Spectral Quark Functions and Quark-Hadron Duality

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- ERA, in Proc. of the Workshop on Lepton Scattering, Hadrons, and QCD, Adelaide, 2001
- ERA+WB, Spectral quark model and low-energy hadron phenomenology, hep-ph/0301202, Phys. Rev. **D67** (2003) 074021

1. **“Partonic quasi-distributions of the pion in chiral quark models”** arXiv:1711.09355 [hep-ph]
2. **“Partonic quasidistributions of the proton and pion from transverse-momentum distributions”** Phys. Rev. D **97**, no. 3, 034031 (2018)
3. **“Nonperturbative partonic quasidistributions of the pion from chiral quark models”** Phys. Lett. B **773**, 385 (2017)
4. **“Transversity relations, chiral and holographic models, and pion wave functions from lattice QCD”** PoS LC **2010**, 041 (2010)
5. **“Gravitational, Electromagnetic, and Transition Form Factors of the Pion”** arXiv:0910.0869 [hep-ph]
6. **“Application of chiral quarks to high-energy processes and lattice QCD”** arXiv:0908.4165 [hep-ph]
7. **“Photon interactions and chiral dynamics”** arXiv:0907.3374 [hep-ph]
8. **“From chiral quark models to high-energy processes”** Acta Phys. Polon. B **40**, 2165 (2009)
9. **“Gravitational and higher-order form factors of the pion in chiral quark models”** Phys. Rev. D **78**, 094011 (2008)
10. **“Pion electromagnetic form factor, perturbative QCD, and large-N(c) Regge models”** Phys. Rev. D **78**, 034031 (2008)
11. **“Generalized parton distributions of the pion”** AIP Conf. Proc. **1030**, 286 (2008)
12. **“Generalized parton distributions of the pion in chiral quark models and their QCD evolution”** Phys. Rev. D **77**, 034023 (2008)

13. **“Pion-photon Transition Distribution Amplitudes in the Spectral Quark Model”** Phys. Lett. B **649**, 49 (2007)

14. **“Confined Chiral Solitons in the Spectral Quark Model”** Phys. Rev. D **76**, 014008 (2007)

15. **“Photon distribution amplitudes and light-cone wave functions in chiral quark models”** Phys. Rev. D **74**, 054023 (2006)

16. **“Application of chiral quark models to high-energy processes”** hep-ph/0410041

17. **“Kwiecinski evolution of unintegrated parton distributions”** hep-ph/0407295

18. **“Low-energy chiral Lagrangian in curved space-time from the spectral quark model”** Phys. Rev. D **70**, 034031 (2004)

19. **“Solution of the Kwiecinski evolution equations for unintegrated parton distributions using the Mellin transform”** Phys. Rev. D **70**, 034012 (2004)

20. **“Impact parameter dependence of the diagonal GPD of the pion from chiral quark models”** hep-ph/0310048

21. **“The Spectral quark model and light cone phenomenology”** hep-ph/0310044

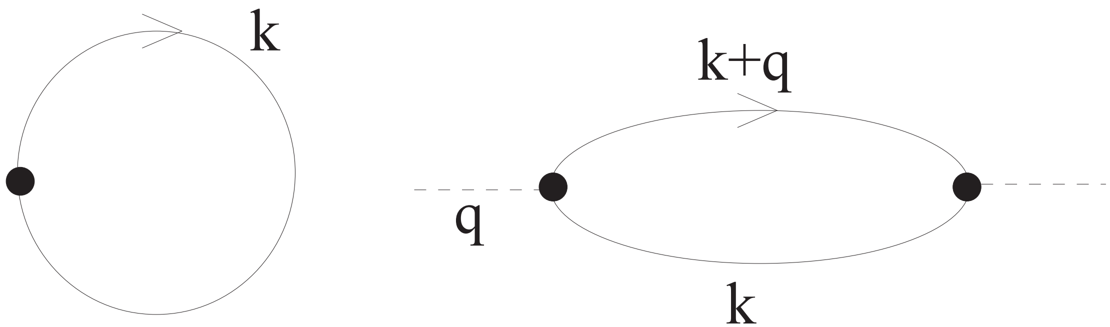
22. **“Impact parameter dependence of the generalized parton distribution of the pion in chiral quark models”** Phys. Lett. B **574**, 57 (2003)

Chiral Quark Model

Determining the Quark content of hadrons requires quark degrees of freedom

Prototype: Nambu-Jona–Lasinio

One-loop (leading-N_c)



The momentum running around the loop is cut, $k < \Lambda$

This is not what we are going to do!

What is the scale of the Chiral Quark Model ?

Constituents of hadrons in a quark model are quarks. They carry 100 % of the momentum in the hadron by relativistic invariance.
In QCD the momentum carried by the (valence) quarks at $Q^2 = 4\mathrm{GeV}^2$ is about 40 %.

$$\frac{\int dx\, xq(x,Q)}{\int dx\, xq(x,Q_0)} = \left(\frac{\alpha(Q)}{\alpha(Q_0)}\right)^{\gamma_1^{(0)}/(2\beta_0)},$$

If we evolve to lower scales we get at LO this corresponds to
$Q_0 = 322 \pm 45$ MeV.

Requirements

(a) Give finite values for hadronic observables
(b) Satisfy the Ward-Takahashi identities, thus reproducing all necessary symmetry requirements
(c) Satisfy the anomaly conditions
(d) Comply to the QCD factorization property, in the sense that the expansion of a correlator at a large Q is a pure twist-expansion, involving only the inverse powers of Q^2, without the $\log Q^2$ corrections
(e) Have the usual dispersion relations

All simultaneously – far from trivial!

The spectral representation

A novel approach, the spectral regularization of the chiral quark model, is based on the Lehmann representation

$$S(p) = \int_C d\omega \frac{\rho(\omega)}{\not{p} - \omega}$$

$\rho(\omega)$ – the spectral function NOT necessarily real or positive, C – a suitable contour in the complex ω plane

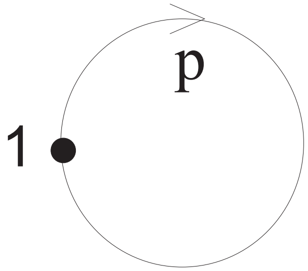
Example: free theory has $\rho(\omega) = \delta(\omega - m)$,

Perturbative QCD yields at LO (Haeri, 1988)

Non-perturbative?

$$\rho(\omega) = \delta(\omega - m) + \text{sign}(\omega)\frac{\alpha_S C_F}{4\pi}\frac{1-\xi}{\omega}\theta(\omega^2 - m^2)$$

Quark condensate



$$\langle \bar{q}q \rangle \equiv -iN_c \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} S(p) = -4iN_c \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \frac{\omega}{p^2 - \omega^2}$$

The integral over p is quadratically divergent, which requires the use of an auxiliary regularization, *removed* at the end

$$\langle \bar{q}q \rangle = -\frac{N_c}{4\pi^2} \int d\omega \omega \rho(\omega) \left[2\Lambda^2 + \omega^2 \log\left(\frac{\omega^2}{4\Lambda^2}\right) + \omega^2 + \mathcal{O}(1/\Lambda) \right]$$

