the scattering amplitude



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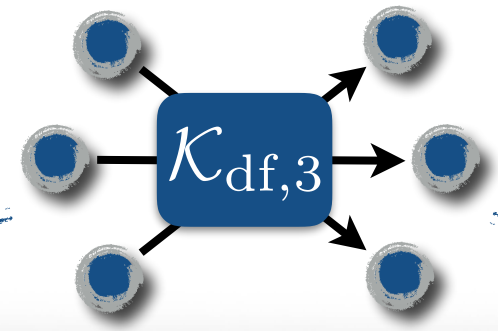

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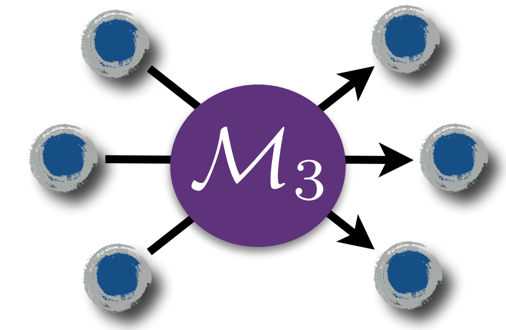

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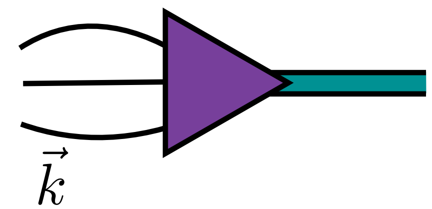
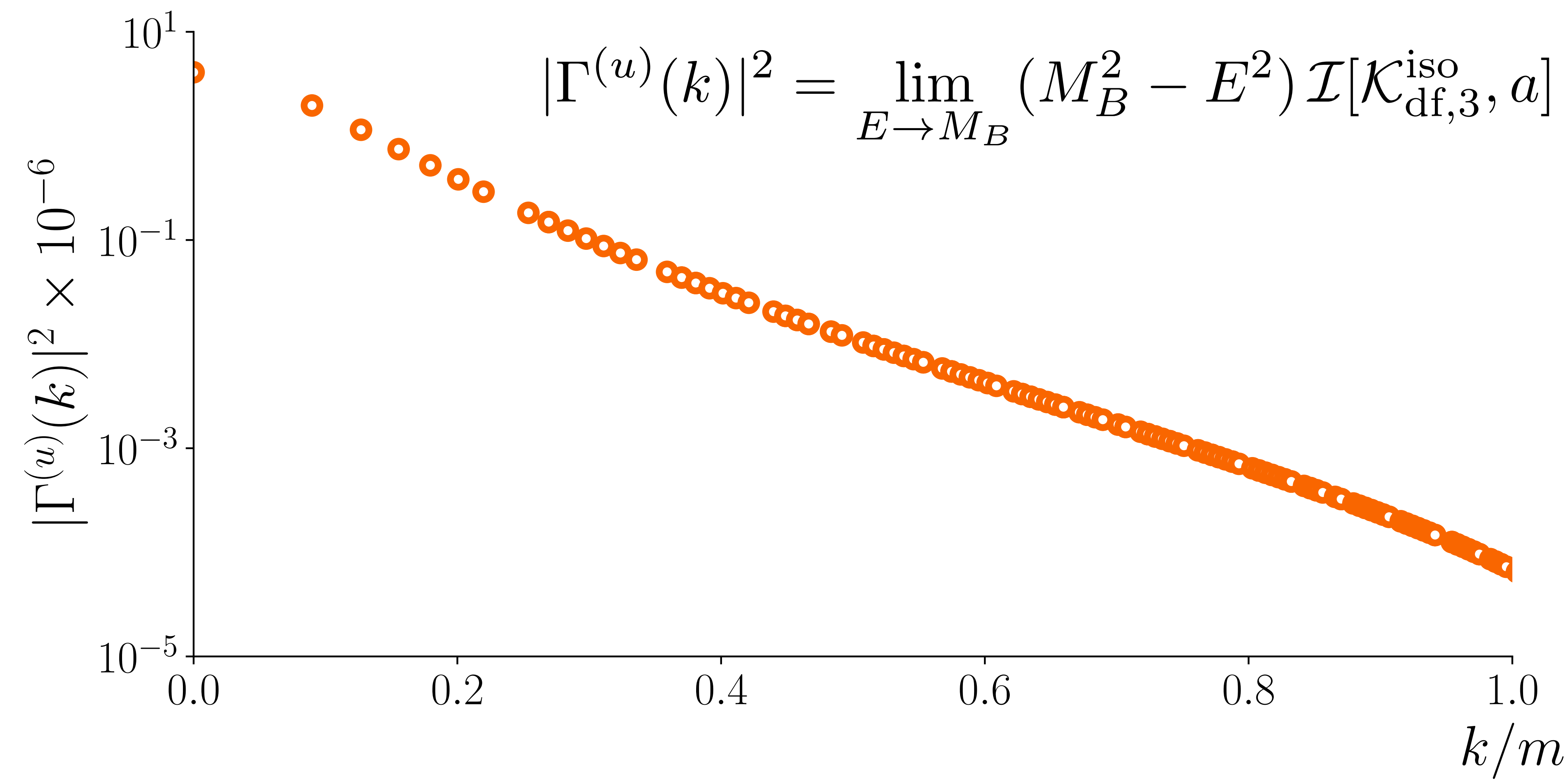
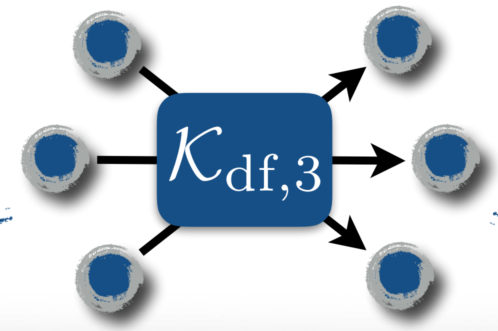


$$|\Gamma^{(u)}(k)|^2 = \lim_{E\to M_B} (M_B^2 - E^2)\,\mathcal{I}[\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}, a]$$



Relation to the scattering amplitude



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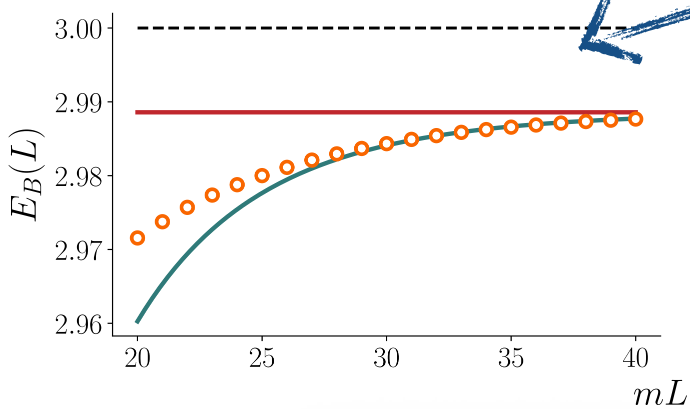

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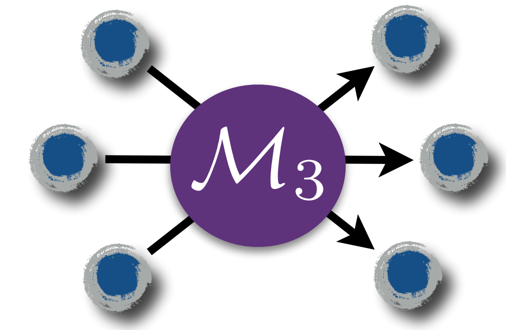
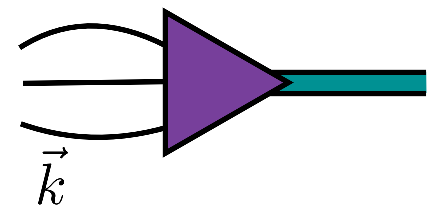
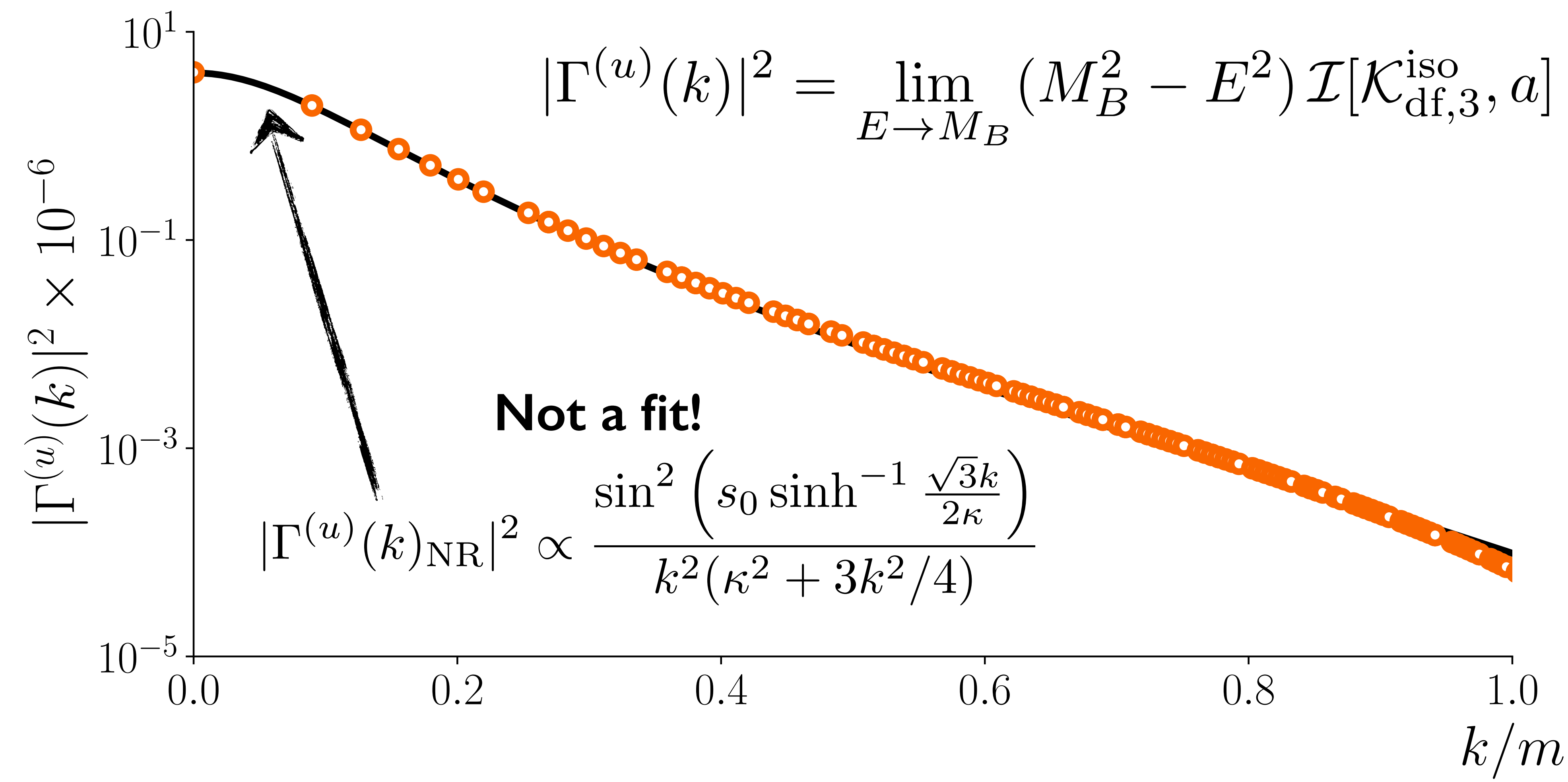


$$\mathcal{M}_3 \sim -\frac{\Gamma(\Omega_3')\overline{\Gamma}(\Omega_3)}{E^2 - M_B^2}$$

Bound state wave function

$\Gamma^u(k) \equiv$ 

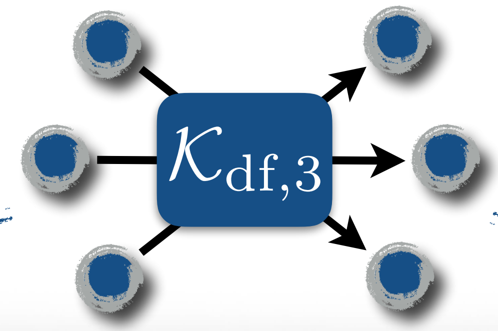


$$|\Gamma^{(u)}(k)|^2 = \lim_{E \to M_B} (M_B^2 - E^2)\, \mathcal{I}[\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}, a]$$

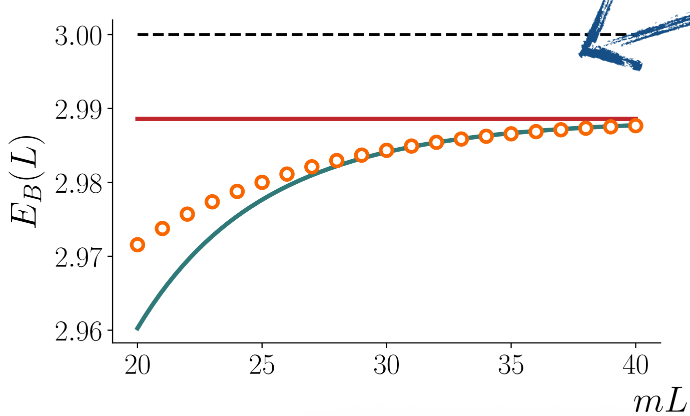
Not a fit!

$$|\Gamma^{(u)}(k)_{\mathrm{NR}}|^2 \propto \frac{\sin^2\left(s_0 \sinh^{-1} \frac{\sqrt{3}k}{2\kappa}\right)}{k^2(\kappa^2 + 3k^2/4)}$$

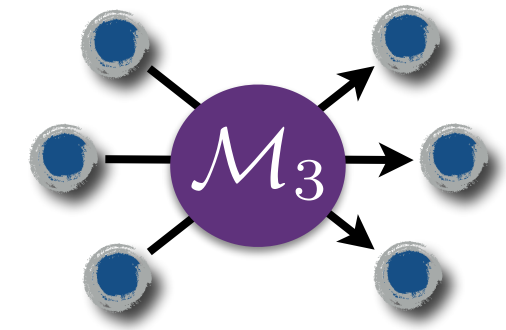
Relation to the scattering amplitude



Discussed so far



What about this?



