Solving the Homogeneous Bethe-Salpeter Equation in Minkowski space

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Some results....

- dFSV PRD 94, 071901(R) (2016): two-fermion bound systems
  EPJC 77, 764 (2017): Light-cone singularities and structure

- FVS PRD 89, 016010 (2014): bound states and LF momentum distributions for two scalars
- FVS PRD 85, 036009 (2012): general formalism for bound and scattering states
- FVS EPJC 75, 398 (2015): scattering lengths for two scalars
- Gutierrez et al PLB 759, 131 (2016): spectra of excited states and LF momentum distributions

Outline

1. Motivations and generalities: BS Amplitude and BS Equation for a two-fermion bound system $\rightarrow \mathcal{L} = g\bar{\psi}\Gamma\psi\ \chi$
2. Nakanishi integral representation (NIR) and the BS Amplitude
3. The exact projection of the BSE onto the hyper-plane $x^+ = 0$ and the NIR of BSA
4. Challenge: Spin dof and BSE
5. A mock pion
6. Conclusions & Perspectives

Motivations and tools

- M: To achieve a fully covariant and, necessarily, non perturbative description of bound systems, with spin degrees of freedom, in Minkowski space, suitable for phenomenological studies. The non perturbative framework established by the Bethe-Salpeter equation (BSE) is our choice
- M: To determine relevant dynamical quantities, e.g. (unpolarized and polarized) light-cone momentum distributions, from the BS amplitude. To this end, the spin-transverse momentum correlations must be described in Minkowski space.
- T: Pivotal role of the Nakanishi Integral Representation (NIR) of the BS amplitude
- T: The fermionic nature of the constituents is suitably managed within the Light-front (LF) framework, making more simple the needed analytical integrations
- T: Standard LAPACK routines for the numerical evaluation of the generalized eigenvalue problem, we formally obtain from BSE

★ Desirable coherent efforts with the Dyson-Schwinger Eq. community for implementing a Minkowskian playground where non perturbative phenomenological studies of the continuous QCD can be carried on

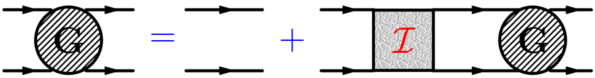
★ Lattice calculations, the recognized non perturbative approach to QCD are performed in the Euclidean space

A snapshot on the BSE

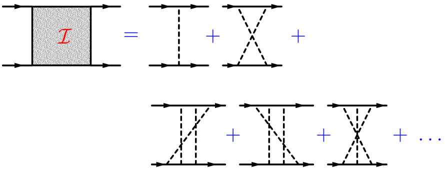
The 4-point Green's Function ($\phi_i \equiv$ scalar fields for simplicity),

$$G(x_1, x_2; y_1, y_2) = < 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 > \quad ,$$

fulfills an integral equation $G = G_0 \; + \; G_0 \; \mathcal{I} \; G$



$\mathcal{I} \;\equiv\;$ interaction kernel, given by the infinite sum of irreducible Feynman graphs. E.g.



$\boxed{\text{Each irr. diagram}}$ generates an infinite set of contributions by $\boxed{\text{iterations}}$

Analyzing

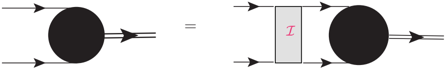
$$G(x_1, x_2; y_1, y_2) =< 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 >$$

close to the bound state(s) pole

$\Rightarrow$ BS Equation

Namely, the integral equation determining the BS amplitude, $\phi(k; p_B, \beta)$, for a bound system is a homogeneous one

$$\phi(k; p_B, \beta) = G_0(k; p_B, \beta) \int d^4k' \, \mathcal{I}(k, k'; p_B) \, \phi(k'; p_B, \beta)$$

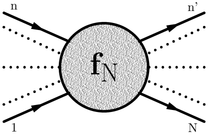


First conclusion: a non perturbative framework, as offered by an integral equation, is necessary for describing a bound state.

Second conclusion: the BS amplitudes get contribution by the infinite set of Fock states, since we have interacting fields! .

Feynman parametric integrals

In the sixties, Nakanishi (PR **130**, 1230 (1963)) proposed an integral representation of N-leg transition amplitudes, based on the parametric formula for the Feynman diagrams.