The finiteness of the result at $\Lambda \to \infty$ requires the conditions

The Spectra
conditions

$$\int d\omega \omega \rho(\omega) = 0, \quad \int d\omega \omega^3 \rho(\omega) = 0$$

and thus

$$\langle \bar{q}q \rangle = -\frac{N_c}{4\pi^2} \int d\omega \log(\omega^2) \omega^3 \rho(\omega)$$

The spectral condition allowed us to rewrite $\log(\omega^2/\Lambda^2)$ as $\log(\omega^2)$, hence no scale dependence is present

Vacuum energy density

The energy-momentum tensor for a purely quark model is defined as

$$\theta^{\mu\nu}(x) = \bar{q}(x)\frac{i}{2}\left\{\gamma^\mu\partial^\nu + \gamma^\nu\partial^\mu\right\}q(x) - g^{\mu\nu}\mathcal{L}(x).$$

At the one-quark-loop level

$$\begin{aligned}
&\langle\theta^{\mu\nu}\rangle = -iN_cN_f\int d\omega\rho(\omega)\int\frac{d^4p}{(2\pi)^4}\times \\
&\mathrm{Tr}\frac{1}{\not p-\omega}\left[\frac{1}{2}\left(\gamma^\mu p^\nu + \gamma^\nu p^\mu\right) - g^{\mu\nu}(\not p-\omega)\right] \\
&= -4iN_cN_f\int d\omega\rho(\omega)\int\frac{d^4p}{(2\pi)^4}\frac{p^\mu p^\nu - g^{\mu\nu}(p^2-\omega^2)}{p^2-\omega^2} \\
&= Bg^{\mu\nu} + \langle\theta^{\mu\nu}\rangle_0, \qquad (1)
\end{aligned}$$

$$B = -iN_cN_f \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \frac{\omega^2}{p^2 - \omega^2}, \qquad (2)$$

where in the subtraction of the free part we have used the

$$\int d\omega \rho(\omega) = 1 \qquad (3)$$

The integral over p is quadratically divergent. B finite implies

$$\int d\omega \omega^2 \rho(\omega) = 0, \quad \int d\omega \omega^4 \rho(\omega) = 0$$

Hence

$$B = -\frac{N_cN_f}{16\pi^2} \int d\omega \omega^4 \log \omega^2 \qquad (4)$$

Thus $\rho(\omega)$ cannot be positive !

According to QCD sum rules

$$B = -\frac{9}{32}\langle\frac{\alpha}{\pi}G^2\rangle = -(224^{+35}_{-70}\mathrm{MeV})^4$$

The negative sign of B enforces

$$\rho_4' > 0$$

Spectral moments

Postulate

$$\rho_0 \equiv \int d\omega \rho(w) = 1,$$

$$\rho_n \equiv \int d\omega \omega^n \rho(\omega) = 0, \quad \text{for } n = 1, 2, 3, \ldots$$

Observables are given by the inverse moments

$$\rho_{-k} \equiv \int d\omega \omega^{-k} \rho(\omega), \quad \text{for } k = 1, 2, 3, \ldots$$

Such a $\rho(\omega)$ exists!

as well as by the "log moments",

$$\rho'_n \equiv \int d\omega \log(\omega^2) \omega^n \rho(\omega), \quad \text{for } n = 2, 3, 4, \ldots$$

Gauge technique and the vertex functions

Delbourgo & West '77

CVC and PCAC imply the Ward-Takahashi identities (WTI)

The gauge technique consists of writing a solution for the unamputated vector and axial vertices

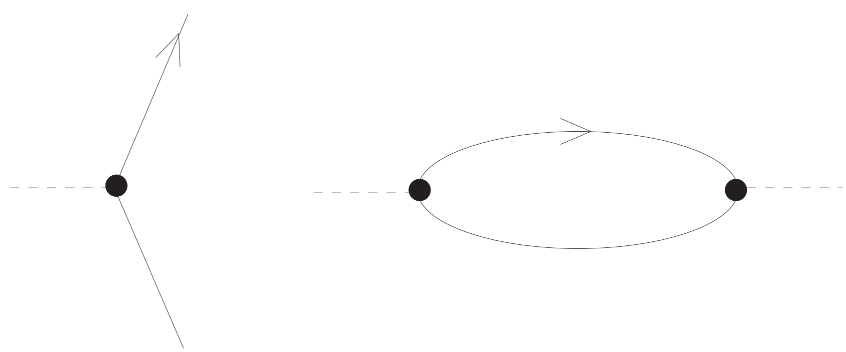
not unique!

$$\Lambda_V^{\mu,a}(p',p) = \int d\omega\rho(\omega)\frac{i}{\not{p}'-\omega}\gamma^\mu\frac{\lambda_a}{2}\frac{i}{\not{p}-\omega}$$

$$\Lambda_A^{\mu,a}(p',p) = \int d\omega\rho(\omega)\frac{i}{\not{p}'-\omega}\left(\gamma^\mu-\frac{2\omega q^\mu}{q^2}\right)\gamma_5\frac{\lambda_a}{2}\frac{i}{\not{p}-\omega}$$

pion pole

Similar for more vertices, one $\rho(\omega)$ for each quark line



$e^+e^- \to$ hadrons

At large s we find

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha_{\text{QED}}^2}{3s}\left(\sum_{i=u,d,\ldots} e_i^2\right)\int d\omega\rho(\omega)$$

This is the proper asymptotic QCD result if

$$\int d\omega\rho(\omega) = 1$$

Pion properties

Finiteness of f_π requires the condition $\rho_2 = 0$. Then

$$f_\pi^2 = -\frac{N_c}{4\pi^2}\int d\omega \log(\omega^2)\omega^2\rho(\omega) \equiv -\frac{N_c}{4\pi^2}\rho_2'$$

The electromagnetic form factor

$$F_\pi^{em}(q^2) = \frac{4N_c}{f_\pi^2}\int dw\rho(\omega)\omega^2 I(q^2,\omega)$$

At low-momenta $F_\pi^{em}(0) = 1$

$$F_\pi^{em}(q^2) = 1 + \frac{1}{4\pi^2 f_\pi^2}\left(\frac{q^2\rho_0}{6} + \frac{q^4\rho_{-2}}{60} + \frac{q^6\rho_{-4}}{240} + \dots\right)$$

The mean squared radius reads (regardless of details of the $\rho(\omega)$)
$\langle r_\pi^2\rangle = \frac{N_c}{4\pi^2 f_\pi^2}$

At large momenta

$$F_\pi^{em}(q^2) \sim \frac{N_c}{4\pi^2 f_\pi^2} \int d\omega \rho(\omega) \{ \frac{2\omega^4}{q^2} \left[\log(-q^2/\omega^2) + 1 \right] + \ldots \}$$

With help of the spectral conditions for $n = 2, 4, 6, ...$ we get

$$F_\pi^{em}(q^2) \quad \sim \quad -\frac{N_c}{4\pi^2 f_\pi^2} \left[\frac{2\rho_4'}{q^2} + \frac{2\rho_6'}{q^4} + \frac{4\rho_8'}{q^6} + ... \right]$$

All spectral conditions needed!