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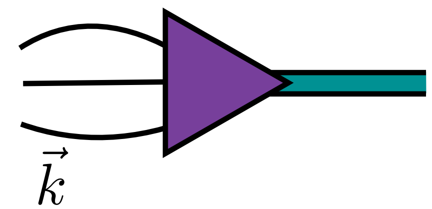
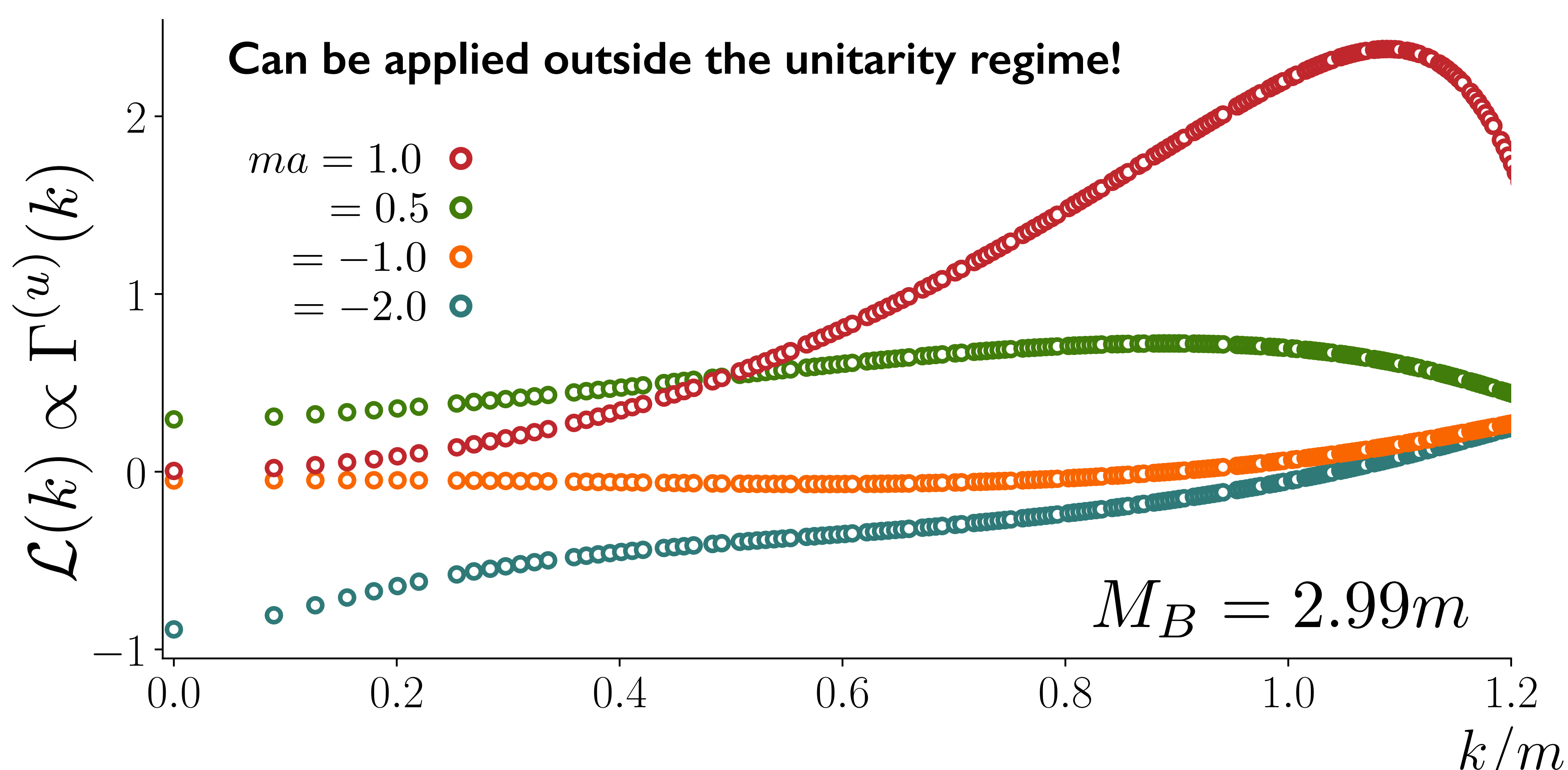
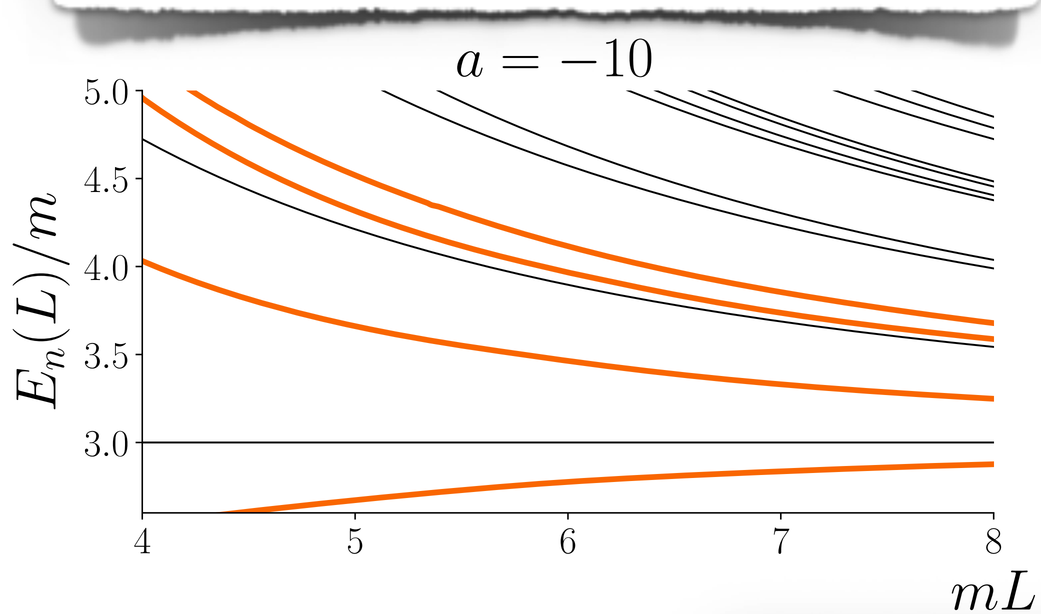




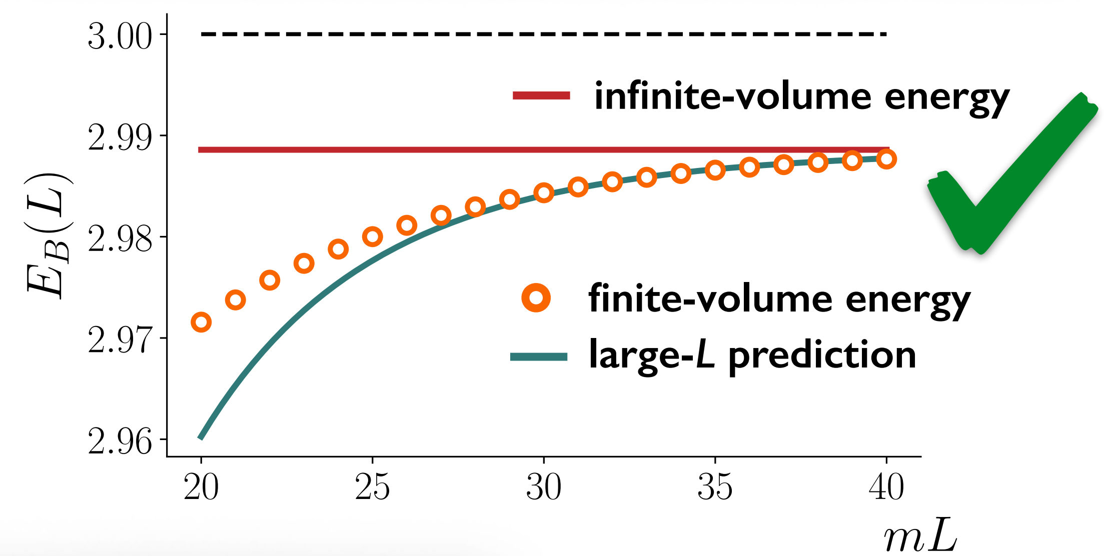
Toy numerics

We are also exploring the result numerically to quantitatively understand the relation between finite- and infinite-volume three-particle physics

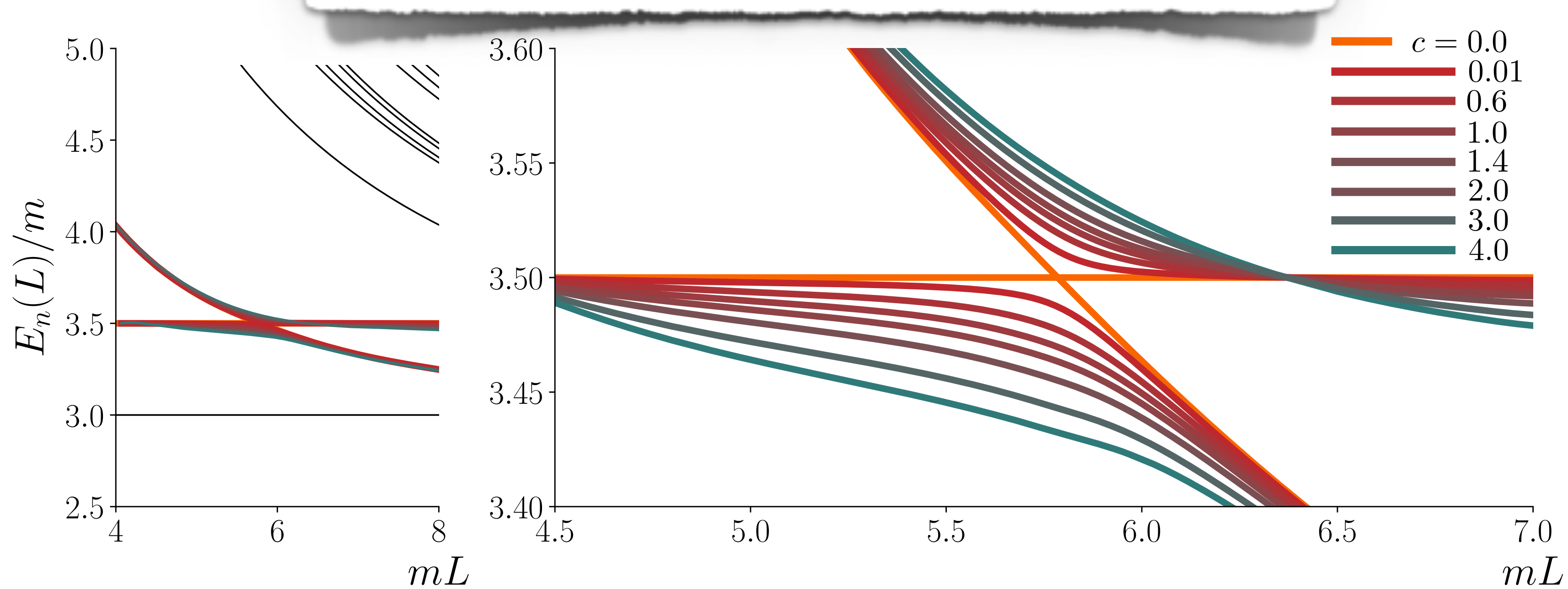
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state

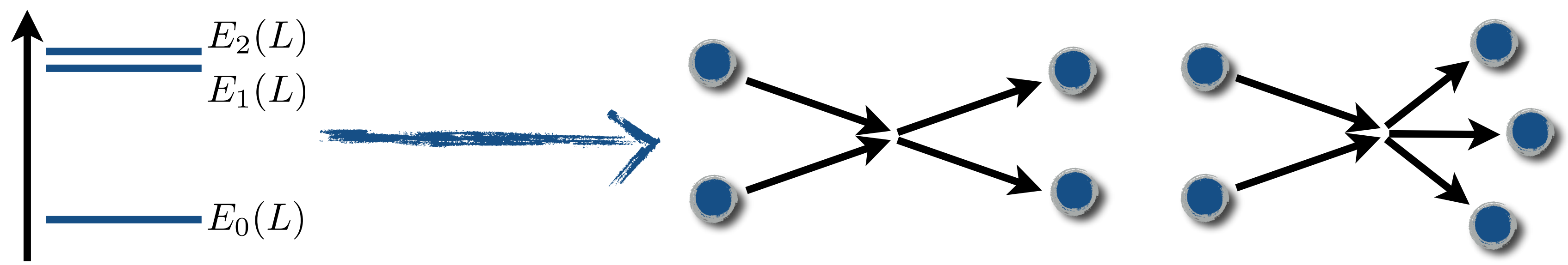


Model of a 3-particle resonance

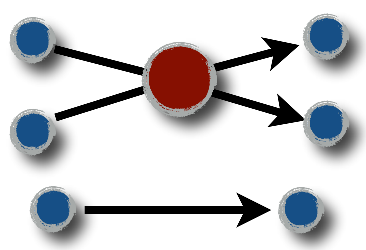


Briceño, MTH, Sharpe *(tomorrow)*

To do list…



- ☐ Remove the restriction on two-particle sub-resonances



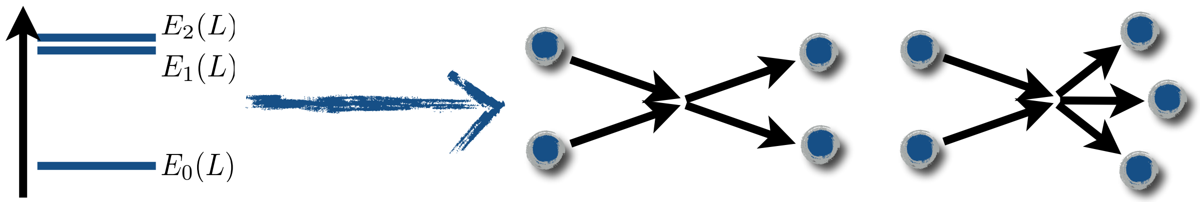
First derivation is complete, checking and writing-up

- ☐ Extend the mapping to multiple species and channels
- ☐ Develop code to explore the relations (especially beyond isotropic)
- ☐ *Understand unphysical artifacts that arise for extreme values of $K_{df,3}$*

- ☐ Derive the formalism for three-particle transition amplitudes
- ☐ **Perform the first three-particle LQCD calculation**

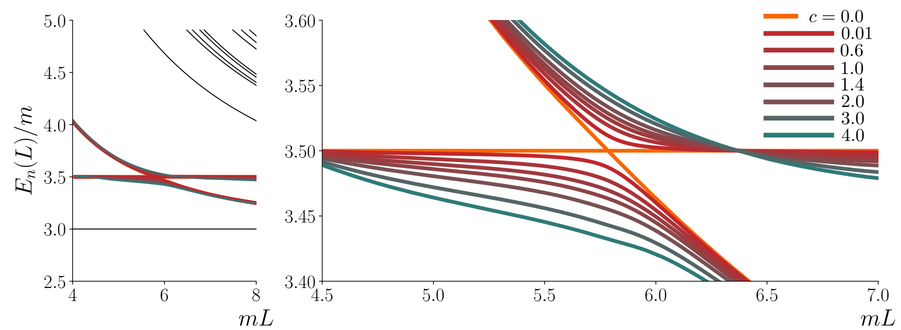
The big picture…

Energies and scattering



Two-particle sector is reaching maturity (see Raul Briceño and John Bulava's talks)

The three-particle frontier is around the corner



The big picture…

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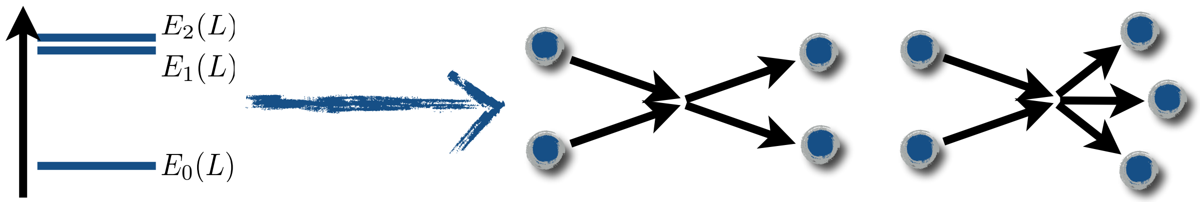

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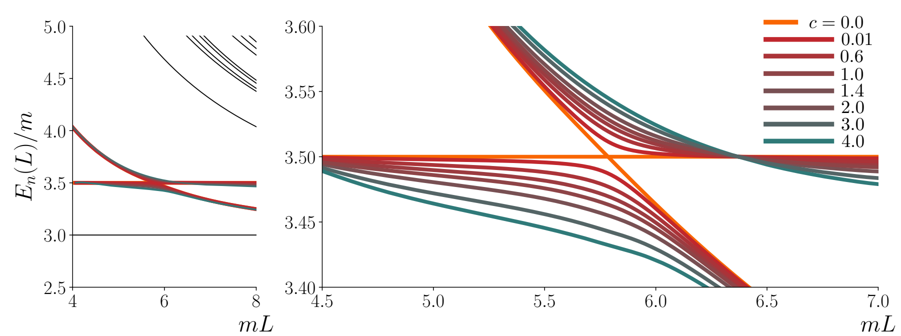
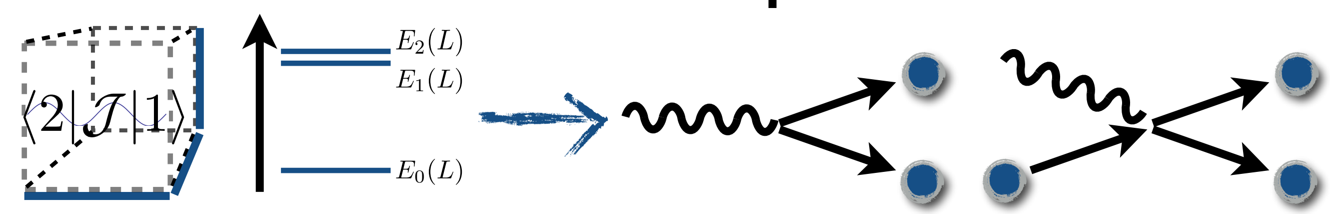
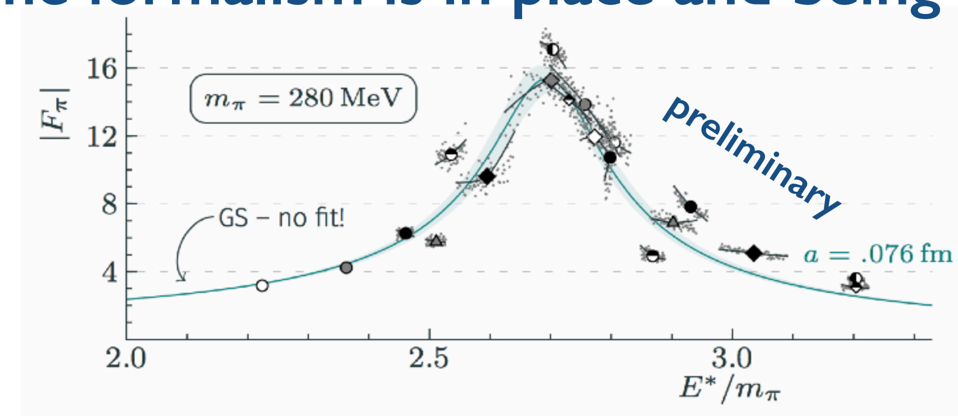


Transition amplitudes



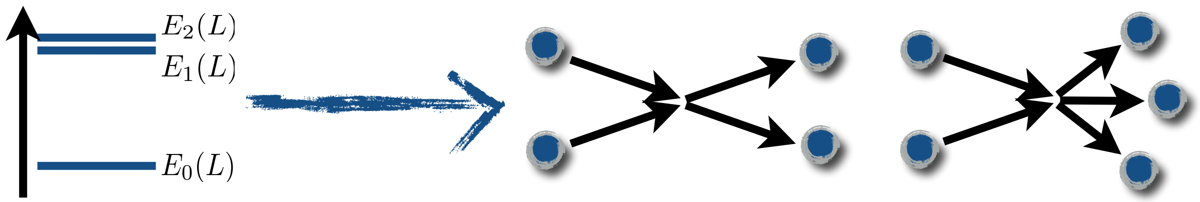
The formalism is in place and being used