In a scalar theory, a generic contribution to the transition amplitude with N external legs is given by

$$f_{\mathcal{G}}(p_1, p_2, ..., p_N) \propto \prod_{r=1}^{k} \int d^4 q_r \; \frac{1}{(\ell_1^2 - m_1^2)(\ell_2^2 - m_2^2) \; \ldots \; (\ell_n^2 - m_n^2)}$$

where one has n propagators and k loops ($\equiv$ number of integration variables).
The label $\mathcal{G} \to (n, k)$

N.B. the dependence upon $\{n, k\}$ is in the denominator

Nakanishi Perturbative-theory Integral Rep.(PTIR) - I

Nakanishi proposal for a compact and elegant expression of the full N-leg amplitude $f_N(s) = \sum_{\mathcal{G}} \ f_{\mathcal{G}}(s)$

Introducing the identity

$$1 \doteq \prod_h \int_0^1 dz_h \delta\left(z_h - \frac{\eta_h}{\beta}\right) \int_0^\infty d\gamma \ \delta\left(\gamma - \sum_l \frac{\alpha_l m_l^2}{\beta}\right)$$

with $\beta = \sum \eta_i(\vec{\alpha})$ and integrating by parts $n - 2k - 1$ times

$$f_{\mathcal{G}}(s) \propto \ \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1 - \sum_h z_h) \ \tilde{\phi}_{\mathcal{G}}(z, \gamma)}{(\gamma - \sum_h z_h s_h)}$$

$\tilde{\phi}_{\mathcal{G}}(z, \gamma) \equiv$ proper weight function
$s \equiv \{s_h\}$ scalars from the ext. momenta

The dependence upon the details of the diagram, (n, k), moves from the denominator $\rightarrow$ the numerator!!

The SAME formal expression for the denominator of ANY diagram $\mathcal{G}$ appears

Nakanishi PTIR - II

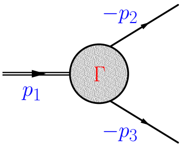
The full N-leg transition amplitude is the sum of infinite diagrams $\mathcal{G}(n, k)$ and it can be formally written as

$$f_N(s) = \sum_{\mathcal{G}} f_{\mathcal{G}}(s) \propto \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1 - \sum_h z_h)\ \phi_N(z, \gamma)}{(\gamma - \sum_h z_h s_h)}$$

where

$$\phi_N(z, \gamma) = \sum_{\mathcal{G}} \tilde{\phi}_{\mathcal{G}}(z, \gamma)$$

Application: 3-leg transition amplitude $\rightarrow$ vertex function for a scalar theory (N.B. for fermions $\rightarrow$ spinor indexes)



$$f_3(s) = \int_0^1 dz \int_0^\infty d\gamma \frac{\phi_3(z, \gamma)}{\gamma - \frac{p^2}{4} - k^2 - zk \cdot p - i\epsilon}$$

with $p = p_1 + p_2$ and $k = (p_1 - p_2)/2$

The expression holds at any order in perturbation-theory !

A vertex function with one leg on mass-shell is related to the BS amplitude. Schematically

$$BSA = G_1 \otimes G_2 \otimes f_3$$

with G_1 and G_2 the propagators of the constituent of the composite system.

The BSE for the celebrated Wick-Cutkosky model, i.e. two massive scalars interacting through a massless scalar can be **exactly solved** by using an integral representation like the one introduced by Nakanishi.

★ The generalization to massive exchange was validate numerically by

- Kusaka et al, PRD **56** (1997) by exploiting the uniqueness of the weight-function
- Carbonell and Karmanov EPJA **27** (2006) 1, and ii) FSV PRD **89** (2014) 016010, by properly integrating both sides of the BSE exploiting Light-front variables, *without resorting to the uniqueness theorem*. Indeed, a successful cross-check was accomplished by using the uniqueness and LF variables by FSV.
- A further generalization was achieved by solving the scattering BSE in the zero-energy limit $\rightarrow$ scattering-state BS amplitude) (FSV, PRD **85** (2012) 036009)

Projecting BSE onto the LF hyper-plane $x^+ = 0$

- NIR contains a great freedom to be exploited, once the weight function is taken as an unknown quantity.
- Even by adopting NIR, BSE still remains a highly singular integral equation
- But NIR makes explicit the analytic structure of the BS amplitude.
- Within the Light-front (LF) framework the *valence component* is obtained by integrating on k^- the BS amplitude) (for simplicity, a two-scalar system)

$$\text{Val. w.f.} = \psi_{n=2}(\xi, k_\perp) = \frac{p^+}{\sqrt{2}}\,\xi\,(1-\xi)\int \frac{dk^-}{2\pi}\,\overbrace{\Phi_b(k,p)}^{\text{BS Amplitude}} =$$

$$= \frac{1}{\sqrt{2}}\xi\,(1-\xi)\,\underbrace{\int_0^\infty d\gamma'\,\frac{g_b(\gamma', 1-2\xi; \kappa^2)}{[\gamma' + k_\perp^2 + \kappa^2 + (2\xi-1)^2\,\frac{M^2}{4} - i\epsilon]^2}}_{\text{NIR}}$$

This is the path for obtaining a more tractable integral equation, exactly equivalent to the original BSE !!