Pure twist expansion, no $\log$s !

Parton Distribution Functions of the Pion

The hadronic tensor for inclusive electroproduction on the pion reads

$$W_{\mu\nu}(p,q) = \frac{1}{2\pi}\mathrm{Im}T_{\mu\nu}(p,q)$$
$$= W_1(q^2, p\cdot q)\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \quad (5)$$
$$+\frac{W_2(q^2,p\cdot q)}{m_P^2}\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right),$$

where the forward virtual Compton scattering amplitude on the pion is defined as (closed quark lines)

$$T_{\mu\nu}(p,q) = i\int d^4x e^{iq\cdot x}\langle\pi(p)|T\left\{J_\mu^{\mathrm{em}}(x)J_\nu^{\mathrm{em}}(0)\right\}|\pi(p)\rangle. \quad (6)$$

and take the Bjorken limit $-q^2 = Q^2 \to \infty$ with $x = Q^2/2pq$ fixed. We take π^+ for definiteness and get

Davidson & ERA in NJL 1995

$$u_\pi(x) = \bar{d}_\pi(1-x) = \theta(x)\theta(1-x),$$

independently of $\rho(\omega)$. One recovers the Bjorken scaling (without log's), the Callan-Gross relation (quarks are spin 1/2), the proper support (relativity), the correct normalization (gauge invariance), and the momentum sum rule.

Another derivation from Quark-Pion scattering amplitude (open quark lines) yields exactly the same result

Non trivial consistency condition

QCD evolution of PDF

The QCD evolution of the constant PDF has been treated in detail by Davidson & ERA at LO and NLO. In particular, the non-singlet contribution to the energy-momentum tensor evolves as

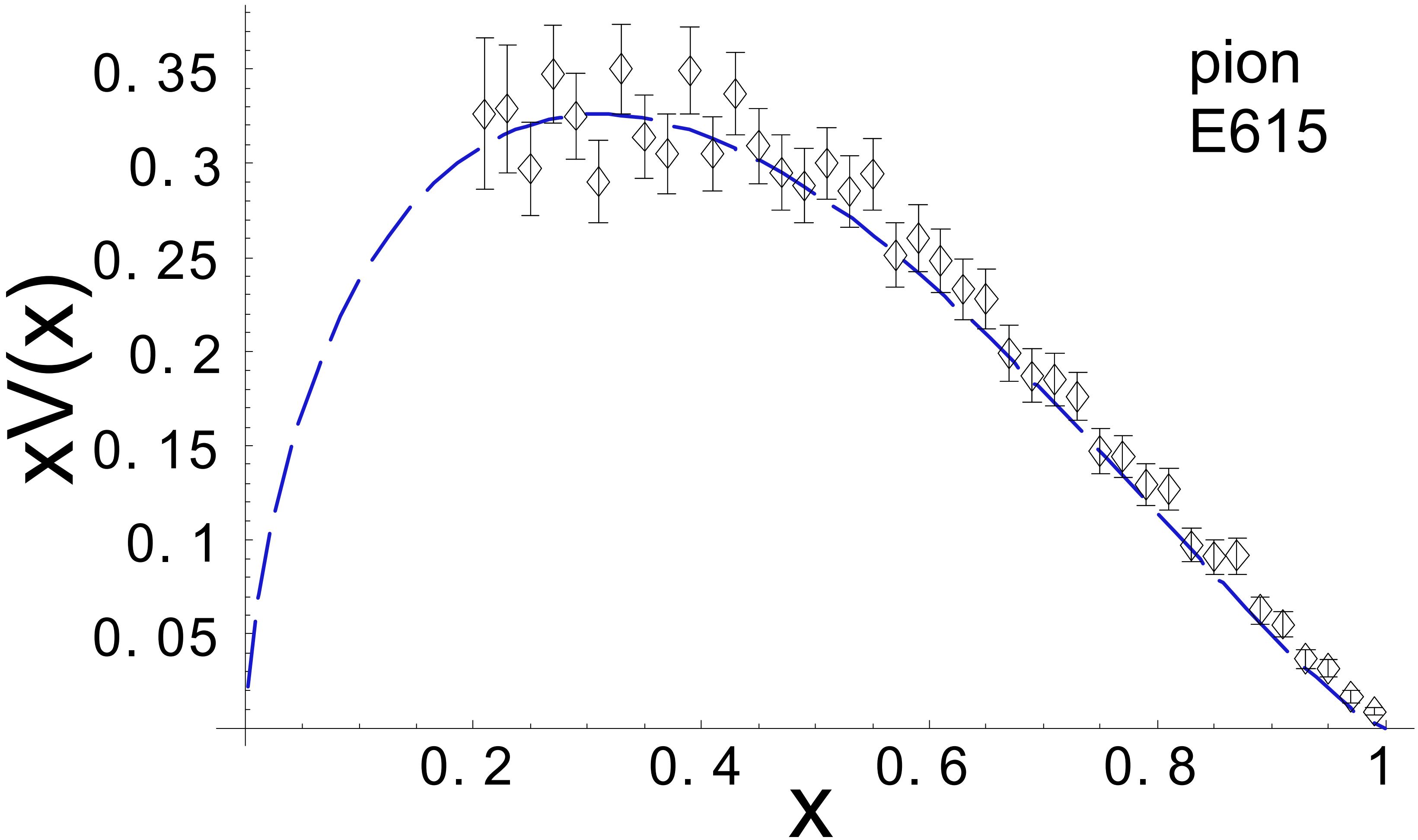
$$\frac{\int dx\, xq(x,Q)}{\int dx\, xq(x,Q_0)} = \left(\frac{\alpha(Q)}{\alpha(Q_0)}\right)^{\gamma_1^{(0)}/(2\beta_0)} ,$$

In has been found that at $Q^2 = 4\mathrm{GeV}^2$ the valence quarks carry $47 \pm 0.02\%$ of the total momentum fraction in the pion. Downward LO evolution yields that at the scale

$$Q_0 = 313^{+20}_{-10}\mathrm{MeV}$$

the quarks carry 100% of the momentum. The agreement of the evolved PDF with the SMRS data analysis is impressive

DGLAP evolution



NLO DGLAP evolution does not change significantly.

Odd-parity processes

$\pi^0 \to \gamma\gamma$

$$F_{\pi\gamma\gamma}(0,0,0) = \frac{1}{4\pi^2 f_\pi}\int d\omega\rho(\omega) = \frac{1}{4\pi^2 f_\pi}$$

which coincides with the QCD result. Not true when the loop momentum is cut!

