Quark mass dependence of $\gamma^{(*)}\pi \to \pi\pi$

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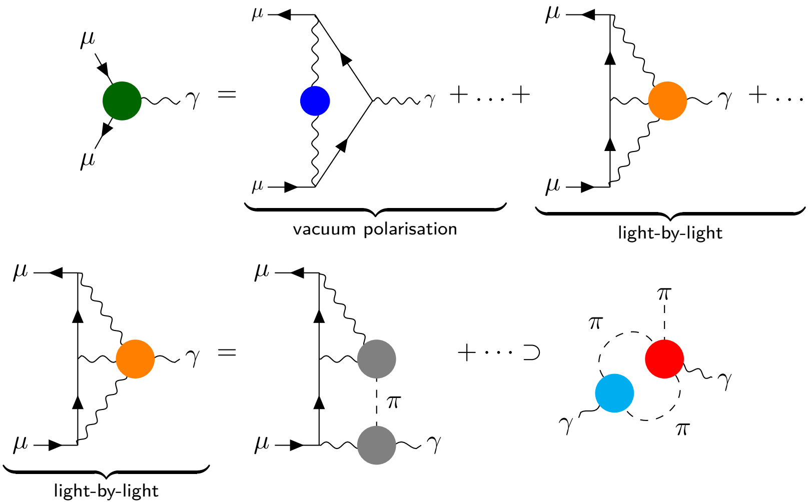
March 15, 2018

Bound states in strongly coupled systems

work in progress with M. Hoferichter, B. Kubis

Content

$g-2$ of μ



[Colangelo et al., Phys. Lett. B **738**(2014)]

Anomaly

anomaly in ChPT:

- odd number of pseudo-Goldstone bosons
- Wess-Zumino-Witten action

[Wess, Zumino, Phys. Lett. B **37**(1971)] [Witten, Nucl. Phys. B **223**(1983)]

structure of amplitude:

$$\mathcal{M}\left(\gamma(q)\,\pi^-(p_1) \to \pi^-(p_2)\,\pi^0(p_0)\right) = i\epsilon_{\mu\nu\alpha\beta}\epsilon^\mu(q)\,p_1^\nu p_2^\alpha p_0^\beta \mathcal{F}(s,t,u)$$
$$s = (q+p_1)^2\,, \qquad t = (p_1-p_2)^2\,, \qquad u = (p_1-p_0)^2$$

at low energies due to anomaly:

$$\mathcal{F}(s,t,u) \to F_{3\pi} = \frac{eN_{\text{colour}}}{12\pi^2 F_\pi^3}$$

tested only at 10 % level

[Hoferichter, Kubis, Sakkas, Phys. Rev. D **86**(2012)]

ρ resonance

resonance properties encoded in pole:

- mass/width from pole position
  - $M_\rho \approx 760\,\mathrm{MeV}$
  - $\Gamma_\rho \approx 140\,\mathrm{MeV}$

  $\Rightarrow \frac{\Gamma_\rho}{M_\rho} \approx 20\,\%$

  [García-Martín et al., Phys. Rev. Lett. **107**(2011)]
  Breit-Wigner not optimal
- coupling via residue:
  model independent extraction of radiative coupling $g_{\rho\pi\gamma}$
- extension to ρ_3 via VMD

[Hoferichter, Kubis, Zanke, Phys. Rev. D **96**(2017)]

Quark mass dependence

- lattice QCD calculations the more expensive the lower M_π
- simulations at physical point become more and more feasible
- simulations of some scattering processes still with unphysical M_π, e.g. $\gamma\pi \to \pi\pi$ with $M_\pi \approx 400\,\text{MeV}$
  [Briceño et al., Phys. Rev. Lett. **115**(2015)]

$\Rightarrow$ need for extrapolation

- for $\pi\pi \to \pi\pi$: unitarised ChPT (e.g. via dispersion relations)
  [Bolton, Briceño, Wilson, Phys. Lett. B **757**(2016)]
- for $\gamma\pi \to \pi\pi$: extrapolation more difficult