N.B. The valence w.f. $\psi_{n=2}(\xi, k_\perp)$ is a generalized Stieltjes transform of the Nakanishi weight funct. $g_b(\gamma', 1-2\xi; \kappa^2)$ (Carbonell, Frederico, Karmanov PLB 769 (2017), 418)

LF projection of the homogeneous BSE

$$\Phi(k,p) = G_0(k,p) \int d^4k' \; \mathcal{K}_{BS}(k,k',p) \; \Phi(k',p) \; \xrightarrow{NIR+LF}$$

$$\int_0^\infty d\gamma' \frac{g_b(\gamma',z;\kappa^2)}{[\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2-i\epsilon]^2} =$$

$$= \int_0^\infty d\gamma' \int_{-1}^{1} dz' \; V_b^{LF}(\alpha;\gamma,z;\gamma',z') \;\; g_b(\gamma',z';\kappa^2).$$

with $V_b^{LF}(\alpha;\gamma,z;\gamma',z')$ determined by the irreducible kernel $\mathcal{I}(k,k',p)$! $\alpha \equiv g^2/16\pi$ for the scalar case.

Ladder approx. for two-scalar and two fermion systems, neither self-energy nor vertex corrections: Carbonell & Karmanov EPJA 27 (2006) 1, EPJA 46 (2010) 387, FSV PRD 89 (2014) 016010, dPFSV PRD 94 (2016) 071901, EPJC 77 (2017) 764
Cross-ladder kernel: Carbonell & Karmanov EPJA 27 (2006) 11), Nogueira et al PRD 95 (2017) 056012

Very good agreement for both eigenvalues (the coupling constants at given binding energies) and eigenvectors, i.e. the Nakanishi weights, among different groups

Wide phenomenology: (i) Scattering lengths in FVS EPJC 75 (2015) 398, (ii) spectra of excited states and LF momentum distributions in Gutierrez et al PLB 759 (2016) 131.

Spin dof and BSE

Two cases:

★ two fermions interacting, in ladder approx. through i) scalar, ii) pseudoscalar, and iii) vector (Feynman gauge) exchanges

★ ★ a fermion-scalar system, with scalar and vector exchanges: preliminary results

BSE for fermions

$$\Phi(k,p) = \ g^2 \ S(p/2+k) \int d^4k' \ F^2(k-k') \ i\mathcal{K}(k,k') \ \Gamma_1 \ \Phi(k',p) \ \bar{\Gamma}_2 \ S(k-p/2)$$

with

$$S(q) = i\frac{\not{q}+m}{q^2-m^2+i\epsilon} \quad , \quad i\mathcal{K} = \frac{1}{(k-k')^2+i\epsilon} \ , \quad F(k-k') = \frac{(\mu^2-\Lambda^2)}{[(k-k')^2-\Lambda^2+i\epsilon]}$$

$$\Gamma_1 = \Gamma_2 = 1 \ (scalar), \ \gamma_5 \ (pseudo), \ \gamma^\mu \ (vector)$$

For the two-fermion case a form factor $F(k-k')$ has been inserted at each interaction vertex, as in Carbonell & Karmanov EPJA 46 (2010) 387.

First step: the covariant decomposition of the BS amplitude:

$$\Phi(k,p) = S_1\ \phi_1(k,p) + S_2\ \phi_2(k,p) + S_3\ \phi_3(k,p) + S_4\ \phi_4(k,p)$$

$\phi_i \equiv$ unknown scalar functions, with well-defined symmetry under the exchange $1 \to 2$, from the symmetry of both $\Phi(k,p)$ and S_i.

NIR applied to ϕ_i !!