See John's talk

The big picture…

Energies and scattering



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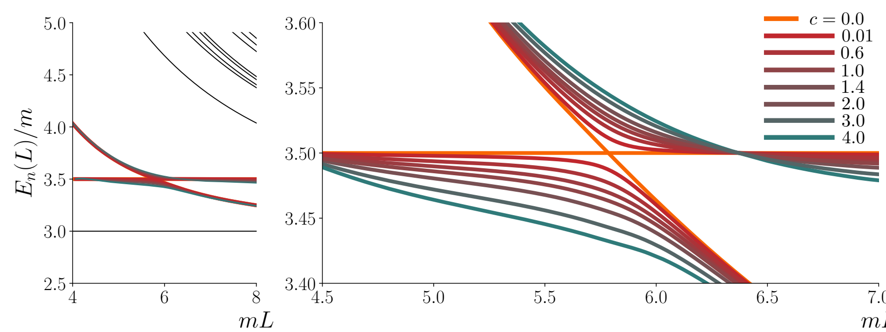
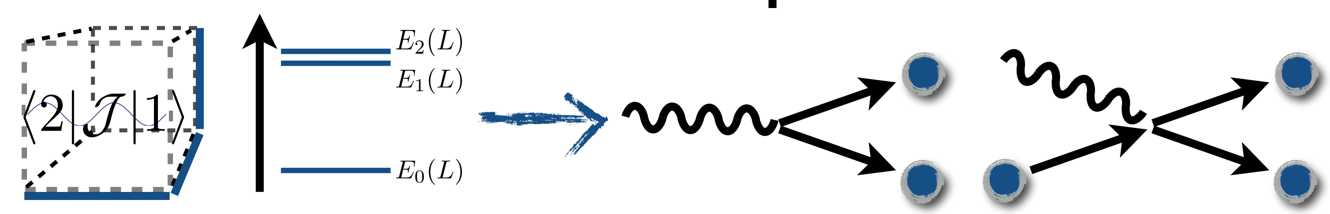


Transition amplitudes



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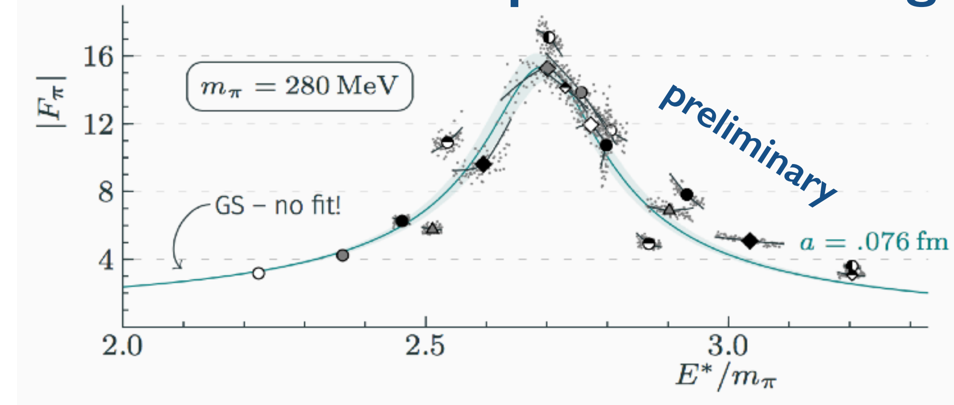
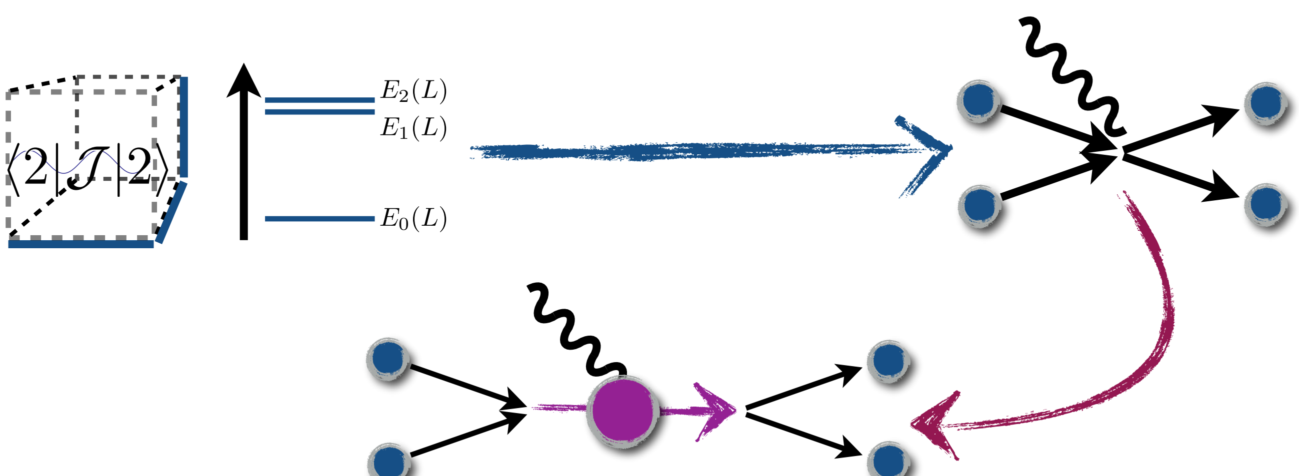


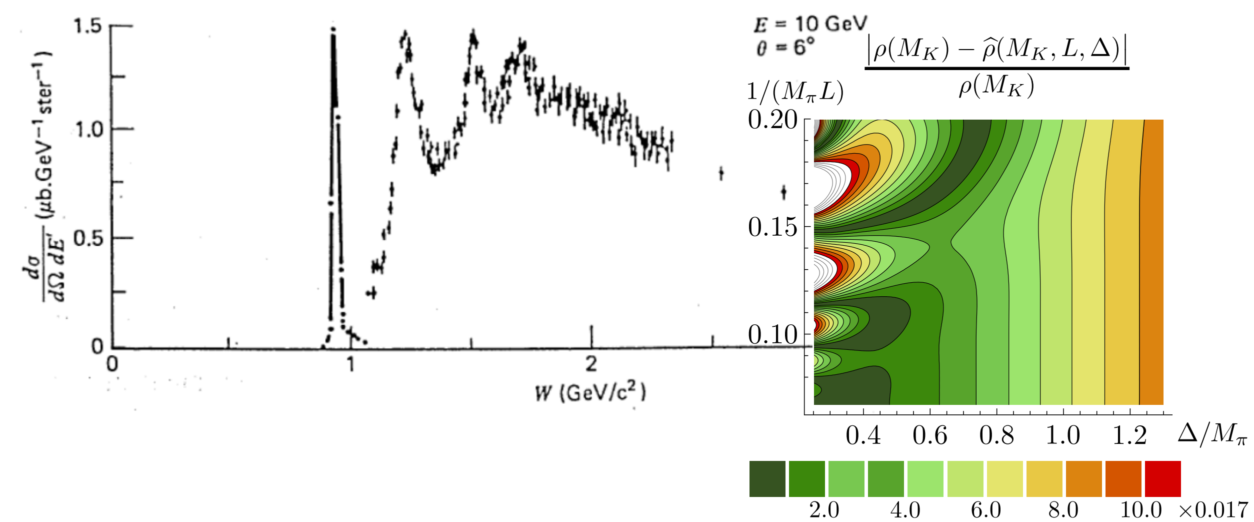
See John's talk

Resonant form factors



Stay tuned for the first resonant form factor from LQCD!

From resonances to high-energy scaling

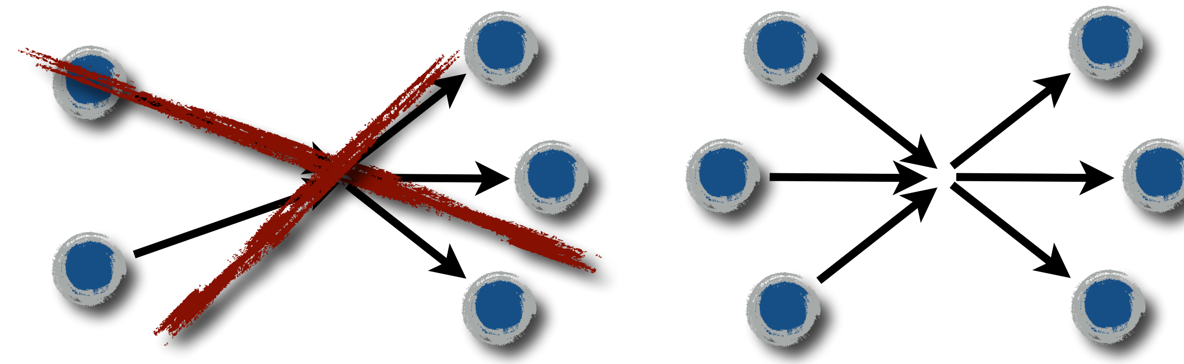


A new idea we are starting to test

Thanks for listening!

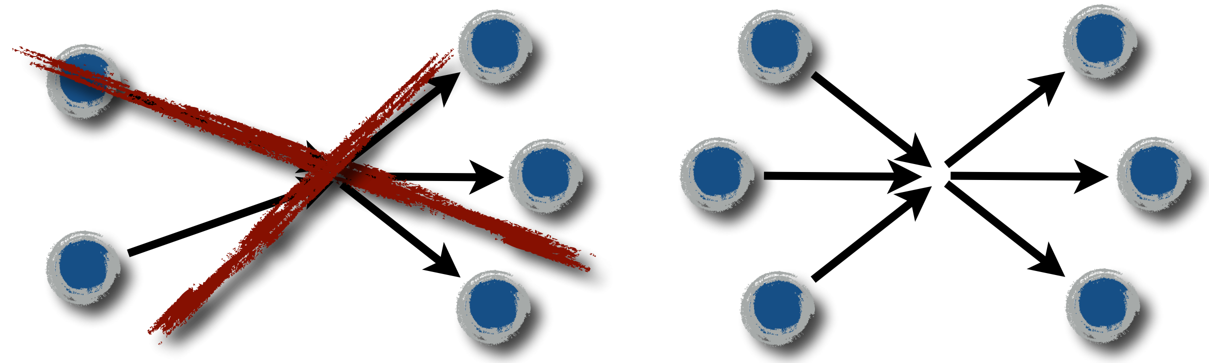
Backup Slides

We begin by considering identical scalar particles



For now we turn off two-to-three scattering using a symmetry

We begin by considering identical scalar particles



For now we turn off two-to-three scattering using a symmetry

Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator with six external legs amputated and projected on shell

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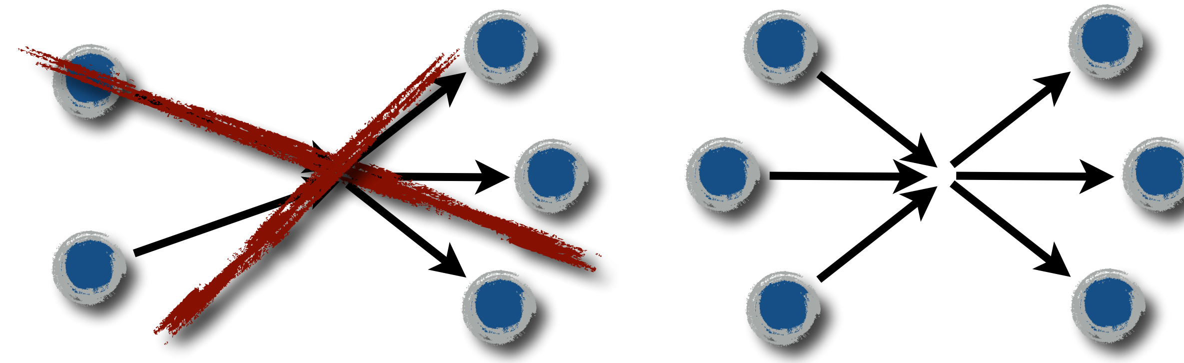

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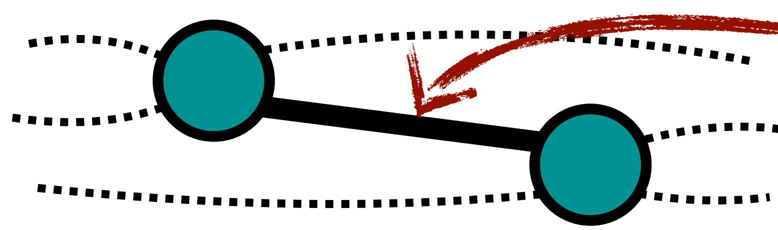
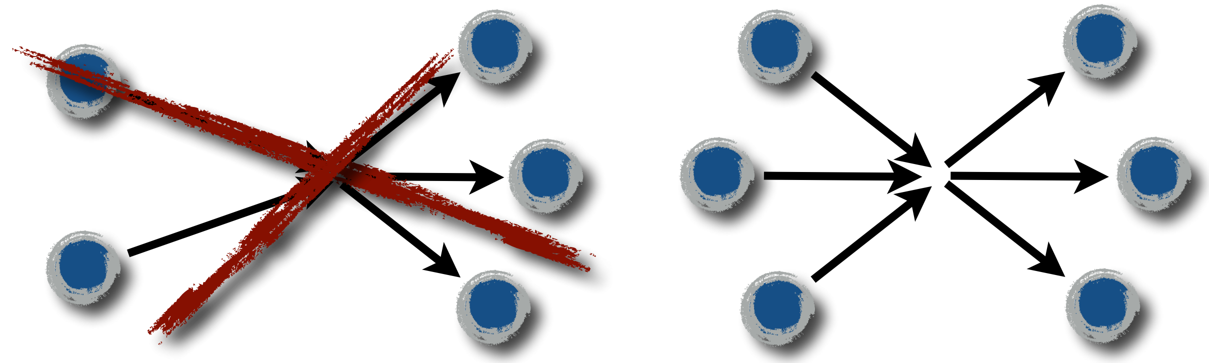
<u>Three-to-three amplitude has kinematic singularities</u>

$i\mathcal{M}_{3\to 3} \equiv$ fully connected correlator with six external legs amputated and projected on shell

$=$  $+ \cdots$

Certain external momenta put this on-shell!

We begin by considering identical scalar particles



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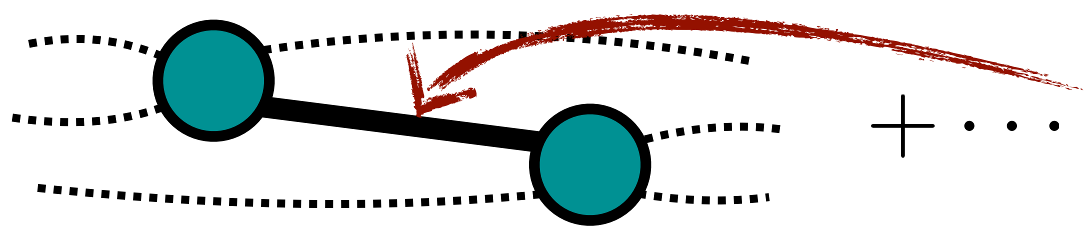
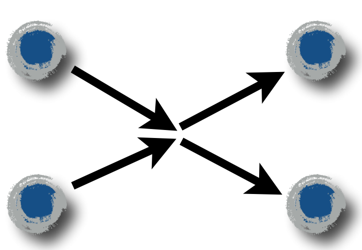
$=$  $+ \cdots$

Certain external momenta put this on-shell!