$\pi\pi \to \pi\pi$: Inverse amplitude method (IAM)

elastic unitarity ($SS^\dagger = 1$, only $\pi\pi$ intermediate states):

$$\mathrm{Im}\left[t(s)\right] = \sigma(s)\left|t(s)\right|^2, \qquad \sigma(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}$$

$$\Rightarrow \qquad \mathrm{Im}\left[\frac{1}{t(s)}\right] = -\sigma(s)$$

unitarity in ChPT only perturbatively:

$$t = t_2 + t_4 + \mathcal{O}\left(p^6\right) \qquad \Rightarrow \qquad \mathrm{Im}\left[\frac{t_4(s)}{t_2^2(s)}\right] = \sigma(s)$$

partial wave dispersion relation + ChPT at subtraction point and left cut:

$$t(s) \approx \frac{t_2(s)^2}{t_2(s) - t_4(s) + \mathrm{P}(s)}$$

[Truong, Phys. Rev. Lett. **67**(1991)]
[Gómez Nicola, Peláez, Ríos, Phys. Rev. D **77**(2008)]

$I = J = 1$ partial waves in $SU(2)$ ChPT

[Gasser, Leutwyler, Annals of Physics **158**(1984)]

- LO:

$$t_2(s) = \frac{s - 4M_\pi^2}{96\pi F^2}$$

- NLO:

$$t_4(s) = \frac{s - 4M_\pi^2}{4608\pi^3 F^4}\left\{s\left(\bar{l} + \frac{1}{3}\right) - \frac{15}{2}M_\pi^2 + (\text{logarithmic terms})\right\} + i\sigma(s)\, t_2^2(s)$$

  - F: pion decay constant in chiral limit
  - $\bar{l} := \bar{l}_2 - \bar{l}_1$: only free parameter at NLO

  both independent of M_π, $F \to F_\pi$ introduces $\bar{l}_4$

  - F_π, $\bar{l}_4$ depend on M_π
  - both approaches differ at $\mathcal{O}(p^6)$ (NNLO)

- ad hoc NNLO polynomial:

$$p_6(s) = \frac{s - 4M_\pi^2}{4608\pi^3 F^4}\frac{1}{16\pi F^2}\left\{\alpha s M_\pi^2 + \beta M_\pi^4 + \gamma s^2\right\}$$

Fit to lattice data

- lattice data: $M_\pi \approx 236\,\text{MeV}$ [Wilson et al., Phys. Rev. D **92**(2015)]
- fit directly energy levels: IAM phases $\rightarrow$ Lüscher's method $\rightarrow$ energy levels
- Roy equations + NNLO ChPT: $\bar{l} = 4.7(6)$ [Colangelo, Gasser, Leutwyler, Nucl. Phys. B **603**(2001)]
- lattice ($N_f = 2 + 1$) + ChPT: $\bar{l}_4 = 4.10(45)$ [FLAG, EPJ C **77**(2017)]

[1] [Bolton, Briceño, Wilson, Phys. Lett. B **757**(2016)]

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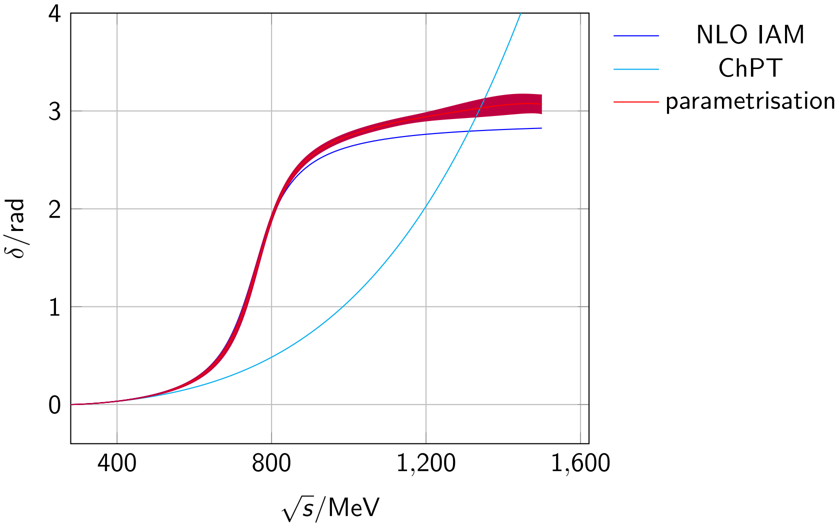
	χ^2/d.o.f.	$\bar{l}$	$\bar{l}_4$	β	γ
only $\bar{l}$	3.2	5.9			
$\bar{l}$ and $\bar{l}_4$ [1]	1.3	7.0(2)	$-1.0(^{+1.1}_{-2.0})$		
$\bar{l}$ and γs^2	1.1	4.9			0.6
$\bar{l}$ and βM_π^4	1.25	7.0		-82.6	