$Tr\{S_i\ S_j\} = \mathcal{N}_i\ \delta_{ij}$ with

$$S_1 = \gamma_5 \quad , \quad S_2 = \frac{\not{p}}{M}\ \gamma_5 \quad , \quad S_3 = \frac{k \cdot p}{M^3}\not{p}\ \gamma_5 - \frac{1}{M}\not{k}\gamma_5 \quad , S_4 = \frac{i}{M^2}\sigma^{\mu\nu} p_\mu k_\nu\ \gamma_5$$

$$\boxed{p = p_1 + p_2 \text{ and } k = (p_1 - p_2)/2}$$

LF projection ⇒ integral-equation system

★ For each ϕ_i, use NIR and apply LF projection

$$\psi_i(\gamma, z) = \int \frac{dk^-}{2\pi}\ \phi_i(k,p) = -\frac{i}{M}\int_0^\infty d\gamma' \frac{g_i(\gamma', z; \kappa^2)}{[\gamma + \gamma' + m^2 z^2 + (1-z^2)\kappa^2 - i\epsilon]^2}$$

- $\gamma \equiv |\mathbf{k}_\perp|^2 \in [0, \infty]$
- $z \equiv 2\xi - 1 \in [-1, 1]$ with $\xi \in [0, 1]$
- $\kappa^2 = 4m^2 - M^2$ with $M = 2m - B$.($B \equiv$ binding energy).

★ ★ The coupled-equation system

$$\psi_i(\gamma, z) = g^2 \sum_j \int_{-1}^{1} dz' \int_0^\infty d\gamma'\ g_j(\gamma', z'; \kappa^2)\ \mathcal{L}_{ij}(\gamma, z, \gamma', z'; p)$$

- $g_j(\gamma', z'; \kappa^2)$ are Nakanishi weights, eigenvectors of the integral-equation system.
- For actual calculations, a suitable basis is: Laguerre(γ) $\times$ Gegenbauer(z).
- The kernel $\mathcal{L}_{ij}(\gamma, z, \gamma', z'; p)$ contains singular contributions produced by integrating on k^- the combination of the numerator of the fermionic propagators and the operators S_i in $\Phi(k', p)$.

Within our LF framework, singular contributions to $\mathcal{L}_{ij}$ can be singled out in a straightforward way, and rigorously evaluated by using well-known results by Yan et al (PRD 7 (1973) 1780 is one of their papers addressing the field theory in the Infinite Momentum frame).

For the two-fermion BSE, singularities have generic form:

$$\mathcal{C}_j = \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} (k^-)^j \; \mathcal{S}(k^-, v, z, z', \gamma, \gamma') \qquad j = 1, 2, 3$$

with $\mathcal{S}(k^-, v, z, z', \gamma, \gamma')$ explicitly calculable

N.B., in the worst case

$$\mathcal{S}(k^-, v, z, z', \gamma, \gamma') \sim \frac{1}{[k^-]^2} \qquad \text{for} \quad k^- \to \infty$$

Then, one cannot close the arc at the ∞ for carrying out the needed analytic integration, but has to deal with a singular behavior on the light-cone, that acquaints meaning in the realm of the distribution functions $\to \delta(x)$

★ The severity of the singularities, i.e. the power j, does depend upon the numerator of the propagators and the structure of the BS amplitude, only

★ ★ The fermion-scalar case is not plagued by singularities of this type.

Numerical comparison: Scalar coupling

	$\mu/m = 0.15$			$\mu/m = 0.50$		
B/m	g^2_{dFSV}(full)	g^2_{CK}		g^2_{dFSV}(full)	g^2_{CK}	g^2_{WR}
0.01	7.844	7.813		25.327	25.23	-
0.02	10.040	10.05		29.487	29.49	-
0.04	13.675	13.69		36.183	36.19	36.19
0.05	15.336	15.35		39.178	39.19	39.18
0.10	23.122	23.12		52.817	52.82	-
0.20	38.324	38.32		78.259	78.25	-
0.40	71.060	71.07		130.177	130.7	130.3
0.50	88.964	86.95		157.419	157.4	157.5
1.00	187.855	-		295.61	-	-
1.40	254.483	-		379.48	-	-
1.80	288.31	-		421.05	-	-

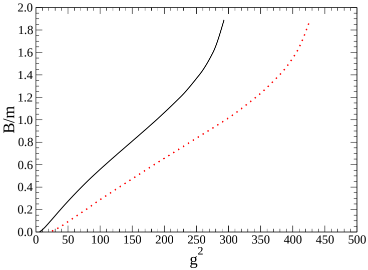
First column: binding energy.
Red digits: coupling constant g^2 for $\mu/m = 0.15$ and 0.50, with the analytical treatment of the fermionic singularities (present work). -
Black digits: results with a numerical treatment of the singularities (Carbonell & Karmanov EPJA **46**, (2010) 387).
Blue digits: results by using the Wick-rotation from Dorkin et al FBS. **42** (2008) 1.