Three-to-three amplitude has more degrees of freedom

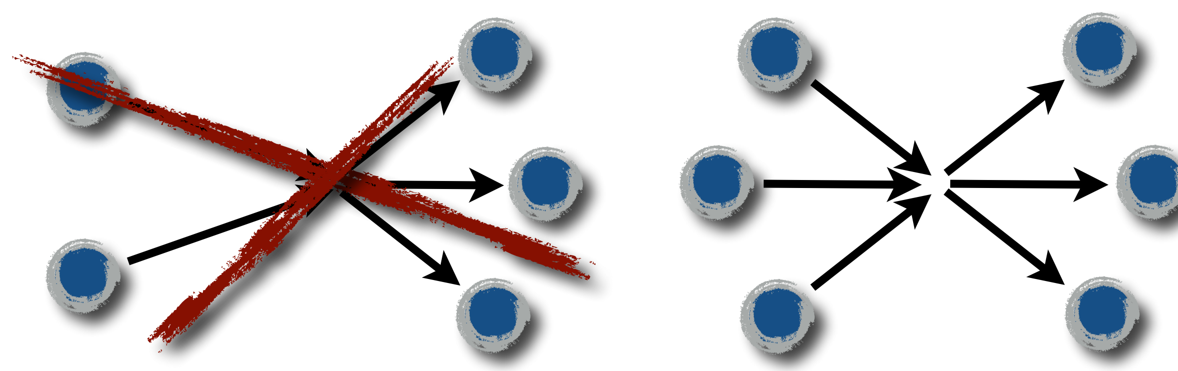


12 momentum components

-10 Poincaré generators

2 degrees of freedom

We begin by considering identical scalar particles



For now we turn off **two-to-three scattering** using a symmetry

Three-to-three amplitude has kinematic singularities

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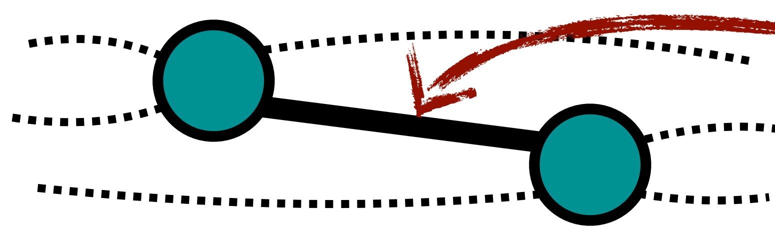
$$=$$  $$+\cdots$$

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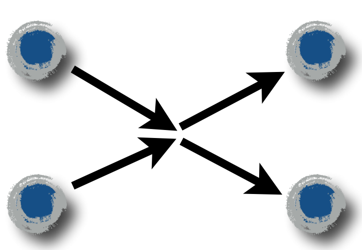

12 momentum components

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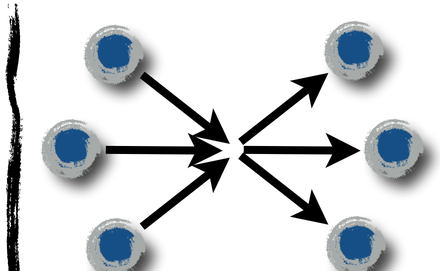


18 momentum components

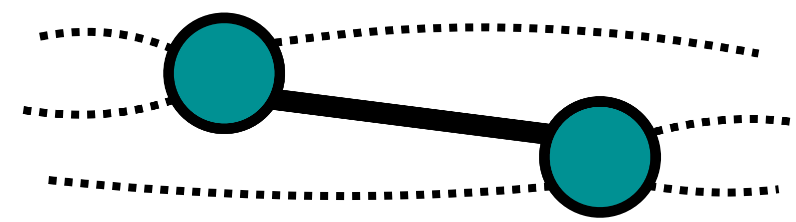
-10 Poincaré generators

8 degrees of freedom

How can we extract a singular, eight-coordinate function using finite-volume energies?

Spectrum depends on a modified quantity with singularities removed

$$\mathcal{K}_{\mathrm{df},3} \not\supset$$

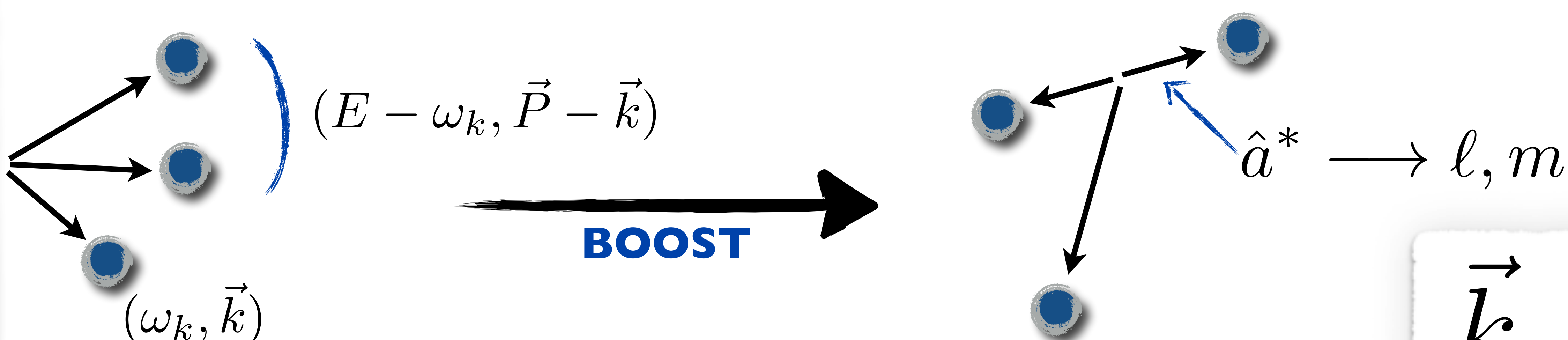


df stands for "divergence free"

Same degrees of freedom as $\mathcal{M}_3$ | Smooth, real function (easier to extract)

Relation to $\mathcal{M}_3$ is known (depends only on on-shell $\mathcal{M}_2$)

Degrees of freedom encoded in an extended matrix space



($\vec{k}$ is restricted to finite-volume momenta)

$$\vec{k}, \ell, m$$

Quantization condition

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

$$\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$$

$F_3 \equiv$ **matrix that depends on geometric functions and** $\mathcal{M}_{2\to2}$**.**

MTH and Sharpe (2014)

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MTH and Sharpe (2014)

(1). **Use two-particle q.c. to constrain $\mathcal{M}_2$ and determine $F_3(E, \vec{P}, L)$.**

$$\det[\mathcal{M}_2^{-1} + F_2] = 0 \longrightarrow \mathcal{M}_2 \longrightarrow F_3(E, \vec{P}, L)$$

Quantization condition

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(2). Use decomposition + parametrization to express $\mathcal{K}_{\mathrm{df},3}(E^*)$ in terms of α_i.

$$\mathcal{K}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) \approx \mathcal{K}_{\mathrm{df},3}[\alpha_1, \cdots, \alpha_N] \longleftarrow$$ **Recall, this is a real, smooth function**

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(3). Use three-particle q.c. with finite-volume energies to determine $\mathcal{K}_{\text{df},3}(E^*)$.

$$\det[\mathcal{K}_{\text{df},3}^{-1} + F_3] = 0 \longrightarrow \boxed{\mathcal{K}_{\text{df},3}(E^*)} ✓$$

Quantization condition

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

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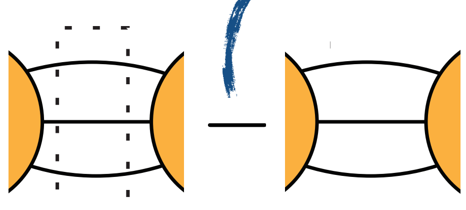
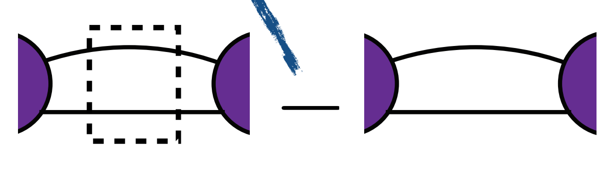
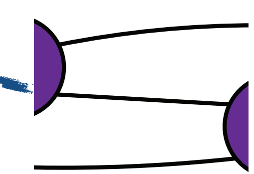
MTH and Sharpe (2014)

All of the complication is buried inside F_3

$$F_3 = \frac{F}{6\omega L^3} - \frac{F}{2\omega L^3}\frac{1}{1+\mathcal{M}_{2,L}G}\mathcal{M}_{2,L}F$$







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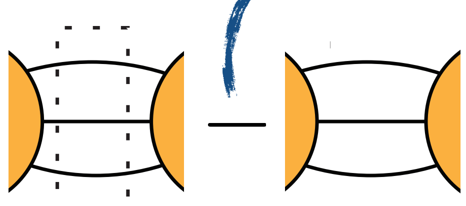
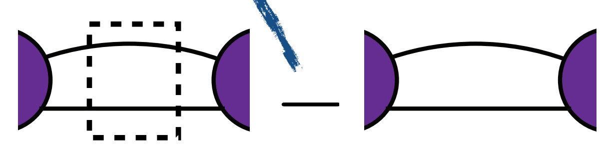
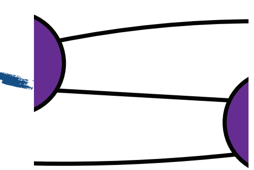