$\Rightarrow$ need to fit full NNLO ChPT IAM (together with J. Ruiz de Elvira)

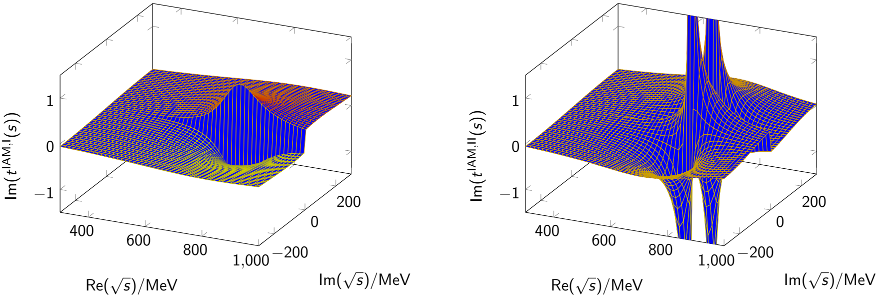
[1] [Bolton, Briceño, Wilson, Phys. Lett. B **757**(2016)]



parametrisation of experimental data from
[Caprini, Colangelo, Leutwyler, EPJ C **72**(2012)]

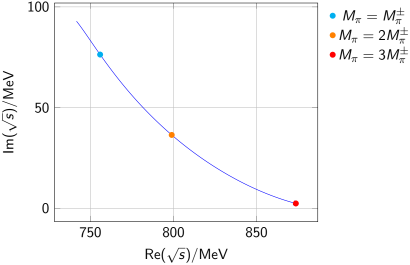
Analytic structure

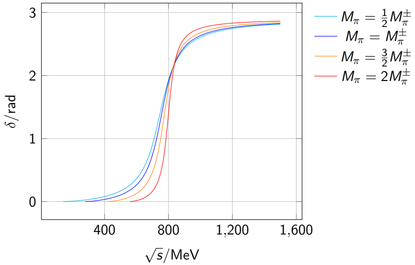
$$t^{\mathrm{II}}(s) = \frac{t(s)}{1 + 2i\sigma\left(s, M_\pi^2\right) t(s)}$$



[Dobado, Peláez, Phys. Rev. D **56**(1997)]

Pion mass dependence

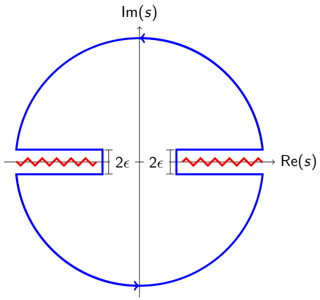




- increasing $M_\pi \Rightarrow$ increase of M_ρ
- increasing $M_\pi \Rightarrow$ decrease of Γ_ρ
  decrease in phase space

[Hanhart, Peláez, Ríos, Phys. Rev. Lett. **100**(2008)]
[Bolton, Briceño, Wilson, Phys. Lett. B **757**(2016)]

Fixed-t dispersion relations



2 variables → 1 variable:

- $s + t + u = 3M_\pi^2 + q^2 =: R$
- fix t
- $\mathcal{F}_s^t(s) := \mathcal{F}(s, t, u^t(s))$

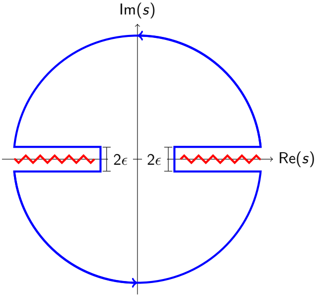
analytic structure of $\mathcal{F}_s^t(s)$:

- $s \geq 4M_\pi^2$: right-hand cut
- $u \geq 4M_\pi^2$: left-hand cut

calculate:

$$\frac{s^n}{2\pi i}\int_{\mathcal{C}} \frac{\mathcal{F}_s^t(\zeta)}{(\zeta - s)\,\zeta^n}\mathrm{d}\zeta$$