$\gamma \to \pi^+\pi^0\pi^-$

$$F(0,0,0) = \frac{1}{4\pi^2 f_\pi^3}\int d\omega\rho(\omega) = \frac{1}{4\pi^2 f_\pi^3}$$

which is the correct result

Pion-photon transition form factor

For two off-shell photons with momenta q_1 and q_2 one defines the asymmetry, A, and the total virtuality, Q^2:

$$A = \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}, \qquad -1 \le A \le 1$$
$$Q^2 = -(q_1^2 + q_2^2)$$

At the soft pion point we find the expansion,

$$F_{\pi\gamma\gamma}(Q^2, A) = -\frac{1}{2\pi^2 f_\pi} \int_0^1 dx \left[\frac{2\rho_2'}{Q^2(1 - A^2(2x-1)^2)} + \dots \right]$$

We can confront this with the standard twist decomposition of the pion transition form factor ,

$$F_{\gamma\gamma\pi}(Q^2, A) = J^{(2)}(A)\frac{1}{Q^2} + J^{(4)}(A)\frac{1}{Q^4} + \dots,$$

Brodsky-Lepage, Praszałowicz-Rostworowski, Dorokhov which yields

$$
\begin{aligned}
J^{(2)}(A) &= \frac{4f_\pi}{N_c}\int_0^1 dx \frac{\varphi(x;Q_0)}{1-(2x-1)^2A^2} \\
J^{(4)}(A) &= \frac{8f_\pi\Delta^2}{N_c}\int_0^1 dx \frac{\varphi_\pi^{(4)}(x)[1+(2x-1)^2A^2]}{[1-(2x-1)^2A^2]^2},
\end{aligned}
$$

with The leading-twist pion distribution amplitude at $Q_0 \sim 320$ MeV

$$\varphi(x;Q_0) = 1$$

This serves as the initial condition for the QCD evolution and $\Delta^2 = -8B/(3f_\pi^2)$.Numerically,

$$\Delta^2 = (0.78 \pm 0.61)\ \mathrm{GeV}^2.$$

An estimate made in a non-local quark model of A. E. Dorokhov *et al.* provides $\Delta^2 = 0.29\ \mathrm{GeV}^2$.
The form of the expansion shows that the all twist distribution amplitudes for the pion are, at the model working scale Q_0, constant

and equal to unity:

$$\varphi_\pi^{(n)}(x) \quad = \quad \theta(x)\theta(1-x) \qquad \text{for} \qquad n = 2, 4, 6, \ldots$$

QCD evolution of PDA

All results of the effective, low-energy model, refer to a soft energy scale, Q_0. In order to compare to experimental results, obtained at large scales, Q, the QCD evolution must be performed. Initial condition:

$$\varphi(x; Q_0) = \theta(x)\theta(1-x).$$

The evolved distribution amplitude reads

$$\begin{aligned} \varphi(x; Q) &= 6x(1-x)\sum_{n=0}^{\infty} C_n^{3/2}(2x-1)a_n(Q) \\ a_n(Q) &= \frac{2}{3}\frac{2n+3}{(n+1)(n+2)}\left(\frac{\alpha(Q^2)}{\alpha(Q_0^2)}\right)^{\gamma_n^{(0)}/(2\beta_0)} \end{aligned}$$

where $C_n^{3/2}$ are the Gegenbauer polynomials, $\gamma_n^{(0)}$ are appropriate anomalous dimensions, and $\beta_0 = 9$.

Results extracted from the experimental data of CLEO provide

$a_2(2.4\text{GeV}) = 0.12 \pm 0.03$, which we use to fix

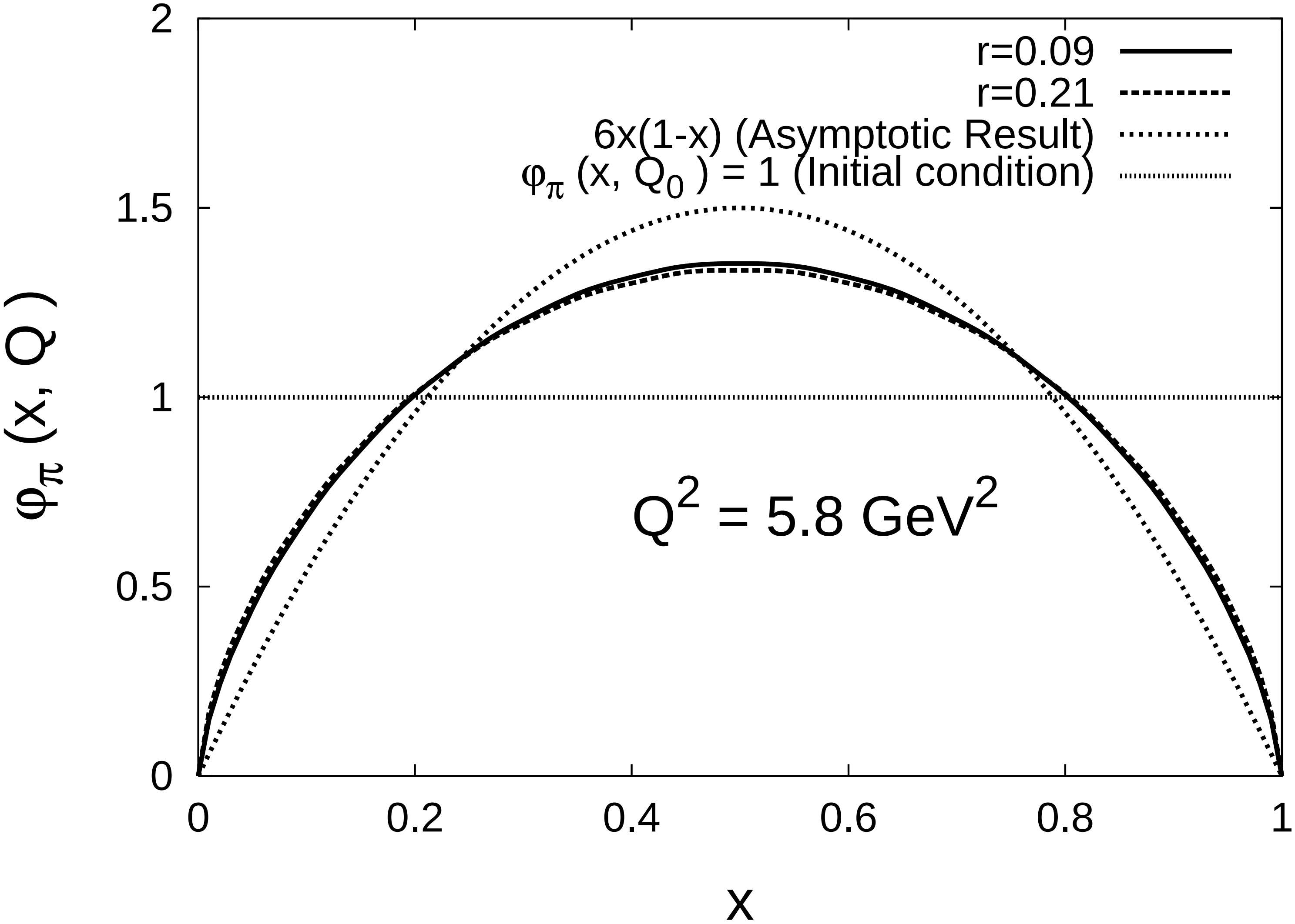
$$\alpha(Q = 2.4\text{GeV})/\alpha(Q_0) = 0.15 \pm 0.06$$