These are all matrices with indices

momentum of one particle

$$\vec{k} = \frac{2\pi\vec{n}}{L}$$

$\otimes$

angular momentum of the other two

$$\ell, m$$

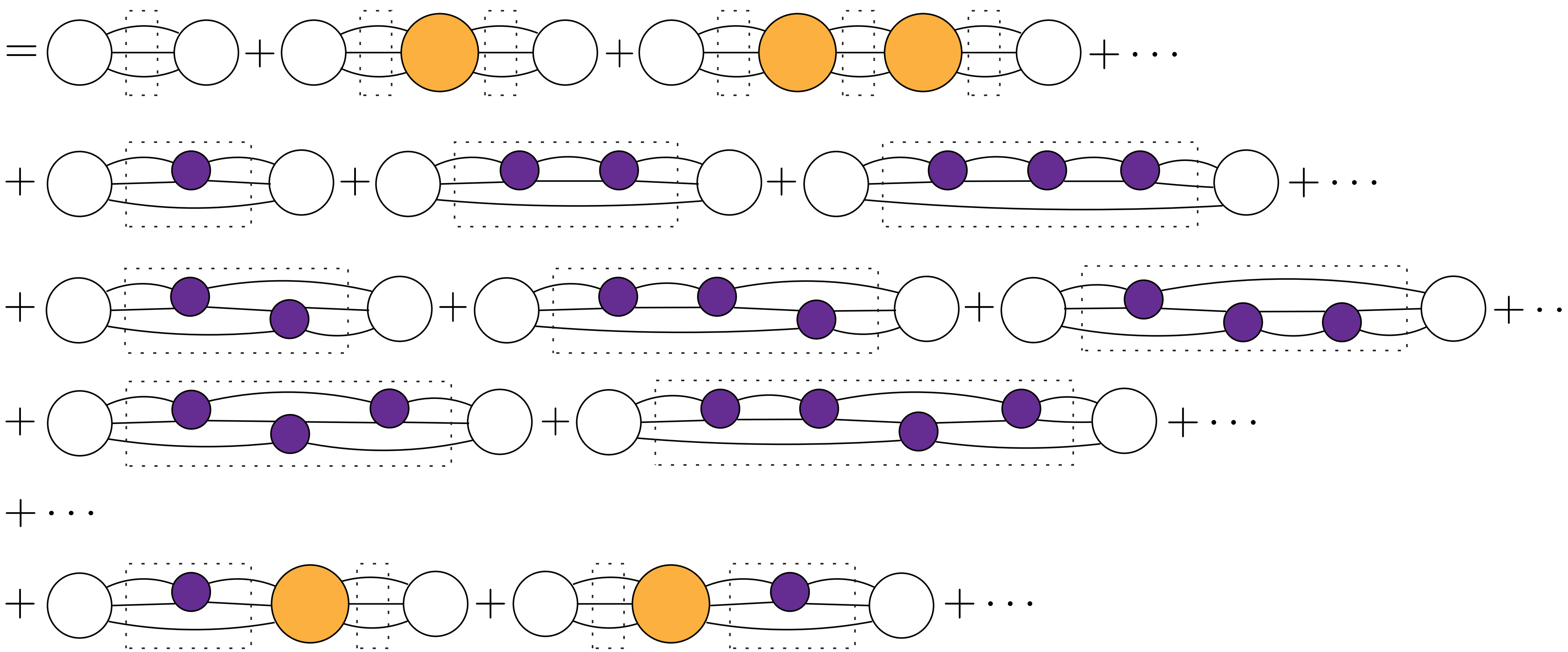
F and G are geometric functions

$\mathcal{M}_{2,L}$ depends on F and $\mathcal{M}_2$

Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3}$

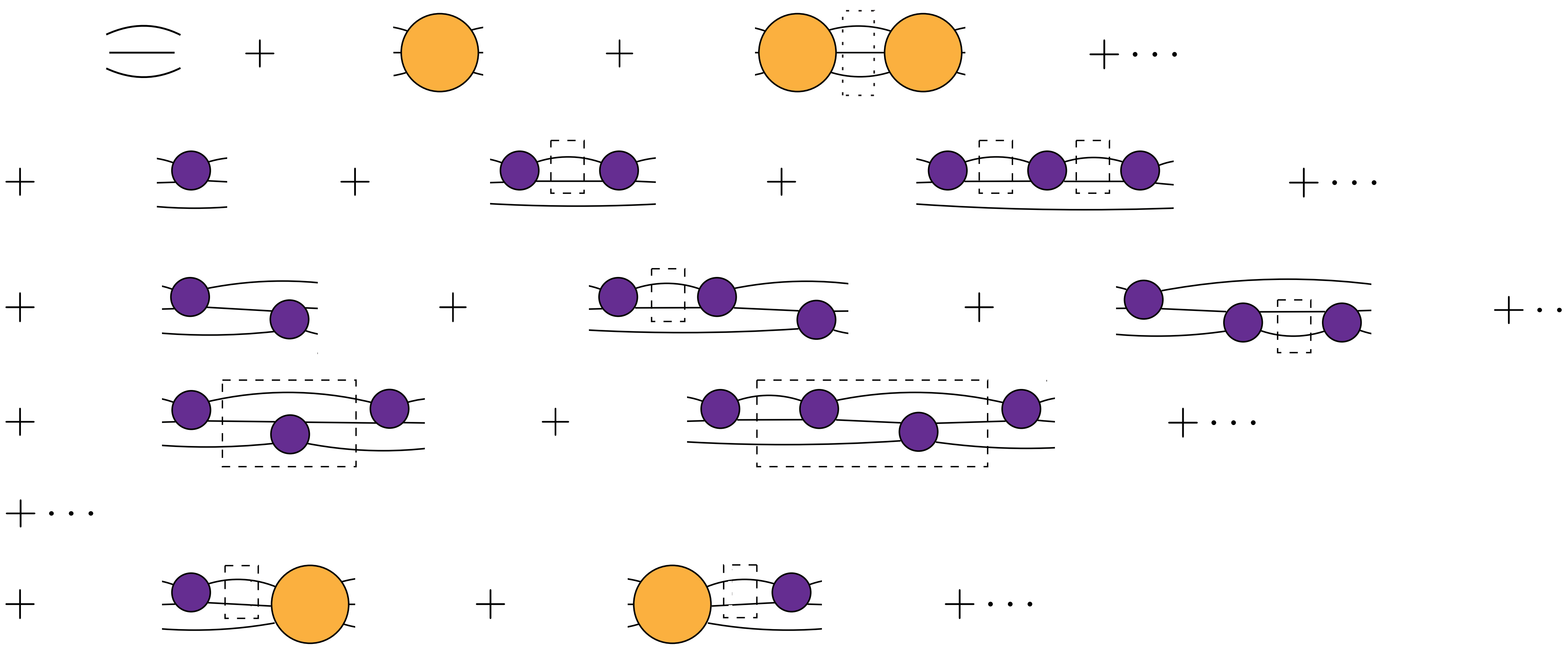
$C_L(E,\vec{P}) =$



Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

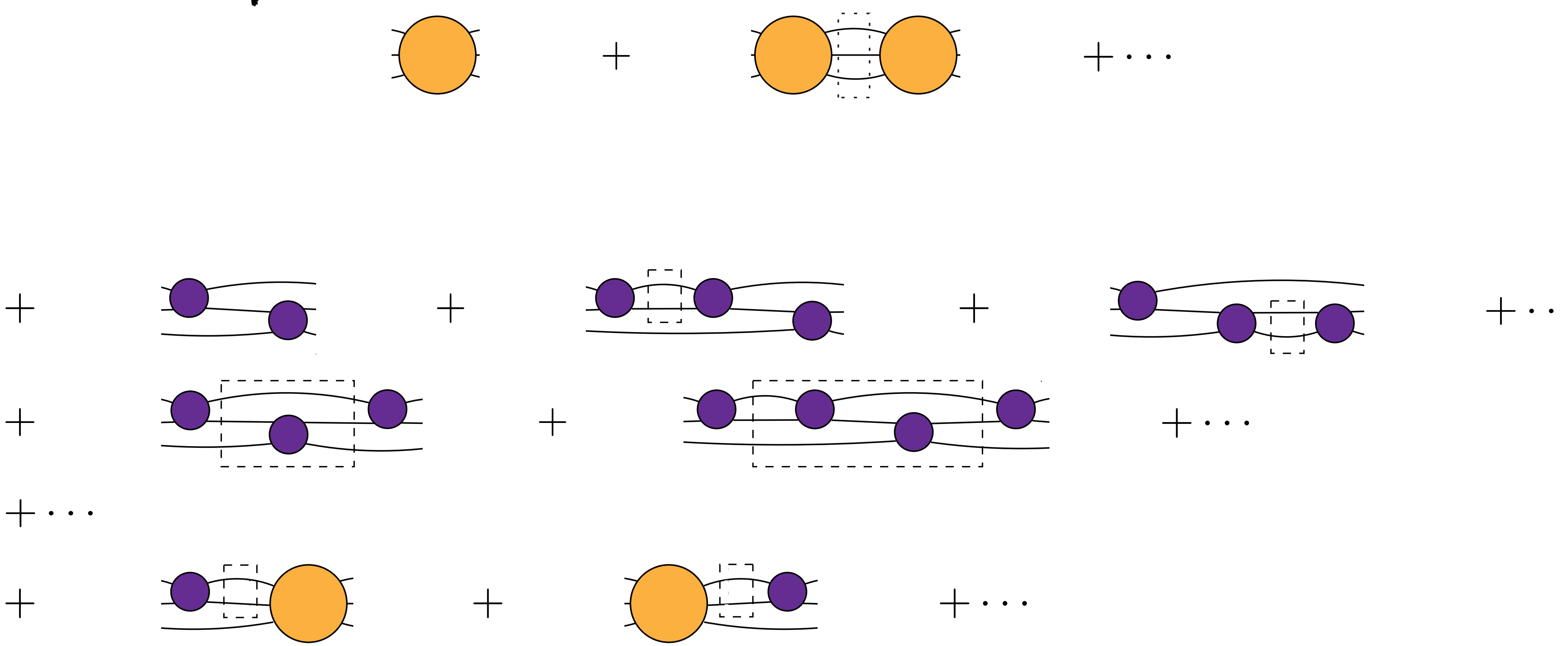
1. Amputate interpolating fields



Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3}$

1. Amputate interpolating fields
2. Drop disconnected diagrams

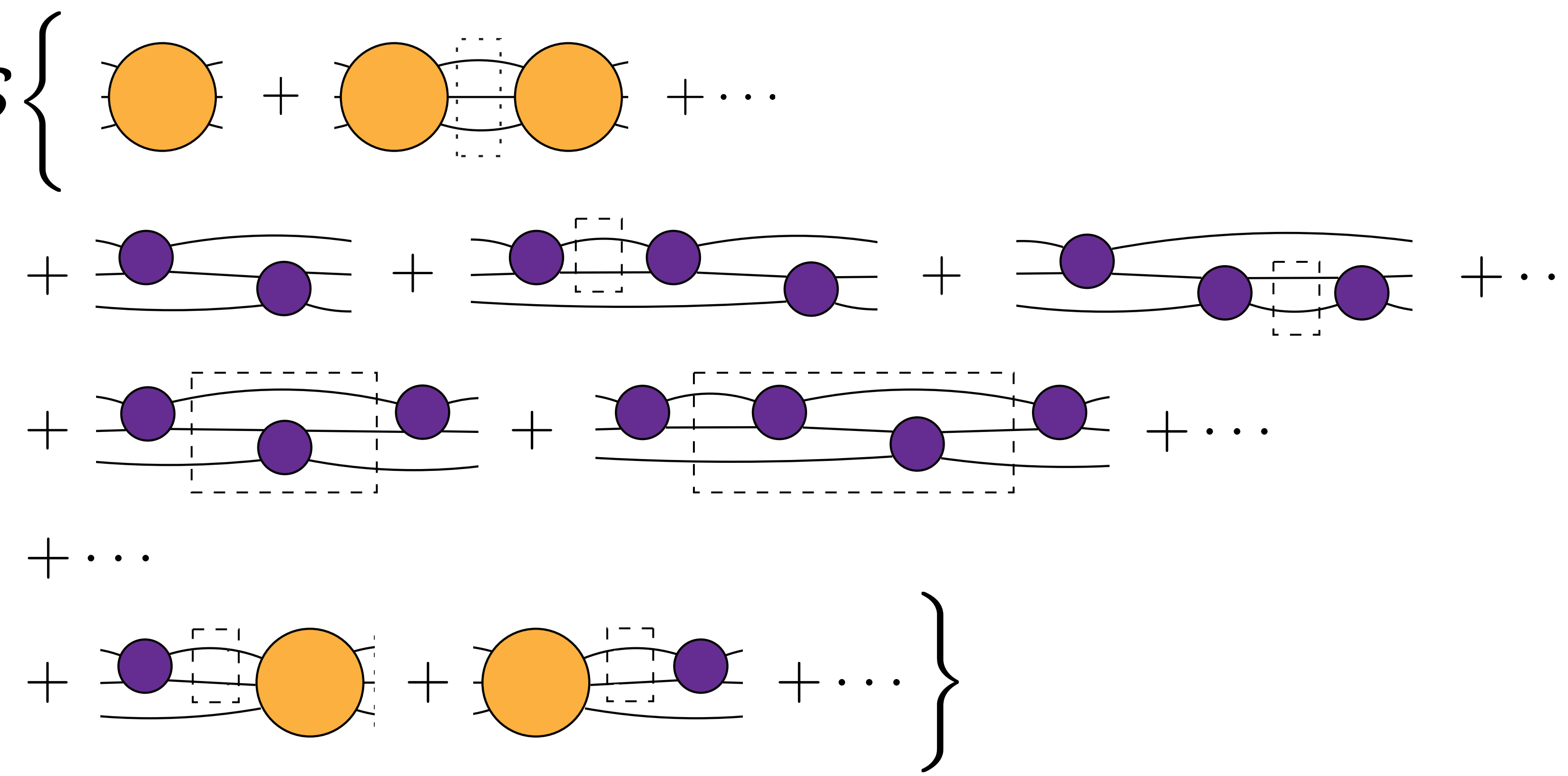


Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3}$

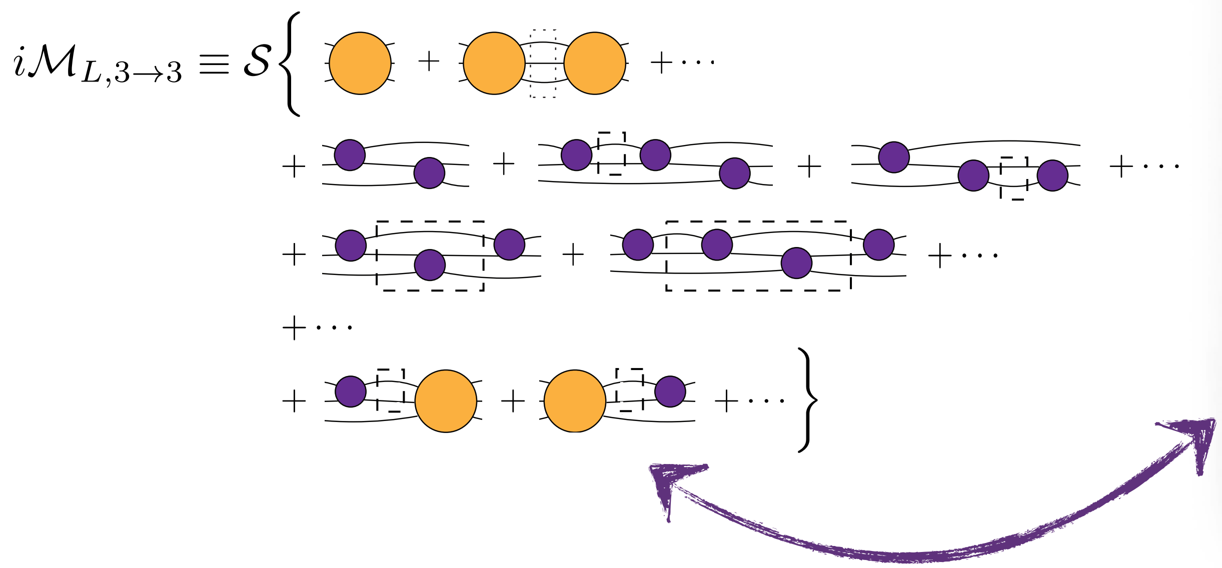
1. Amputate interpolating fields
2. Drop disconnected diagrams
3. Symmetrize

$i\mathcal{M}_{L,3\to3} \equiv \mathcal{S}$



Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

Combined with our earlier analysis this gives a matrix equation



$$\mathcal{M}_{L,3} = \mathcal{S}\left[\mathcal{D}_L + \mathcal{L}_L \frac{1}{\mathcal{K}_{\mathrm{df},3}^{-1} + F_3}\mathcal{R}_L\right]$$

$$\mathcal{L}_L = \mathcal{X} F_3, \quad \mathcal{R}_L = F_3 \mathcal{X},$$

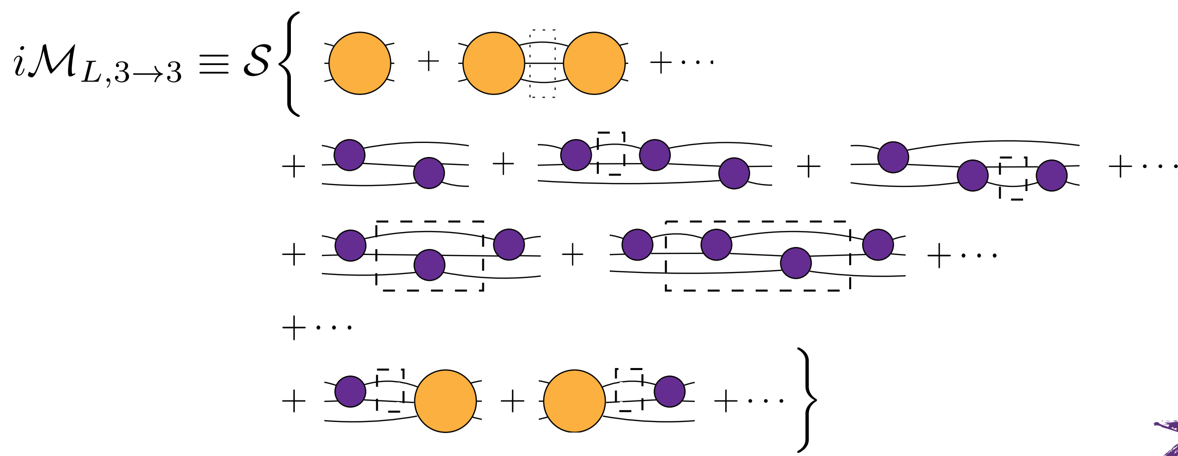
$$\mathcal{D}_L = -\mathcal{X}\left[F_3 - F_3\big|_{G\to 0}\right]\mathcal{X}$$

with the "amputation matrix" $\mathcal{X} = \left(\frac{F}{2\omega L^3}\right)^{-1}$

MTH and Sharpe (2015)

Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

Combined with our earlier analysis this gives a matrix equation



$$\mathcal{M}_{L,3} = \mathcal{S}\left[\mathcal{D}_L + \mathcal{L}_L \frac{1}{\mathcal{K}_{\mathrm{df},3}^{-1} + F_3} \mathcal{R}_L\right]$$

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With this analytic relation in hand we can…

(a) Set $E \to E + i\epsilon$, **(b) Send** $L \to \infty$, **(c) Send** $\epsilon \to 0^+$.

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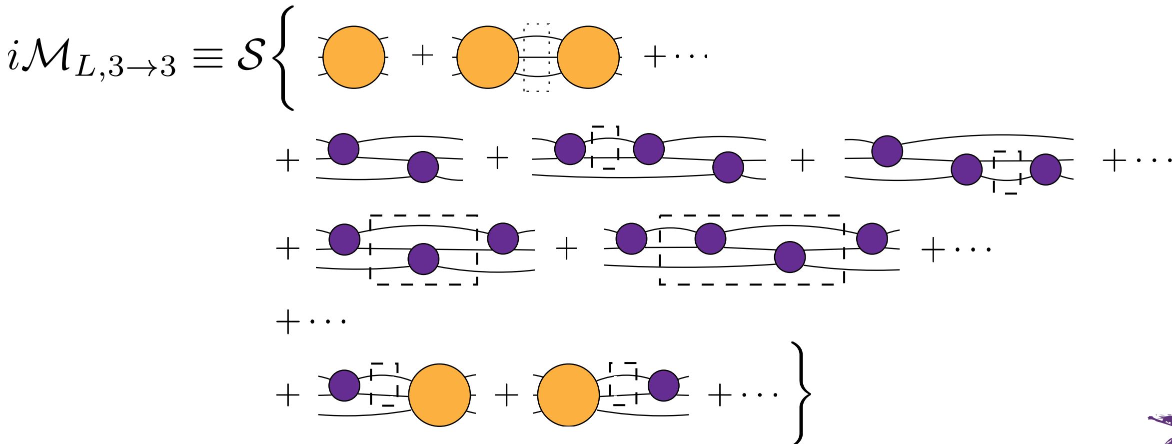

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Leads to an integral equation for the scattering amplitude

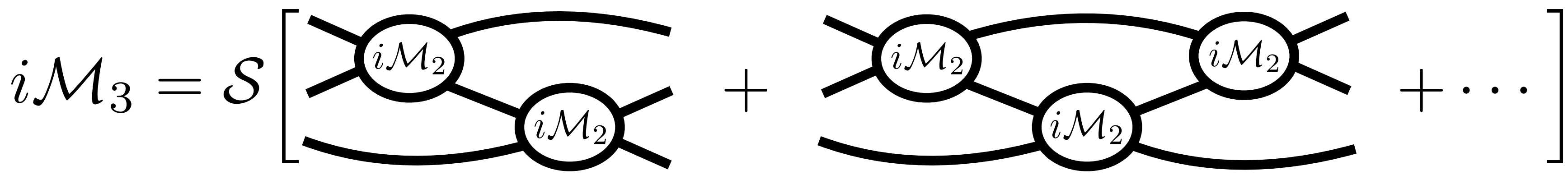
$$\mathcal{M}_3(E^*) = \mathcal{I}\left[\mathcal{K}_{\mathrm{df},3}(E^*), \mathcal{M}_2\right]$$

Fixed total energy, manifestly convergent, on-shell only, no reference to EFT, takes care of unitarity and singularities, useful independent of finite-volume physics?

MTH and Sharpe (2015)

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

- **Provides a useful benchmark: Deviations measure three-particle physics**
- **Meaning for three-to-three scattering is clear**



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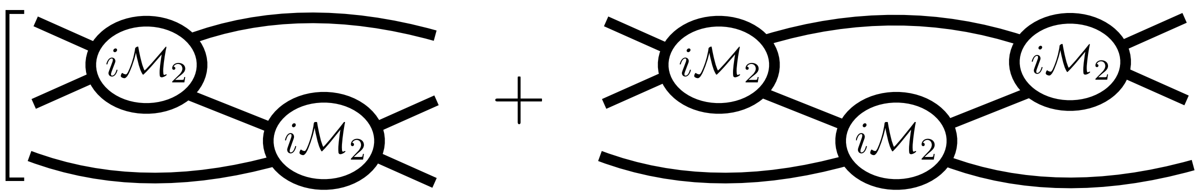
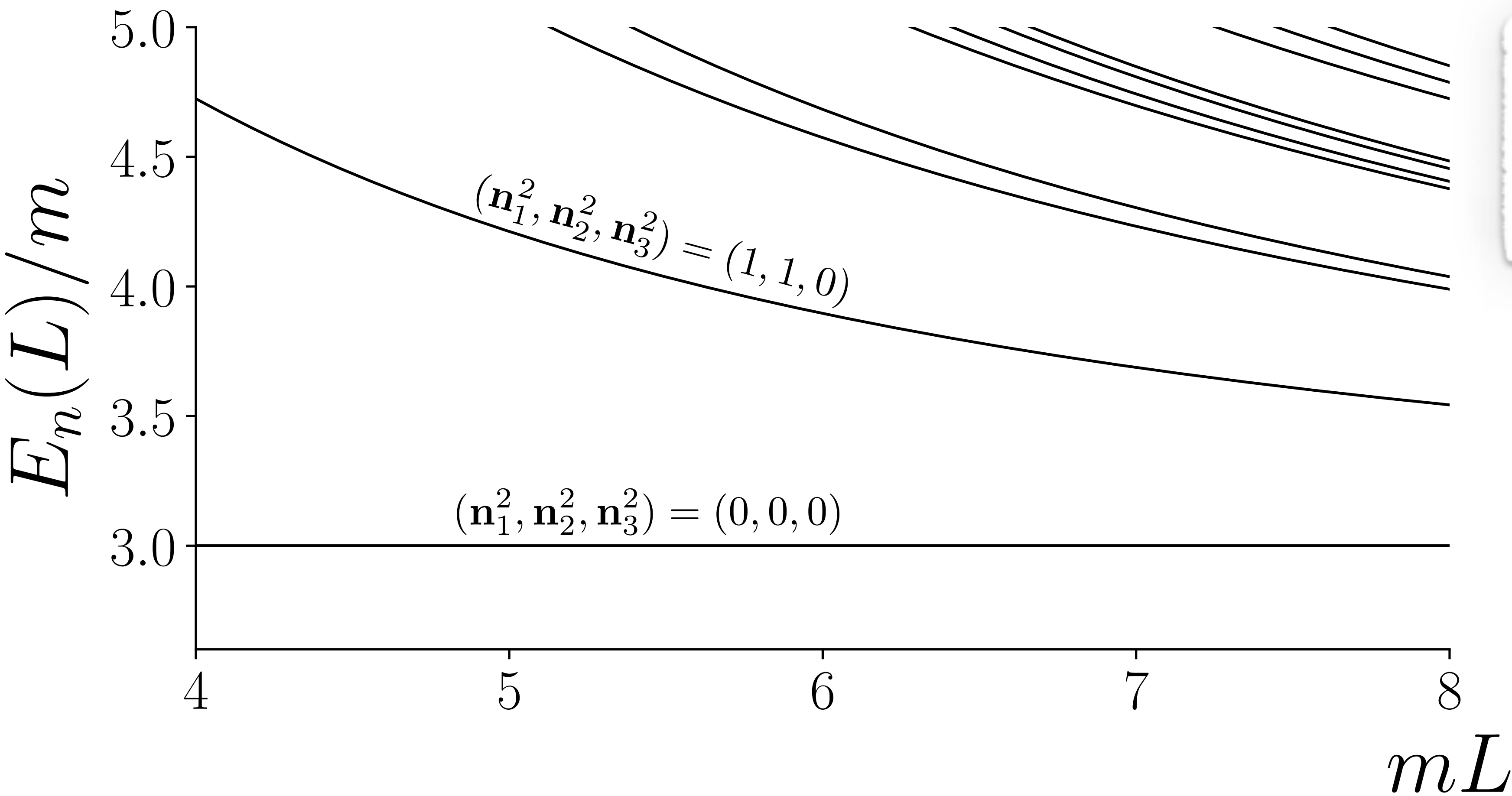