Numerical comparison: Pseudo-Scalar coupling

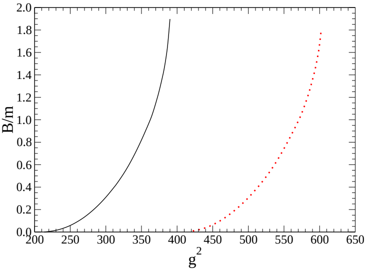
	$\mu/m = 0.15$			$\mu/m = 0.50$	
B/m	g^2_{dFSV}(full)	g^2_{CK}		g^2_{dFSV} (full)	g^2_{CK}
0.01	225.7	224.8		422.6	422.3
0.02	233.2	232.9		430.5	430.1
0.04	243.1	243.1		440.9	440.4
0.05	247.1	247.0		444.9	444.3
0.10	262.1	262.1		460.4	459.9
0.20	282.9	282.9		482.1	480.7
0.40	311.7	311.8		513.3	515.2
0.50	322.9	323.1		525.8	525.9
1.00	362.3	-		570.9	-
1.40	380.1	-		591.8	-
1.80	388.7	-		602.1	-

Solid lines: $\mu/m = 0.15$ - Dotted lines: $\mu/m = 0.5$

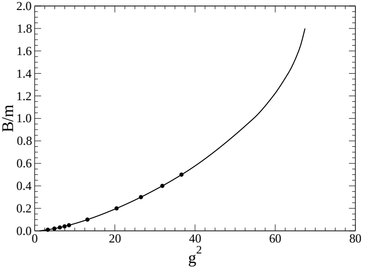
Scalar



Pseudoscalar



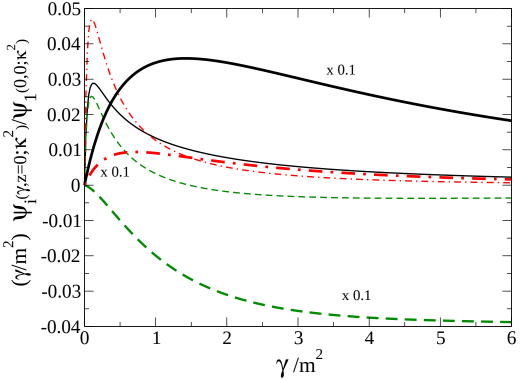
Massless vector exchange



Full dots: g^2 from Carbonell & Karmanov EPJA **46**, (2010) 387, with a numerical treatment of the singularities.

N.B. A critical value g_{crit} is clearly approached for $B/m \to 2$ (cf G. Baym PR 117 (1960) 886)

Vector coupling and high-momentum tails: $\gamma \equiv |\mathbf{k}_\perp|^2$



LF amplitudes ψ_i times γ/m^2 at fixed $z = 0$ ($\xi = 1/2$), for the massless-vector coupling.

$B/m = 0.1$ (thin lines) and 1.0 (thick lines).

——— : $(\gamma/m^2)\ \psi_1$.
— —: $(\gamma/m^2)\ \psi_2$.
— • : $(\gamma/m^2)\ \psi_4$.
$\psi_3 = 0$ for $z = 0$ (odd function)

Power one is expected for the pion valence amplitude from dimensional arguments by X. Ji et al, PRL 90 (2003) 241601 (cf also Brodsky & Farrar (PRL 31 (1973) 1153) for the counting rules of exclusive amplitudes)

For scalars $\phi(\gamma, z) \sim 1/[\gamma]^2$ (FSV PRD 89 (2014) 016010)

A mock pion

A fermion-antifermion 0^- system bound through a massive-vector exchange, with a fermion mass $m = 200$ MeV , hence for $m_\pi = 150$ MeV one gets $B/m = 1.25$). The effective gluon mass has been fixed to $\mu = 300$ MeV

Light-front valence amplitudes

$$\psi_i(\gamma, z) = \int \frac{dk^-}{2\pi}\ \phi_i(k,p) = -\frac{i}{M}\int_0^\infty d\gamma' \frac{g_i(\gamma', z; \kappa^2)}{[\gamma + \gamma' + m^2 z^2 + (1-z^2)\kappa^2 - i\epsilon]^2}$$