At LO this corresponds to $Q_0 = 322 \pm 45$ MeV

Now we can predict

$$\begin{aligned} a_4(2.4\text{GeV}) &= 0.06 \pm 0.02 \;\; (\exp : -0.14 \pm 0.03 \mp 0.09) \\ a_6(2.4\text{GeV}) &= 0.02 \pm 0.01 \end{aligned}$$

Encouraging, with leading-twist and LO QCD evolution!



r=0.09
r=0.21
6x(1-x) (Asymptotic Result)
φπ (x, Q0) = 1 (Initial condition)
Q2 = 5.8 GeV2
φπ (x, Q)
x
0
0.2
0.4
0.6
0.8
1
0.5
1.5
2

$k_\perp$-unintegrated parton distribution can be shown to be equal to

$$u_\pi(x, k_\perp) = \frac{N_c}{4\pi^3 f_\pi^2} \int d\omega \rho(\omega) \frac{\omega^2}{k_\perp^2 + \omega^2} \theta(x)\theta(1-x),$$

hence at Q_0 one has an interesting relation

$$q(x, k_\perp) = \bar{q}(1-x, k_\perp) = \Psi(x, k_\perp).$$

At $k_\perp = 0$ we have

$$q(x, 0_\perp) = \frac{N_c}{4\pi^3 f_\pi^2}.$$

Finally, via integrating with respect to $k_\perp$ the following identity between the PDF and the PDA is obtained at the scale Q_0:

$$q(x) = \varphi(x)$$

The first moment of the PDF is responsible for the The momentum sum rule, We find , $\int_0^1 dx\, xq(x) = \int_0^1 dx\, x\bar{q}(x) = \frac{1}{2}$, is satisfied. Actually, this quarks carry all momentum property is a simple consequence of the crossing property $\bar{q}(x) = q(1-x)$ and the normalization condition.

Further results/predictions

Gasser-Leutwyler coefficients: Leading-N_c quark model values

Magnetic permeability of the vacuum, χ

$$\langle 0|\bar{q}(0)\sigma_{\alpha\beta}q(0)|\gamma^{(\lambda)}(q)\rangle = ie_q\chi\,\langle\bar{q}q\rangle\left(q_\beta\varepsilon_\alpha^{(\lambda)} - q_\alpha\varepsilon_\beta^{(\lambda)}\right)$$

$$\chi = \frac{N_c}{4\pi^2}\rho_1'/\langle\bar{q}q\rangle$$

First log-moment

Tensor susceptibility of the vacuum

$$\Pi = i\langle 0|\int d^4z\,T\{\overline{q}(z)\sigma^{\mu\nu}q(z),\overline{q}(0)\sigma_{\mu\nu}q(0)\}|0\rangle = -12f_\pi^2$$

Broniowski, Polyakov & Goeke '98

Résumé

Spectral condition	Physical significance
zeroth moment	normalization
$\rho_0 = 1$	proper normalization of the quark propagator
	preservation of anomalies
	proper normalization of the pion distribution amplitude
	proper normalization of the pion structure function
	reproduction of the large-N_c quark-model values of the Gasser-Leutwyler coefficients
positive moments	finiteness/pure twist
$\rho_1 = 0$	finiteness of the quark condensate, $\langle \bar{q} q \rangle$
	vanishing quark mass at asymptotic Euclidean momenta,
$\rho_2 = 0$	finiteness of the vacuum energy density, B
	finiteness of the pion decay constant, f_π
$\rho_3 = 0$	finiteness of the quark condensate, $\langle \bar{q} q \rangle$
$\rho_4 = 0$	finiteness of the vacuum energy density, B
$\rho_n = 0, \; n = 2, 4 \ldots$	absence of logs in the twist expansion of vector amplitudes
$\rho_n = 0, \; n = 5, 7 \ldots$	finiteness of nonlocal quark condensates, $\langle \bar{q} (\partial^2)^{(n-3)/2} q \rangle$
	absence of logs the twist expansion of the scalar pion form factor

Spectral condition	Physical significance
negative moments	values of observables
$\rho_{-2} > 0$	positive quark wave-function normalization at vanishing momentur
$\rho_{-1}/\rho_{-2} > 0$	positive value of the quark mass at vanishing momentum, $M(0) >$
ρ_{-n}	low-momentum expansion of correlators
log-moments	values of observables
ρ'_1	magnetic permeability of the vacuum
$\rho'_2 < 0$	$f_\pi^2 = -N_c/(4\pi^2)\rho'_2$
$\rho'_3 > 0$	negative value of the quark condensate, $\langle \bar{q}q \rangle = -N_c/(4\pi^2)\rho'_3$
$\rho'_4 > 0$	negative value of the vacuum energy density, $B = -N_c/(4\pi^2)\rho'_4$
$\rho'_5 < 0$	positive value of the squared vacuum virtuality of the quark, $\lambda_q^2 = -\rho'_5/\rho'_3$
ρ'_n	high-momentum (twist) expansion of correlators

Meson dominance model

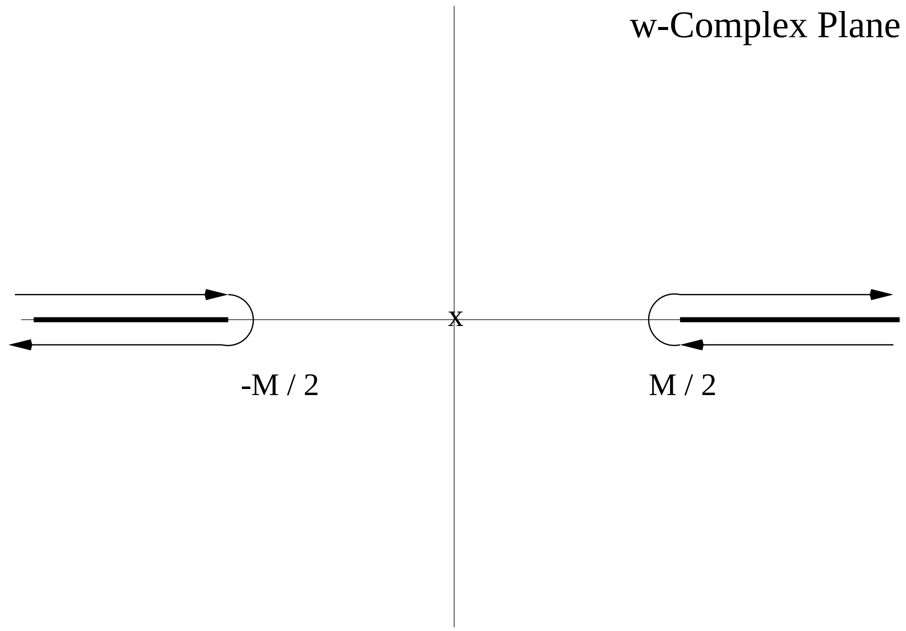
Explicit example of $\rho(\omega)$! Vector-meson dominance (VMD) of the pion form factor is assumed (works up to $t \sim 2$ GeV^2)

$$F_V(t) = \frac{M_V^2}{M_V^2 + t} \equiv -\frac{4N_c}{(4\pi)^2 f_\pi^2} \int d\omega \rho(\omega) \times \int_0^1 dx \log \left[\omega^2 + x(1-x)t\right]$$

with $M_V = m_\rho$. Given this, we get the part of $\rho(\omega)$ responsible for the even moments

$$\rho_V(\omega) = \frac{1}{2\pi i} \frac{3\pi^2 M_V^3 f_\pi^2}{4N_c} \frac{1}{\omega} \frac{1}{(M_V^2/4 - \omega^2)^{5/2}}.$$

The function $\rho_V(\omega)$ has a single pole at the origin and branch cuts starting at $\pm M_V/2$.