$+\cdots\Big]$

Aside: Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)



$$E_n^{\text{non-int}}(L) = \omega_1 + \omega_2 + \omega_3$$

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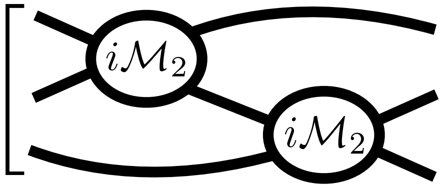
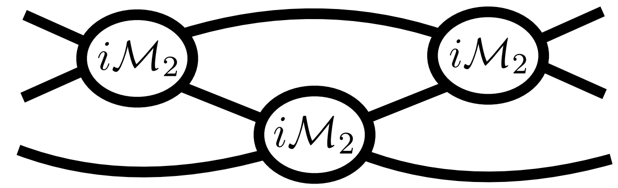
$i\mathcal{M}_3 = \mathcal{S}\Big[$  $+$  $+\cdots\Big]$

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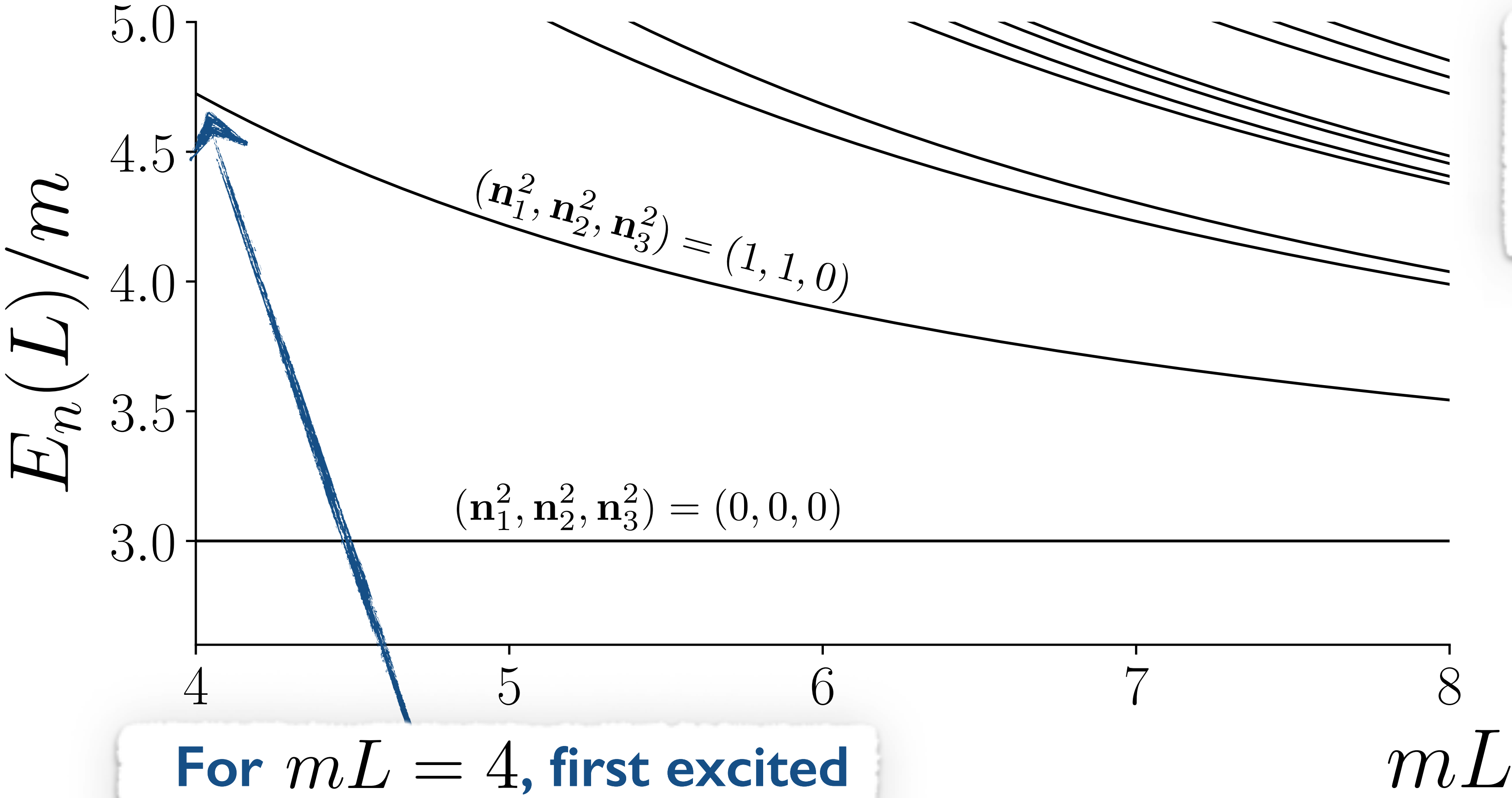

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$$\frac{p^2}{m^2} = \left(\frac{2\pi}{mL}\right)^2 \approx 2.46$$

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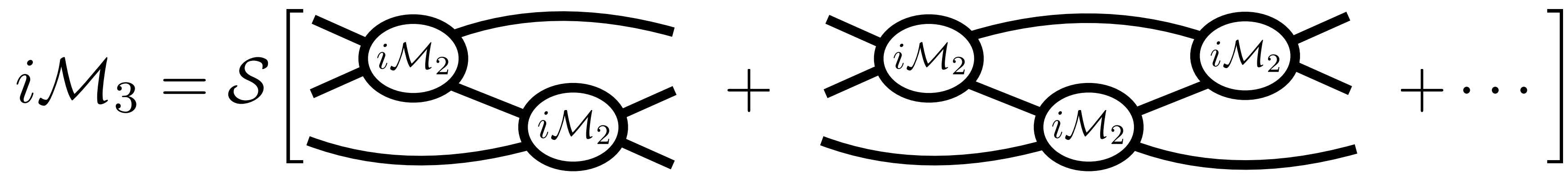

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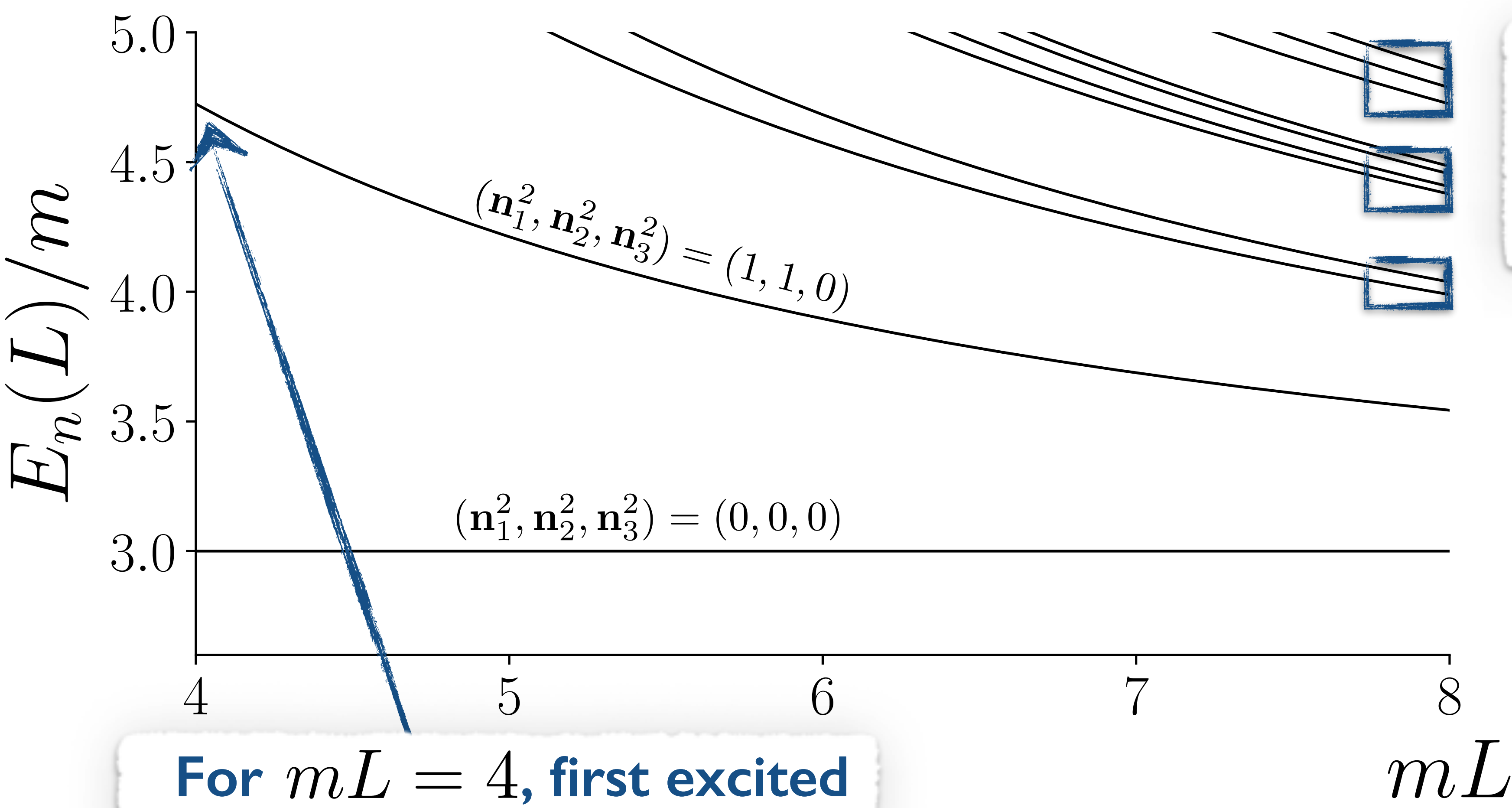

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Why are these states clustered?

Accidental NR degeneracy!

$$E_n^{\mathrm{NR}}(L) = 3m + \frac{2\pi^2}{L^2}(\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2)$$

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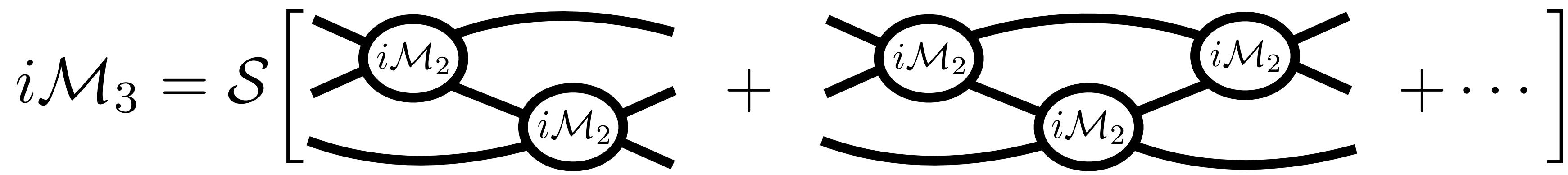

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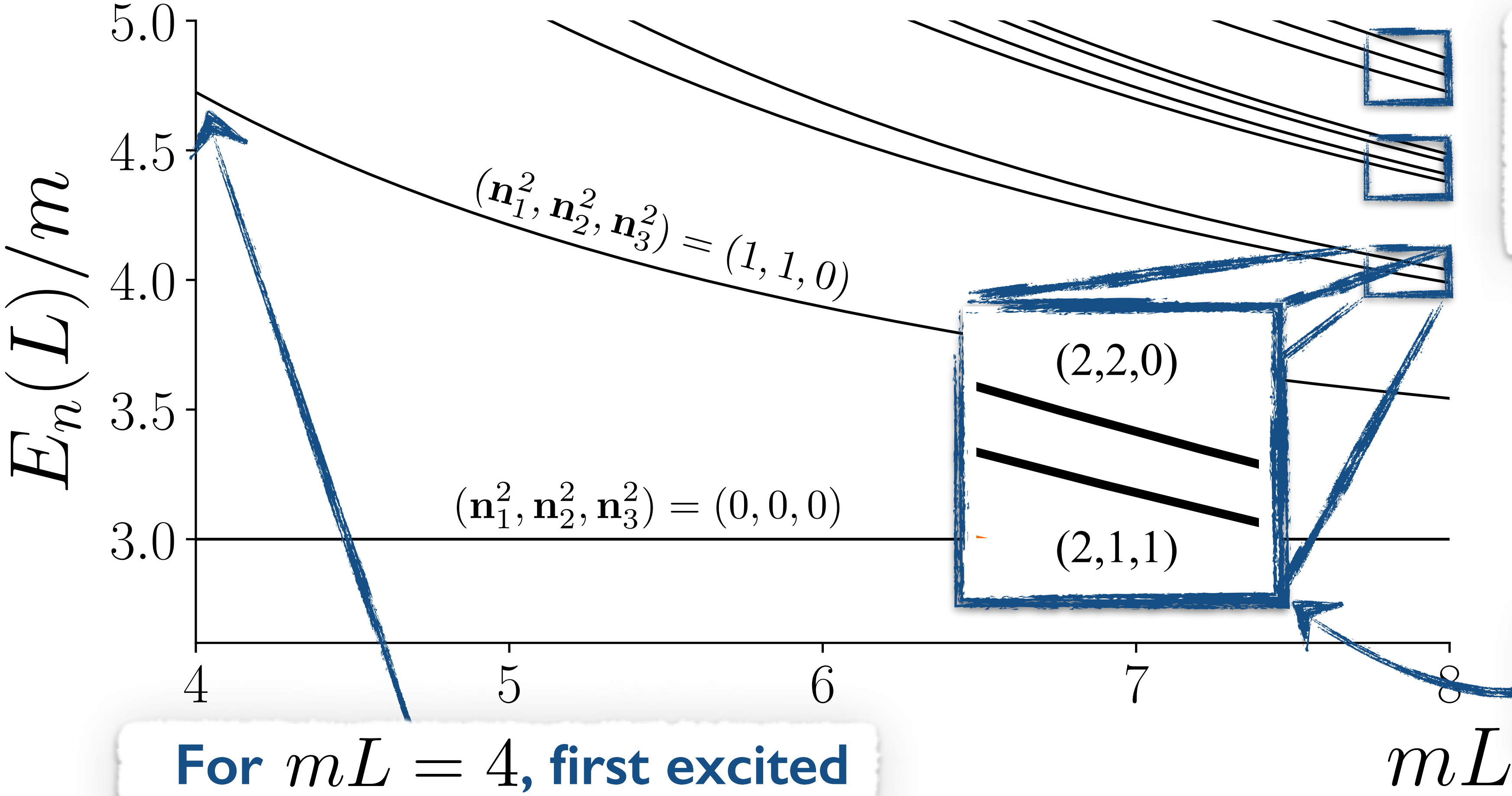

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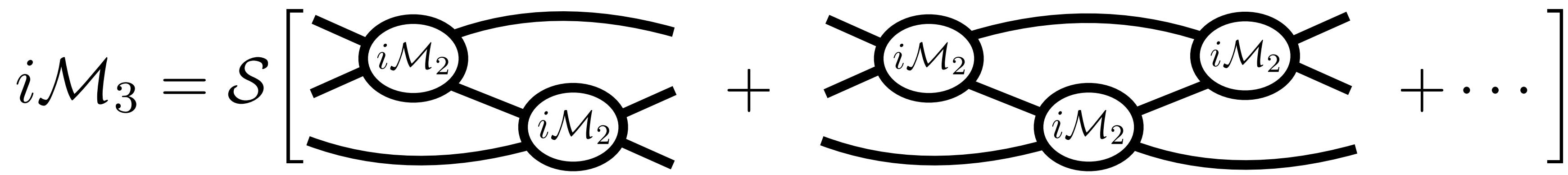