Fixed-t dispersion relations



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$$\mathcal{F}_s^t(s) = P_{n-1}^t(s) + \frac{s^n}{2\pi i}\int_{s^{\text{thr}}}^{\infty} \frac{\text{disc}_x\left[\mathcal{F}_s^t(x)\right]}{(x - s)\,x^n}\,\mathrm{d}\,x + \frac{(u^t(s))^n}{2\pi i}\int_{u^{\text{thr}}}^{\infty} \frac{\text{disc}_x\left[\mathcal{F}_u^t(x)\right]}{(x - u^t(s))\,x^n}\,\mathrm{d}\,x$$

Reconstruction theorem

- discontinuities via partial wave expansion
- truncate (here: only f_1)

$$\mathcal{F}(s,t,u) = \mathcal{B}(s) + \mathcal{B}(t) + \mathcal{B}(u)$$
$$\mathcal{B}(\zeta) = \text{const.} + \frac{\zeta^n}{2\pi i}\int_{4M_\pi^2}^{\infty} \frac{\text{disc}\,[f_1(x)]}{(x-\zeta)\,x^n}\,\mathrm{d}\,x$$

- applicable for all s, t, u
- correct cuts in all channels
- study $\mathcal{B}$ (only right cut) instead of $\mathcal{F}$
- invariant if $\mathcal{B}(\zeta) \mapsto \mathcal{B}(\zeta) + \lambda(3\zeta - R)$ with $R = s + t + u = 3M_\pi^2 + q^2$
- possible to incorporate higher partial waves
- applicable to other processes (e.g. $\eta' \to \eta\pi\pi$, cf. talk by T. Isken)

[Stern, Sazdjian, Fuchs, Phys. Rev. D **47**(1993)]
[Ananthanarayan et al., EPJ C **19&22**(2001)]
[Novotný, Zdráhal, Phys. Rev. D **78**(2008)]

Inhomogeneous Omnès problem

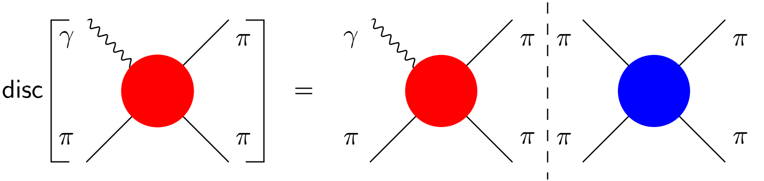


figure by T. Isken

elastic unitarity $\Rightarrow$ Watson's theorem:

$$\mathrm{disc}\left(f_1(x)\right) = 2if_1(x)\exp\left[-i\delta(x)\right]\sin\left[\delta(x)\right]$$

$$\delta = \pi\pi \text{ scattering phase } (I = J = 1)$$

hat function:

$$\widehat{\mathcal{B}}(s) := f_1(s) - \mathcal{B}(s) = \frac{3}{2}\int_{-1}^{1}\left(1 - z^2\right)\mathcal{B}\left(t(s,z)\right)\mathrm{d}z$$

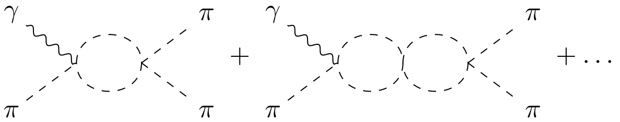
$$\mathrm{disc}\left(f_1(x)\right) = 2i\left[\mathcal{B}(x) + \widehat{\mathcal{B}}(x)\right]\exp\left[-i\delta(x)\right]\sin\left[\delta(x)\right]$$

Inhomogeneous Omnès problem

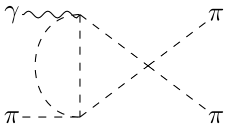
$$\mathcal{B}(s) = \Omega(s)\left[\sum_{k=0}^{m-1} c_k s^k + \frac{s^m}{\pi}\int_{4M_\pi^2}^{\infty} \frac{\widehat{\mathcal{B}}(x)\sin[\delta(x)]}{|\Omega(x)|(x-s)x^m}\mathrm{d}x\right]$$

$$\Omega(s) = \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\delta(x)}{x(x-s)}\mathrm{d}\,x\right], \qquad \widehat{\mathcal{B}}(s) = \frac{3}{2}\int_{-1}^{1}\left(1-z^2\right)\mathcal{B}(t(s,z))\,\mathrm{d}z$$

right cut only:



right cut + crossed channel cuts:



Inhomogeneous Omnès problem

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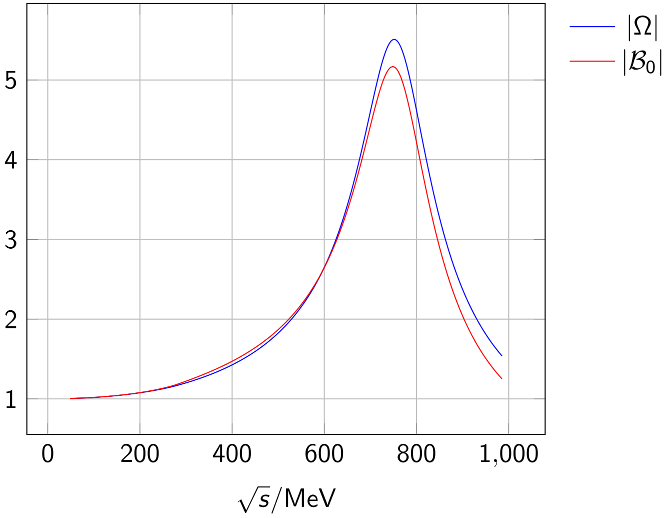
$$\Omega(s) = \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\delta(x)}{x(x-s)}\,\mathrm{d}\,x\right], \qquad \widehat{\mathcal{B}}(s) = \frac{3}{2}\int_{-1}^{1}\left(1-z^2\right)\mathcal{B}(t(s,z))\,\mathrm{d}z$$

- δ unknown at high energies ($\delta \to \pi$ useful)
- $m = 1$: one subtraction constant
- $m = 3$: ~~three~~ two subtraction constants (polynomial shifts)
- $\widehat{\mathcal{B}}$ linear in $\mathcal{B}$:

$$\mathcal{B}(s) = \sum_{k=0}^{m-1} c_k \mathcal{B}_k(s)$$

  - basis functions $\mathcal{B}_k$ depend on δ only
  - $\mathcal{B}_k$ can be computed numerically (iteratively)



if $\widehat{\mathcal{B}} = 0$: $\mathcal{B}_0 = \Omega$: important to incorporate left cut

Pion mass dependence

$$\mathcal{B}(s, M_\pi) = \sum_{k=0}^{m-1} c_k(M_\pi)\mathcal{B}_k(s, M_\pi)$$

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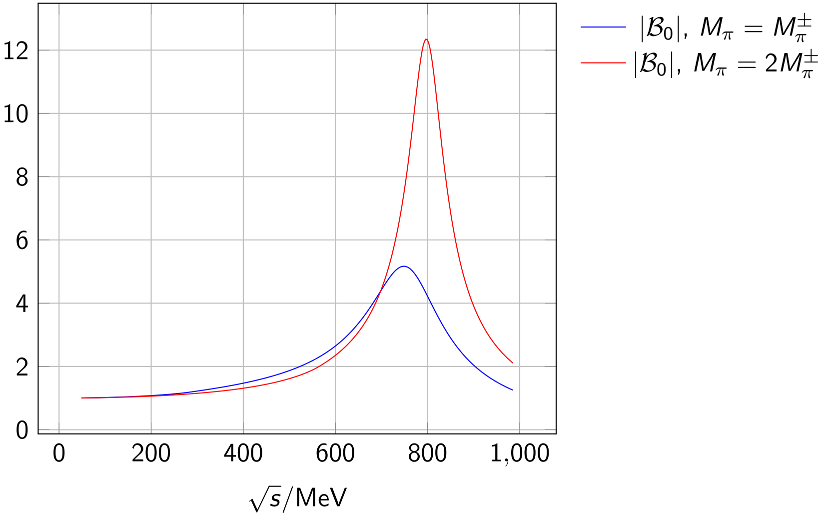
basis functions:

$$\mathcal{B}_k(s, M_\pi) = \Omega(s, M_\pi)\left[s^k + \frac{s^m}{\pi}\int_{4M_\pi{}^2}^{\infty} \frac{\widehat{\mathcal{B}}_k(x, M_\pi)\sin[\delta(x, M_\pi)]}{|\Omega(x, M_\pi)|(x-s)x^m}\mathrm{d}x\right]$$

$$\widehat{\mathcal{B}}_k(s, M_\pi) = \frac{3}{2}\int_{-1}^{1}(1-z^2)\,\mathcal{B}_k(t(s, z, M_\pi), M_\pi)\,\mathrm{d}z$$