The condition $\rho_0 = 1$ gives $M_V^2 = 24\pi^2 f_\pi^2 / N_c$ (matching quark models to VMD) The positive even moments fulfill the spectral conditions Miracle!

$$\rho_{2n} = 0, \qquad n = 1, 2, 3 \ldots$$

The log-moments and negative even moments are finite

For the case of the scalar spectral function (controlling odd moments) we proceed heuristically, by proposing its form in analogy to ρ_V $\rho = \rho_V + \rho_S$

$$\rho_S(\omega) = \frac{1}{2\pi i} \frac{-48\pi^2 \langle \bar{q} q \rangle}{N_c M_S^4 (1 - 4\omega^2/M_S^2)^{5/2}}$$

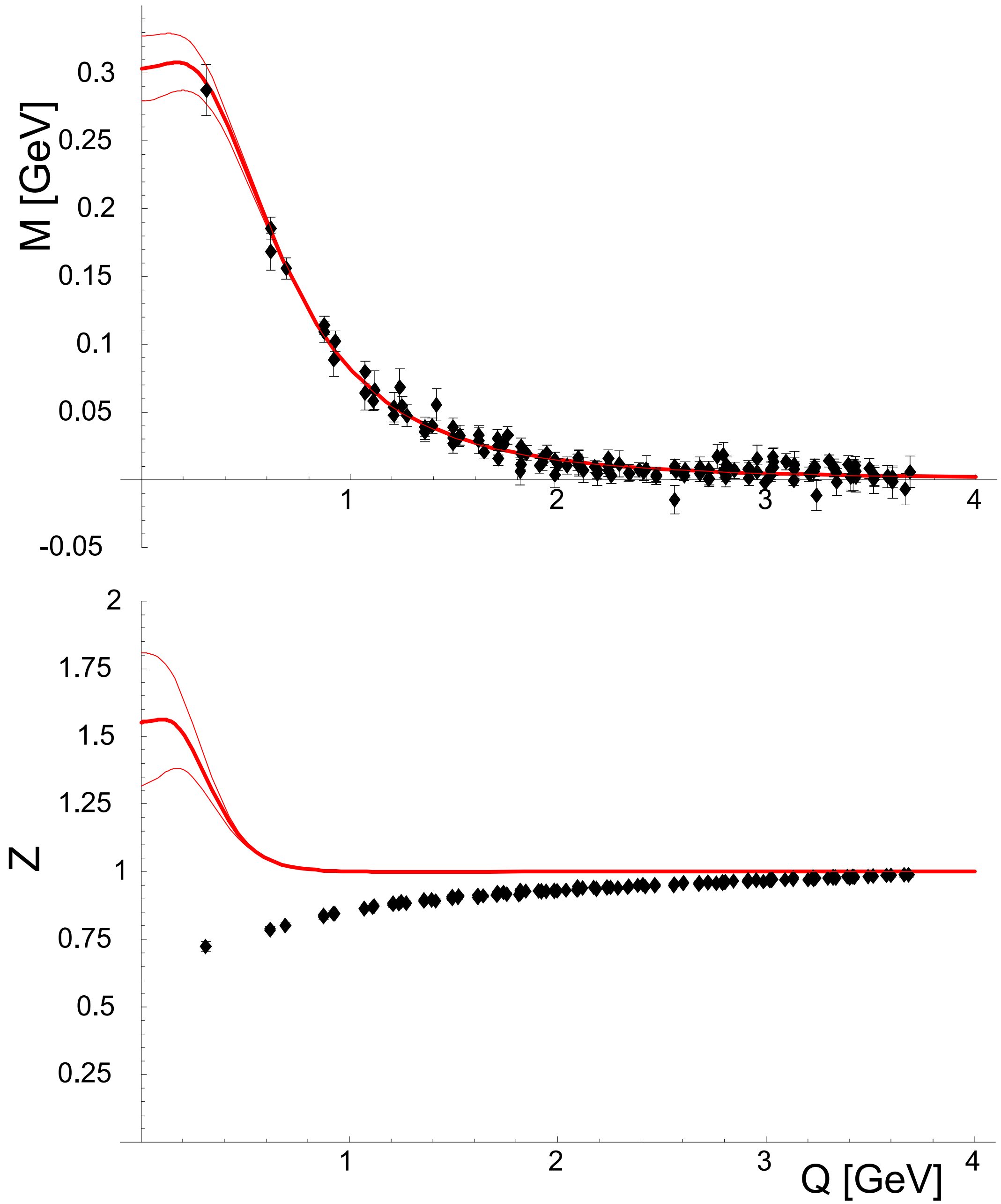
M_S is a scale parameter. The analytic structure similar to $\rho_V(\omega)$, except for the absence of the pole at $\omega = 0$. Odd positive moments vanish!

The quark propagator (from meson properties

$$
\begin{aligned}
S(p) &= A(p)\not{p} + B(p) = Z(p)\frac{\not{p} + M(p)}{p^2 - M^2(p)} \\
A(p^2) &\equiv \int_C d\omega \frac{\rho_V(\omega)}{p^2 - \omega^2} = \frac{1}{p^2}\left[1 - \frac{1}{(1 - 4p^2/M_V^2)^{5/2}}\right] \\
B(p^2) &\equiv \int_C d\omega \frac{\omega \rho_S(\omega)}{p^2 - \omega^2} = \frac{48\pi^2 \langle \bar{q} q \rangle}{M_S^4 N_c (1 - 4p^2/M_S^2)^{5/2}}
\end{aligned}
$$

No poles in the whole complex plane! Only branch cuts starting at $p^2 = M_{V,S}^2/4$. The absence of poles is sometimes called "the analytic confinement"

Poles would lead to cuts in form f.



data:

Bowman, Heller,

& Williams '02

$M(Q^2)$ decreases as $1/Q^3$ at large Euclidean momenta, which is favored by the recent lattice calculations. The fit results in

$$\begin{aligned} M_S &= 970 \pm 21 \text{ MeV}, \\ M(0) &= 303 \pm 24 \text{ MeV} \end{aligned}$$

with $\chi^2/\text{DOF} = 0.72$. The corresponding value of $\langle \bar{q}q \rangle$ is

$$\langle \bar{q}q \rangle = -(243.0^{+0.1}_{-0.8} \text{ MeV})^3$$

Effective action and consistency conditions

The vacuum to vacuum transition amplitude in the presence of external bosonic (s, p, v, a) and fermionic $(\eta, \bar{\eta})$ fields of a chiral quark model Lagrangian can be written as a path integral as

$$Z[s, p, v, a, \eta, \bar{\eta}] \quad = \quad \langle 0|\mathrm{T}\exp\left\{ i \int d^4x \left[\bar{q}\Big(\not{v} + \not{a}\gamma_5 - (s + i\gamma_5 p) \Big) q + \bar{\eta} q + \bar{q}\eta \right] \right\} |0\rangle$$