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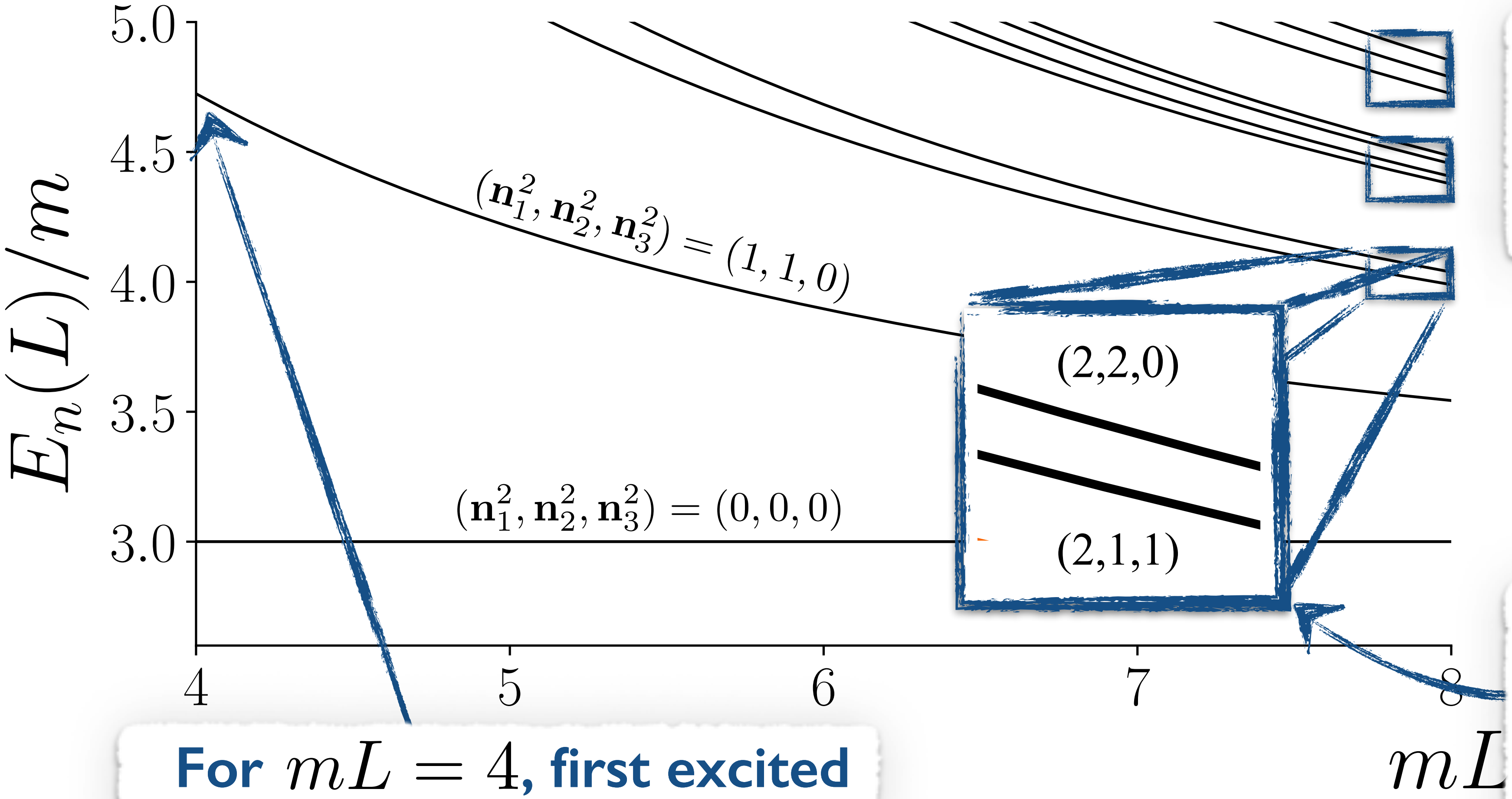

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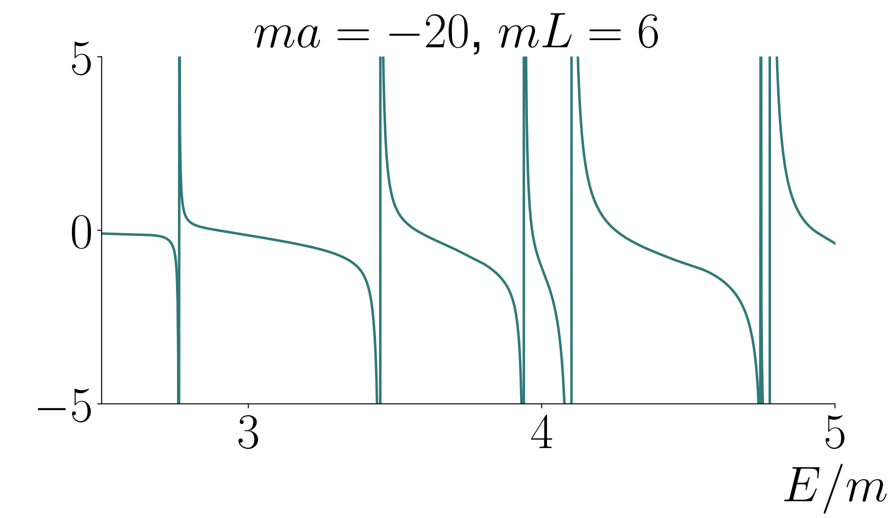
In fact we have already seen these clusters



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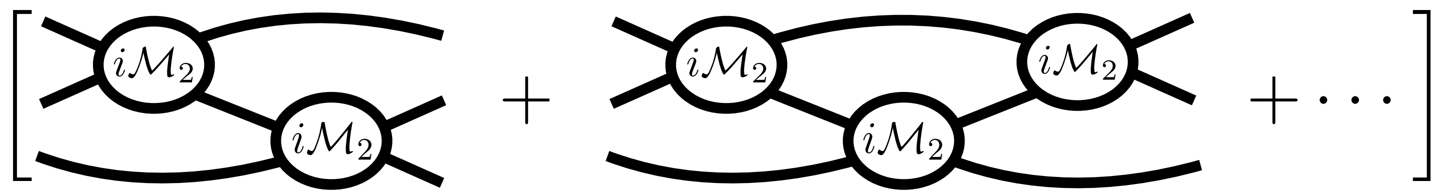
$i\mathcal{M}_3 = \mathcal{S}$ 

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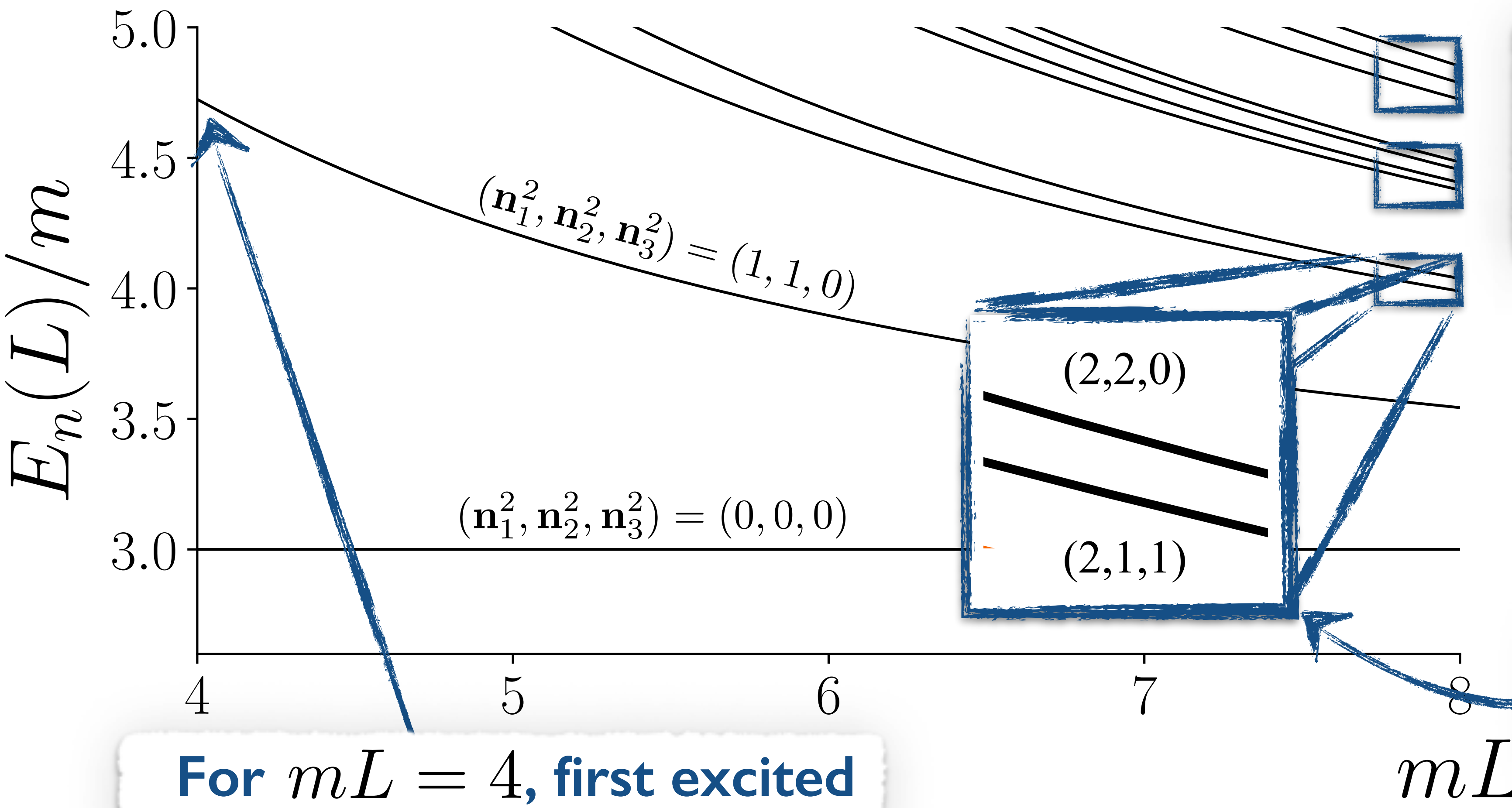

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Is it safe to say that we need a relativistic approach?

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

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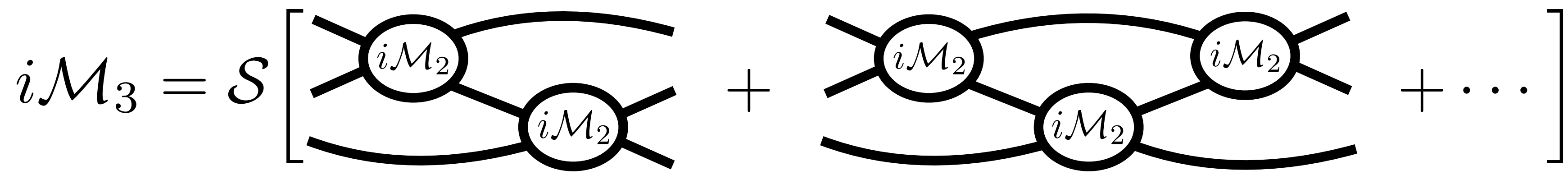

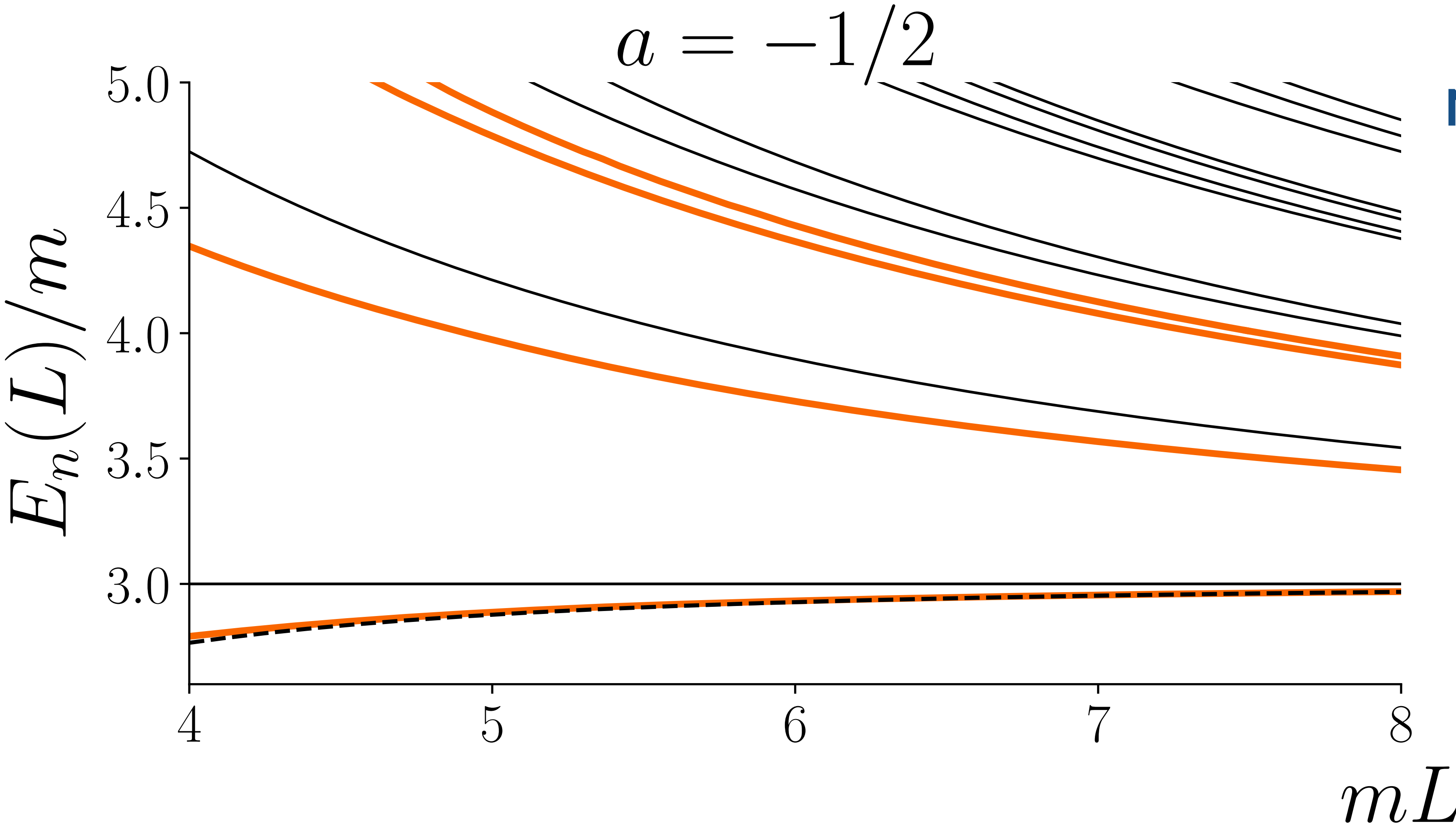

Now we turn on the interactions

$$1/F_3^{\mathrm{iso}}(E, L, a) = 0$$

$$F_3^{\mathrm{iso}} = \frac{1}{L^3} \sum_{\vec{k},\vec{p}} \left[\frac{\widetilde{F}_s}{3} - \widetilde{F}_s \frac{1}{\mathcal{H}} \widetilde{F}_s \right]_{kp}$$

$$\mathcal{H} = 1/(2\omega\mathcal{K}_2) + \widetilde{F}_s + \widetilde{G}_s$$

known functions

- - - - - - $1/L$ **expansion**

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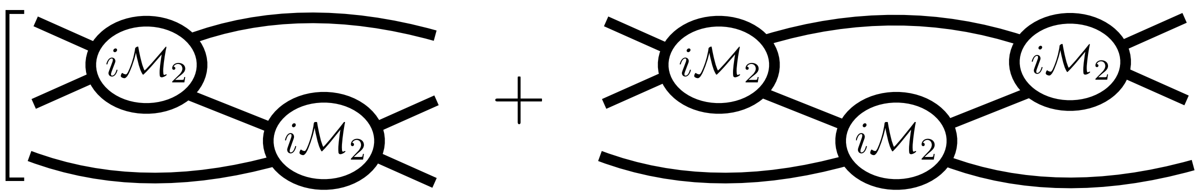
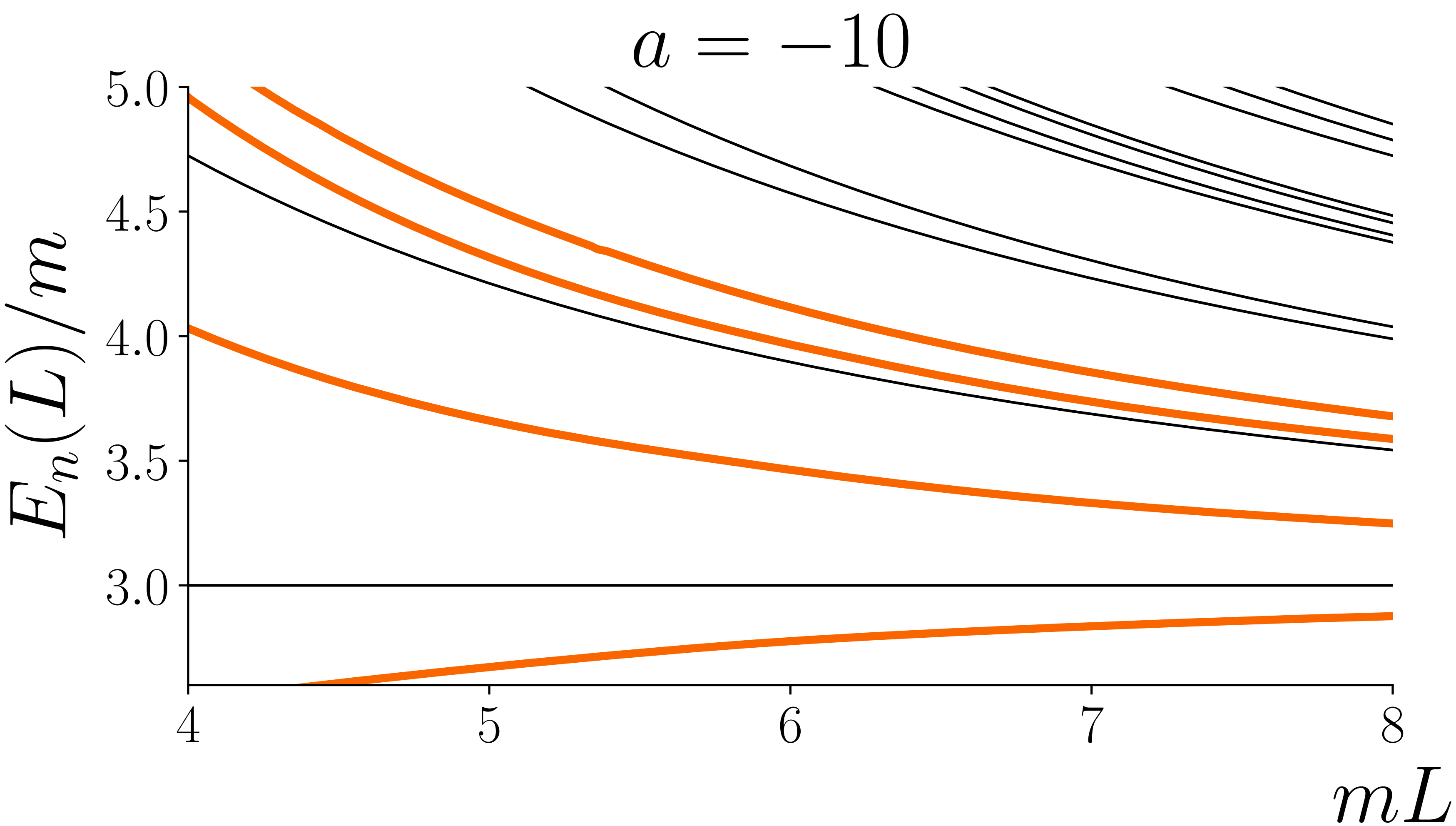
$i\mathcal{M}_3 = \mathcal{S}\Big[$  $+\cdots\Big]$