Omnès function:

$$\Omega(s, M_\pi) = \exp\left[\frac{s}{\pi}\int_{4M_\pi{}^2}^{\infty}\frac{\delta(x, M_\pi)}{x(x-s)}\,\mathrm{d}\,x\right]$$

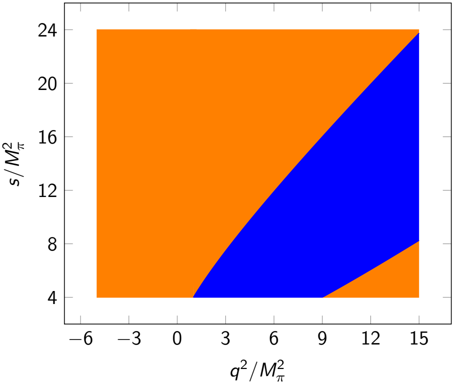


12
10
8
6
4
2
0
0
200
400
600
800
1,000
$\sqrt{s}$/MeV
$|\mathcal{B}_0|,\ M_\pi = M_\pi^\pm$
$|\mathcal{B}_0|,\ M_\pi = 2M_\pi^\pm$

Virtual photon

- amplitude holomorphic in q^2
- decay via crossing symmetry:

$$\mathcal{M}\left(\gamma(q) \to \pi^-(p_2)\,\pi^0(p_0)\,\pi^+(p)\right) = \mathcal{M}\left(\gamma(q)\,\pi^-(-p) \to \pi^-(p_2)\,\pi^0(p_0)\right)$$



analytic continuation via contour integral (→ Khuri-Treiman equations):

$$\widehat{\mathcal{B}}_k(s) = \frac{3}{2}\int_{-1}^{1}\left(1-z^2\right)\mathcal{B}_k\left(t\left(s,z;q^2\right)\right)\mathrm{d}z$$

[Bronzan, Kacser, Phys. Rev. **132**(1963)] [Niecknig, Kubis, Schneider, EPJ C **72**(2012)]

Subtraction constants

intermediate states in $\gamma \to \pi\pi\pi$ lead to discontinuity in q^2

$$c_k\left(q^2, M_\pi^2\right) = \sum_{j=0}^{n-1} d_j\left(M_\pi^2\right)\left(q^2\right)^j + \frac{\left(q^2\right)^n}{2\pi i}\int_{\nu^{\text{thr}}}^{\infty} \frac{\text{disc}_x\left(c_k\left(x, M_\pi^2\right)\right)}{\left(x - q^2\right)x^n}\text{d}x$$

	M/MeV	Γ/MeV	decay channels	Γ_i/Γ/%
ω	782.65(12)	8.49(8)	$\pi^+\pi^-\pi^0$	89.2(7)
			$\pi^0\gamma$	8.40(22)
ϕ	1019.460(16)	4.247(16)	K^+K^-	48.9(5)
			$K_L^0 K_S^0$	34.2(4)
			$\pi^+\pi^-\pi^0$	15.32(32)

[Particle Data Group, Chin. Phys. C **40**(2016 and 2017 update)]

- discontinuity via Breit-Wigner [Hoferichter et al., EPJ C **74**(2014)]
- pion mass dependence of d_j e.g. via matching to ChPT

Summary

framework:

1. fit IAM to LQCD data for $\pi\pi \to \pi\pi$ to fix δ
2. use δ to compute basis function numerically
3. determine subtraction constants via fit to LQCD data for $\gamma\pi \to \pi\pi$
4. extrapolate to physical point

Summary

framework:

1. fit IAM to LQCD data for $\pi\pi \to \pi\pi$ to fix δ
2. use δ to compute basis function numerically
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4. extrapolate to physical point

approximations:

- isospin symmetry
- only $\pi\pi$ intermediate multi-particle states
- IAM uses ChPT to given order (in particular at left cut)
- δ unknown at high energies
- $\text{disc}\left[f_J\left(s\right)\right] = 0, \qquad \forall J \geq 3$
- model building for q^2 dependence of subtraction constants