The consistency of the calculation requires the following trivial identity for the generating functional

$$\langle \bar{q}(x) q(x) \rangle = i \frac{1}{Z} \frac{\delta Z}{\delta s(x)} \Big|_0 = \lim_{x' \to x} (-i)^2 \frac{1}{Z} \frac{\delta^2 Z}{\delta \eta(x) \bar{\eta}(x')} \Big|_0$$

where $\big|_0$ means external sources set to zero.

$$Z[\eta,\bar{\eta},s,p,\ldots] = \int DU e^{-i\langle\eta,S[U,s,p,v,a]\eta\rangle} e^{i\Gamma[U,s,p,v,a]}$$

where the propagator and effective actions are given by

$$\langle x'|S[U,s,p,v,a]_{aa'}|x\rangle = \int d\omega\rho(\omega)\langle x|(\mathbf{D})^{-1}_{aa'}|x'\rangle$$

and

$$\Gamma[U,s,p,v,a] = -\mathrm{i}N_c \int d\omega\rho(\omega)\mathrm{Tr}\log\left(i\mathbf{D}\right),$$

respectively and the Dirac operator is given by

$$i\mathbf{D} \quad = \quad i\partial\!\!\!/ - \omega U^5 - \hat{m}_0 + \left(v\!\!\!/ + a\!\!\!/\gamma_5 - s - i\gamma^5 p\right)$$

With $U^5 = U^{\gamma_5}$, and $U = u^2 = e^{i\sqrt{2}\Phi/f}$.

Gasser-Leutwyler-Donoghue coefficients

$$
\begin{aligned}
L_3 &= -2L_2 = -4L_1 = -\frac{N_c}{(4\pi)^2}\frac{\rho_0}{6}, \\
L_4 &= L_6 = 0, \\
L_5 &= 6L_{13} = -\frac{N_c}{(4\pi)^2}\frac{\rho_1'}{2B_0}, \\
L_8 &= \frac{N_c}{(4\pi)^2}\left[\frac{\rho_2'}{4B_0^2} - \frac{\rho_1'}{4B_0} - \frac{\rho_0}{24}\right], \\
L_9 &= -2L_{10} = \frac{N_c}{(4\pi)^2}\frac{\rho_0}{3}, \\
L_{12} &= -2L_{11} = -\frac{N_c}{(4\pi)^2}\frac{\rho_0}{6}
\end{aligned}
\tag{7}
$$

Dual relations

The quark model cannot be better than the large N_c expansion.
Compare to the single resonance approximation

$$2L_1^{\rm SRA} = L_2^{\rm SRA} = \frac{1}{4}L_9^{\rm SRA} = -\frac{1}{3}L_{10}^{\rm SRA} = \frac{f^2}{8M_V^2}, \tag{8}$$

$$L_5^{\rm SRA} = \frac{8}{3}L_8^{\rm SRA} = \frac{f^2}{4M_S^2}, \tag{9}$$

$$L_3^{\rm SRA} = -3L_2^{\rm SRA} + \frac{1}{2}L_5^{\rm SRA}, \tag{10}$$

$$2L_{13}^{\rm SRA} = 3L_{11}^{\rm SRA} + L_{12}^{\rm SRA} = \frac{f^2}{4M_{f_0}^2}, \tag{11}$$

$$L_{12}^{\rm SRA} = -\frac{f^2}{2M_{f_2}^2}, \tag{12}$$

$$\rho_1'^{\,\rm SRA} = \frac{8\pi^2\langle\bar{q}q\rangle}{N_c M_S^2}, \tag{13}$$

$$\rho_2'^{\,\mathrm{SRA}} = -\frac{4\pi^2 f^2}{N_c} = -\frac{M_V^2}{6}, \qquad (14)$$

Assuming that L_8 and L_{10} are consistent with WSR one gets

$$2L_1 = L_2 = -\frac{1}{2}L_3 = \frac{1}{2}L_5 = \frac{2}{3}L_8 = \frac{1}{4}L_9 = -\frac{1}{3}L_{10} = \frac{N_c}{192\pi^2}$$

This also implies the set of mass dual relations,

$$M_A = M_P = \sqrt{2}M_V = \sqrt{2}M_S = 4\pi\sqrt{3/N_c}f_\pi \,. \qquad (15)$$

Other predictions

Pion transition form factor:

$$F_{\pi\gamma\gamma}(Q^2, A) = \frac{2f_\pi}{AN_c}\frac{1}{Q^2}\log\left[\frac{2M_V^2 + (1+A)Q^2}{2M_V^2 + (1-A)Q^2}\right]$$

$$+ \quad \frac{16 f_\pi M_V^2}{N_c[4M_V^4 + 4Q^2M_V^2 + (1-A^2)Q^4]}$$

Pion light-cone wave function and PDF:

$$\Psi(x, k_\perp) = q(x, k_\perp) = \frac{3M_V^3}{16\pi(k_\perp^2 + M_V^2/4)^{5/2}}\theta(x)\theta(1-x)$$

The average transverse momentum squared is equal to

$$\langle k_\perp^2 \rangle \equiv \int d^2k_\perp \, k_\perp^2 \Psi(x, k_\perp) = \frac{M_V^2}{2}$$

which numerically gives $\langle k_\perp^2 \rangle = (544 \text{ MeV})^2$ (at Q_0). The estimates from QCD sum rules yield smaller values: one gets $(316\text{MeV})^2$ or $(333 \pm 40\text{MeV})^2$

Quark propagator in the coordinate representation:

$$A(z) = \frac{48 + 24M_V\sqrt{-z^2} - 6M_V^2 z^2 + M_V^3(-z^2)^{3/2}}{96\pi^2 z^4}\exp(-M_V\sqrt{-z^2}/2)$$
$$B(z) = \langle\overline{q}q\rangle/(4N_c)\exp(-M_S\sqrt{-z^2}/2)$$

Nonlocal quark condensate:

$$Q(z) = \exp(-M_S\sqrt{-z^2}/2)$$

Magnetic permeability of the vacuum: (at Q_0):

$$\chi = \frac{2}{M_S^2}$$

After evolution $\chi(1\ \mathrm{GeV}) = 3.3\ \mathrm{GeV}^2$ in agreement with other estimates

c.f. Ball, Braun & Kivel '03

Summary

(a) Assumptions: generalized spectral representation, one quark loop (large N_c), gauge technique (WTI), **spectral conditions** (finiteness)
(b) Symmetries, anomalies, normalization, pure twist expansion, preserved
(c) **Dynamics encoded in moments**, the approach itself is non-dynamical
(d) Specific relations follow, since all observables are expressed in moments of the spectral function
(e) The method is technically very simple (computations are short) and predictive (lots of applications)
(f) What does not work (at the moment): second Weinberg sum rule (modify vertices?, freedom)
(g) Applicable to **high-energy** processes. Very reasonable results follow after evolution
(h) Interesting particular realization of the spectral method: the **meson-dominance model**
(i) **Analytic confinement** in the sense of the absence of poles in the quark propagator
(j) Surprisingly good $M(Q^2)$ vs. lattice results, $Z(Q^2)$ could be better
(k) Specific predictions of the VMD model for unintegrated PDF, non-local quark condensate, ...