Can also accommodate large *a*

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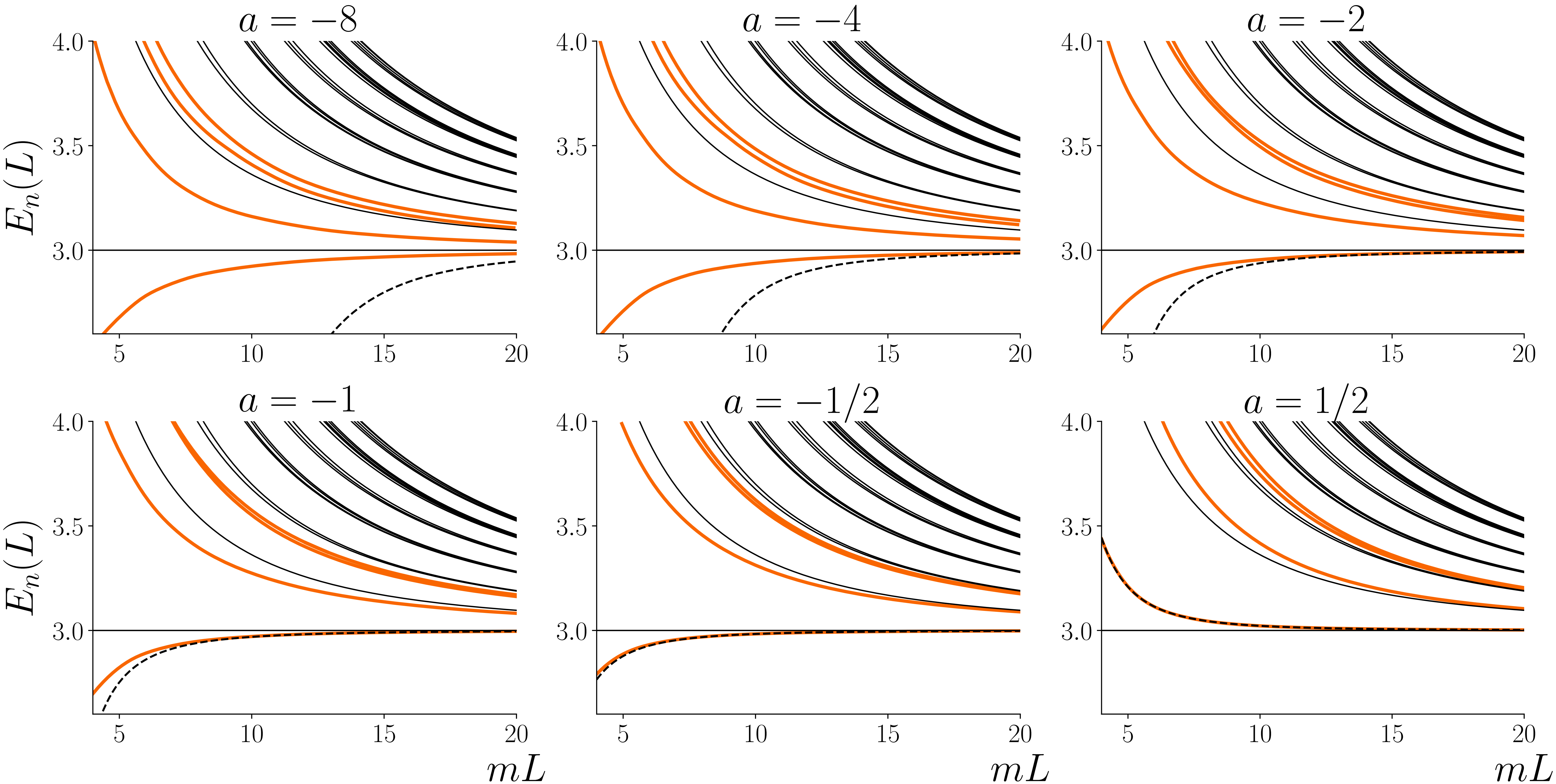
$\mathcal{K}^{\rm iso}_{\rm df,3}(E) = 0$ solutions

Straightforward to vary *a* and to study large volumes



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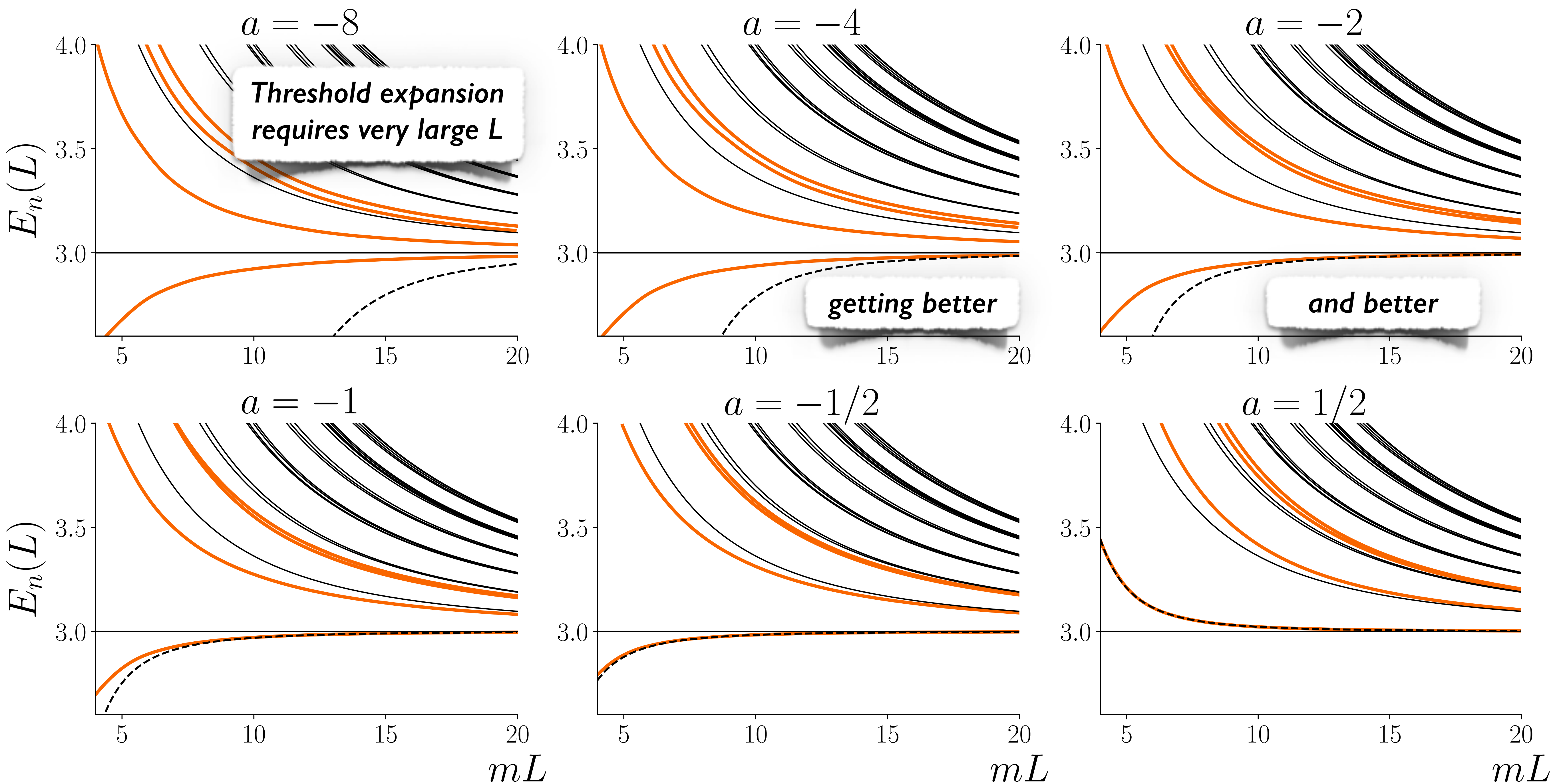

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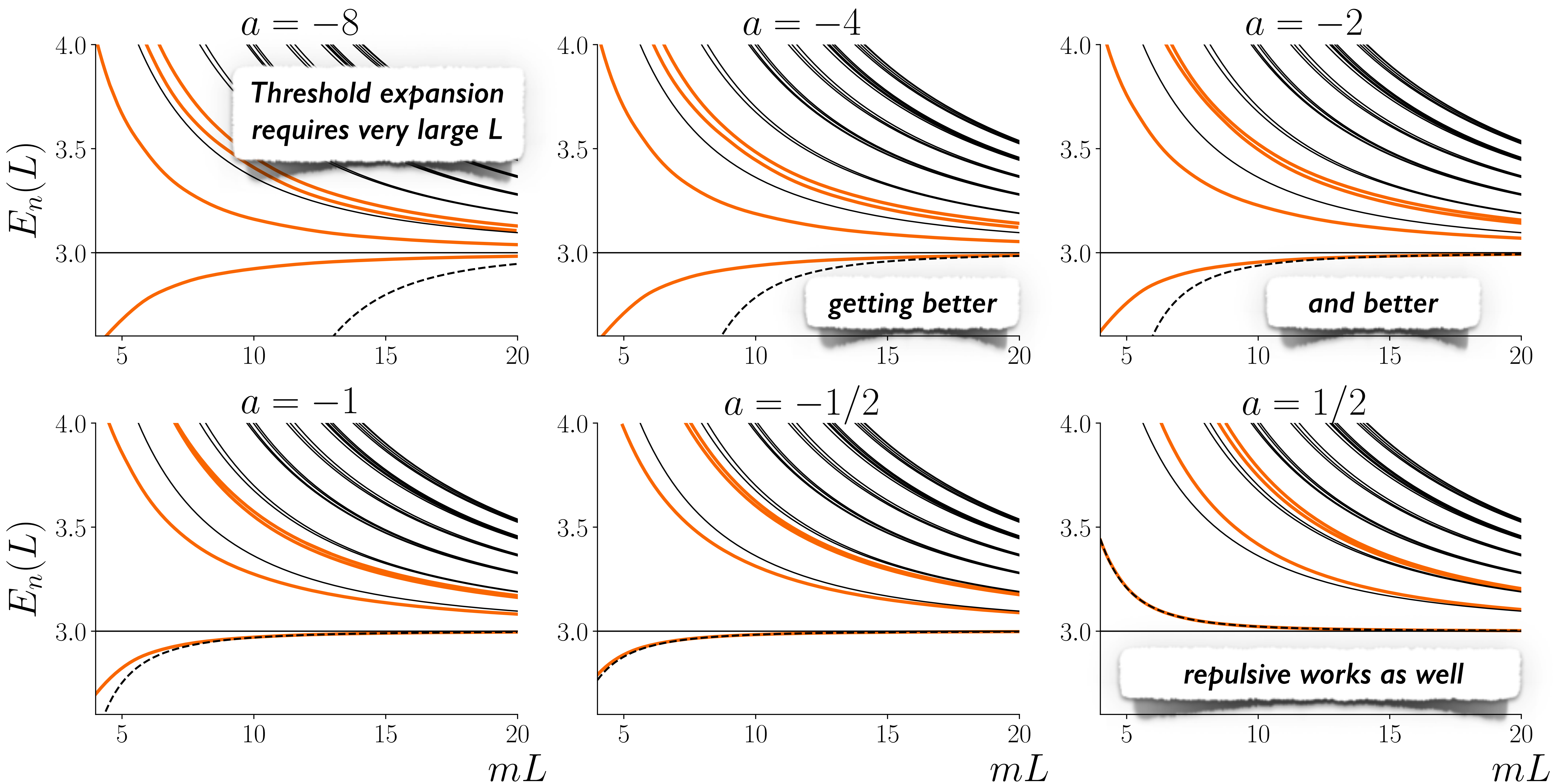

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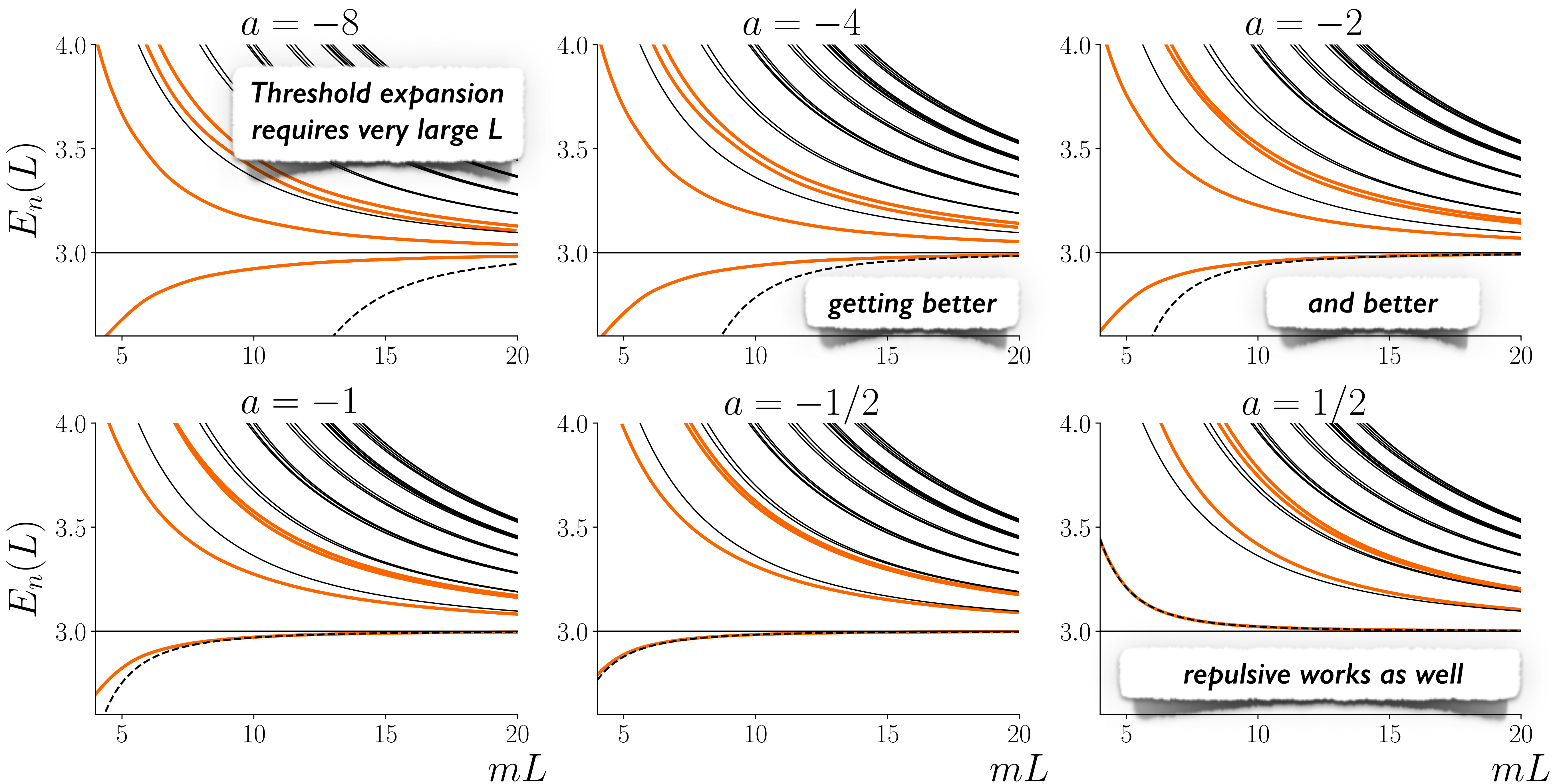

But, to avoid poles in $\mathcal{K}_2$, we must require $a < 1/m$