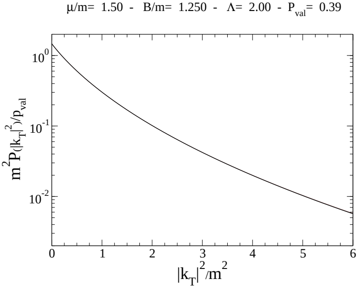
where

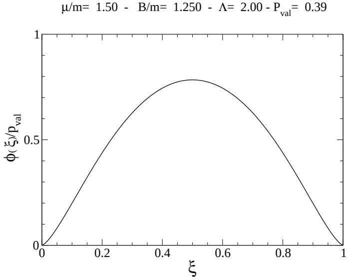
$$\phi_i(k,p) = \int_0^\infty d\gamma' \int_{-1}^{1} dz'\ \frac{g_i(\gamma', z'; \kappa^2)}{\left[k^2 + z' k\cdot p - \gamma' - \kappa^2 + i\epsilon\right]^3}$$

N.B. $\kappa^2 = 4m^2 - M^2$, $\gamma = |k_T|^2$, $z \ \rightarrow\ \xi \in [0,1]$

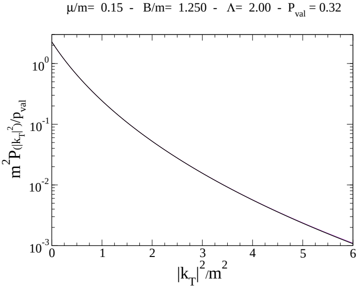
$|\psi_i(|k_T|^2, \xi)|^2 \rightarrow$ longitudinal and transverse-momentum distributions, for the valence component

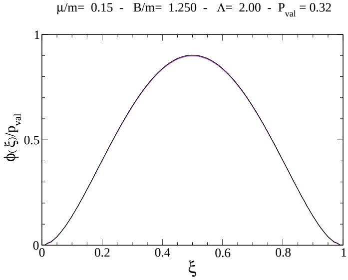
$\alpha_S = 32.4$





$\alpha_S = 5.2$





Preliminary result for a fermion-scalar bound system

The covariant decomposition of the BS amplitude for a $(1/2)^+$ bound system, composed by a fermion and a scalar, reads

$$\Phi(k,p) = \Big[S_1 \; \phi_1(k,p) + S_2 \; \phi_2(k,p) \Big] \; U(p,s)$$

with $U(p,s)$ a Dirac spinor, $S_1(k) = 1$, $S_2(k) = \not{k}/M$,and $M^2 = p^2$

A first check: scalar coupling $\alpha^s = \lambda^s_F \lambda^s_S/(8\pi m_S)$, for $m_F = m_S$ and $\mu/\bar{m} = 0.15, \; 0.50$

$B/\bar{m}$	$\alpha^s_M(0.15)$	$\alpha^s_{WR}(0.15)$	$\alpha^s_M(0.50)$	$\alpha^s_{WR}(0.50)$
0.10	1.5057	1.5057	2.6558	2.6558
0.20	2.2969	2.2969	3.2644	3.6244
0.30	3.0467	3.0467	4.5354	4.5354
0.40	3.7963	3.7963	5.4505	5.4506
0.50	4.5680	4.5681	6.4042	6.4043
0.80	7.2385	7.2387	9.8789	9.8794
1.00	9.7779	9.7783	13.7379	13.7380

First column: the binding energy in unit of $\bar{m} = (m_S + m_F)/2$..
Second and fourth columns: coupling constant α_M, obtained by solving the BSE in Minkowski space, for given $B/\bar{m}$.
Third and fifth columns: Wick-rotated results, α_{WR}.

Conclusions & Perspectives

- A systematization of the technique for solving the fermionic BSE with spin dof has been given.
- The LF framework has well-known advantages in performing analytical integrations, and in the case of systems with spin dof its effectiveness has been shown in its full glory.
- Dynamical quantities can be addressed in Minkowski space: the space where the processes evolve!!
- Our numerical investigations, performed in ladder approximation at the present stage, confirm both the robustness of the Nakanishi Integral Representation for the BS amplitude,that can be applied to any analytical BS kernel, and encourage to extend the technique to the Dyson-Schwinger Eq. (work in progress)

Besides the valence momentum distributions, other dynamical quantities to be investigated

- Pion transverse-momentum distributions and electromagnetic form factor (work in progress)
- Fragmentation functions?