NEXT

Baryons = Quarks + "Spectral" Diquarks

Glueballs = "Spectral Gluons"

BACKUP SLIDES

In the perturbative phase with no spontaneous symmetry breaking, where $\rho(\omega) = \rho(-\omega) = \delta(\omega)$, we have $\langle \bar{q} q \rangle = 0$.
With the accepted value of

$$\langle \bar{q} q \rangle = \simeq -(243 \text{ MeV})^3$$

we infer the value of the third log-moment. The negative sign of the quark condensate shows that

$$\int d\omega \log(\omega^2) \omega^3 \rho(\omega) > 0.$$

The vector and axial-vector currents of QCD are:

$$J_V^{\mu,a}(x) = \bar{q}(x) \gamma^\mu \frac{\lambda_a}{2} q(x), \quad J_A^{\mu,a}(x) = \bar{q}(x) \gamma^\mu \gamma_5 \frac{\lambda_a}{2} q(x)$$

CVC and PCAC:

$$\partial_\mu J_V^{\mu,a}(x) = 0, \quad \partial_\mu J_A^{\mu,a}(x) = \bar{q}(x) \hat{M}_0 i \gamma_5 \frac{\lambda_a}{2} q(x)$$

A number of results are then obtained essentially for free:

- Pions arise as Goldstone bosons, with standard current-algebra properties for free!
- At high energies parton-model features, such as the spin-1/2 nature of hadronic constituents, are recovered

The vector and axial unamputated vertex functions:

$$\Lambda^{\mu,a}_{V,A}(p',p) = \int d^4x d^4x' \langle 0| T\left\{ J^{\mu,a}_{V,A}(0) q(x') \bar{q}(x) \right\} |0\rangle e^{ip'\cdot x' - ip\cdot x}$$

WTI

$$(p'-p)_\mu \Lambda^{\mu,a}_V(p',p) = S(p')\frac{\lambda_a}{2} - \frac{\lambda_a}{2} S(p)$$

$$(p'-p)_\mu \Lambda^{\mu,a}_A(p',p) = S(p')\frac{\lambda_a}{2}\gamma_5 + \gamma_5 \frac{\lambda_a}{2} S(p)$$

“Transverse ambiguity”

The above ansätze fulfil the WTI’s. They are determined up to *transverse pieces*.
This ambiguity appears in all effective models. Current conservation fixes only the longitudinal pieces. Example:

$$j_\mu = \bar{\psi}\left(f_1\gamma_\mu + i f_2 \sigma_{\mu\nu} q^\nu\right)\psi$$

The condition $q^\mu j_\mu = 0$ does not constrain the f_2-term, since $\sigma_{\mu\nu} q^\nu q^\mu = 0$ from antisymmetry.

Vertices with two currents

Vertices with two currents, axial or vector, are constructed similarly. The vacuum polarization is

$$
\begin{aligned}
i\Pi_{VV}^{\mu a,\nu b}(q) &= \delta^{ab}\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)\bar{\Pi}_{VV}(q) = \int d^4x e^{-iq\cdot x}\langle 0|T\left\{J_V^{\mu a}(x)J_V^{\nu b}(0)\right\}|0\rangle \\
&= -N_c\int d\omega\rho(\omega)\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\not p-\not q-\omega}\gamma_\mu\frac{\lambda_a}{2}\frac{i}{\not p-\omega}\gamma_\nu\frac{\lambda_b}{2}\right]
\end{aligned}
$$

transverse

$$
\begin{aligned}
\bar{\Pi}_{VV}(q) &= \ldots \\
I(q^2,\omega) &= -\frac{1}{(4\pi)^2}\int_0^1 dx\log\left[\omega^2+x(1-x)q^2\right]
\end{aligned}
$$

Dispersion relation

The twice-subtracted dispersion relation holds:

$$\bar{\Pi}_V(q^2) = \frac{q^4}{\pi}\int_0^\infty \frac{dt}{t^2}\,\frac{\mathrm{Im}\bar{\Pi}_V(t)}{t-q^2-i0^+}$$

This is in contrast to quark models formulated in the Euclidean space, where the usual dispersion relations do not hold

The pion decay constant, defined as

$$\langle 0\,|\,J_A^{\mu a}(x)|\,\pi_b(q)\rangle = i f_\pi q_\mu \delta_{a,b} e^{iq\cdot x},$$

can be computed from the axial-axial correlation function. The result is

Vacuum energy density

$$\langle\theta^{\mu\nu}\rangle = -iN_cN_f\int d\omega\rho(\omega)\int\frac{d^4p}{(2\pi)^4}\times$$
$$\mathrm{Tr}\frac{1}{\not p-\omega}\left[\frac{1}{2}(\gamma^\mu p^\nu+\gamma^\nu p^\mu)-g^{\mu\nu}(\not p-\omega)\right]=Bg^{\mu\nu}+\langle\theta^{\mu\nu}\rangle_0,$$

where $\langle\theta^{\mu\nu}\rangle_0$ is the energy-momentum tensor for the free theory, evaluated with $\rho(\omega)=\delta(\omega)$, and B (bag constant) is the vacuum energy density:

$$B=-iN_cN_f\int d\omega\rho(\omega)\int\frac{d^4p}{(2\pi)^4}\frac{\omega^2}{p^2-\omega^2},$$

The conditions that have to be fulfilled for B to be finite are

$$\rho_2=0,\quad \rho_4=0$$

Then

$$B = -\frac{N_c N_f}{16\pi^2}\rho_4' \equiv -\frac{3N_c}{16\pi^2}\int d\omega \log(\omega^2)\omega^4\rho(\omega)$$

for $N_f = 3$.

The even conditions (here quadratic and quartic) imply that $\rho(\omega)$ *cannot be positive definite*; otherwise the even moments could not vanish!

Pion-quark coupling

Near the pion pole ($q^2 = 0$) we get

$$\Lambda_A^{\mu,a}(p+q,p) \to -\frac{q^\mu}{q^2}\Lambda_\pi^a(p+q,p),$$

where

$$\Lambda_\pi^a(p+q,p) = \int d\omega\rho(\omega)\frac{i}{\not p + \not q - \omega}\frac{\omega}{f_\pi}\gamma_5\lambda_a\frac{i}{\not p - \omega}$$

We recognize in our formulation the Goldberger-Treiman relation for quarks:

$$g_\pi(\omega) = \frac{\omega}{f_\pi}$$