Spares

Reconstruction theorem: sketch of derivation

1. fixed-t DR $\mathcal{F}_s^t$, fixed-u DR $\mathcal{F}_s^u$, fixed-s DR $\mathcal{F}_t^s$
2. partial wave expansion (for simplicity disc $f_J(x) = 0$ for all $J \geq 3$)

$$\begin{aligned}\mathcal{F}(s,t,u) &= \sum_{J=0}^{\infty} f_{2J+1}(s)\, P'_{2J+1}(z(s,t,u)) \\ &= f_1(s) + \ldots\end{aligned}$$

3. express discontinuities in DRs via PWs
4. symmetrise amplitude

$$\begin{aligned}\mathcal{F}(s,t,u) &= \mathcal{B}(s) + \mathcal{B}(t) + \mathcal{B}(u) \\ \mathcal{B}(\zeta) &= \text{const.} + \frac{\zeta^n}{2\pi i} \int_{4M_\pi^2}^{\infty} \frac{\text{disc}\,[f_1(x)]}{(x-\zeta)\, x^n}\, \mathrm{d}\, x\end{aligned}$$

[Stern, Sazdjian, Fuchs, Phys. Rev. D **47**(1993)]
[Ananthanarayan et al., EPJ C **19&22**(2001)]
[Novotný, Zdráhal, Phys. Rev. D **78**(2008)]

Subtraction constants: details

$$c_k\left(q^2, M_\pi^2\right) = \sum_{j=0}^{n-1} d_j\left(M_\pi^2\right)\left(q^2\right)^j + \frac{\left(q^2\right)^n}{2\pi i}\int_{\nu^{\text{thr}}}^{\infty} \frac{\text{disc}_x\left(c_k\left(x, M_\pi^2\right)\right)}{\left(x-q^2\right)x^n}\text{d}x$$

Breit-Wigner for ω [Hoferichter et al., EPJ C **74**(2014)] :

$$\text{disc}_{q^2}\left(c_k\left(q^2, M_\pi^2\right)\right) = 2i\text{Im}\left[\frac{r}{M_\omega^2 - i\sqrt{q^2}\Gamma_\omega - q^2}\right]$$

M_π dependence of:

- M_ω like M_π dependence of M_ρ [Bijnens, Gosdzinsky, Phys. Lett. B **388**(1996)]
- M_ρ via IAM (approx. linear in M_π)
- $\Gamma_{\omega\to3\pi}$ via dispersive framework [Niecknig, Kubis, Schneider, EPJ C **72**(2012)] well described by kinematic terms only (cf. talk by T. Isken)
- $\Gamma_{\omega\to\pi^0\gamma}$ via effective Lagrangian [Klingl, Kaiser, Weise, Z. Phys. A **356**(1996)]
- d_j via ChPT/resonance saturation

LECs

$$F_\pi = F + \frac{M_\pi^2}{16\pi^2 F_\pi}\bar{l}_4$$

$$\bar{l}_4 = 16\pi^2 l_4^r(\mu) - \ln\left(\frac{M_\pi^2}{\mu^2}\right)$$

here: $\mu = 770\,\text{MeV}$

[Gasser, Leutwyler, Annals of Physics **158**(1984)]

Quantum numbers

	I	J	P	C	G	M/MeV	Γ/MeV
γ	0,1	1	$-$	$-$		0	
π^0	1	0	$-$	$+$	$-$	134.9770(5)	
$\pi^\pm$	1	0	$-$		$-$	139.570 61(24)	
$\rho^\pm, \rho^0$	1	1	$-$	$-$	$+$	≈ 770	≈ 149
ω	0	1	$-$	$-$	$-$	782.65(12)	8.49(8)
ϕ	0	1	$-$	$-$	$-$	1019.460(16)	4.247(16)

[Particle Data Group, Chin. Phys. C **40**(2016 and 2017 update)]

Symmetry and partial wave decomposition of $\mathcal{F}$

G invariance

$\Rightarrow$ only isoscalar photons

$\Rightarrow$ only isospin 1 waves

$+$ Bose symmetry in final $\pi\pi$ state

$\Rightarrow$ only odd partial waves

C invariance $+$ crossing symmetry

$\Rightarrow \mathcal{F}(s,t,u) = \mathcal{F}(u,t,s)$

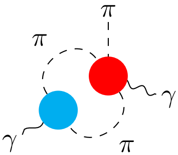
isospin invariance

$\Rightarrow \mathcal{F}(s,t,u) = \mathcal{F}(t,s,u)$

altogether:

- $\mathcal{F}$ completely symmetric
- only odd partial waves

Connection to doubly-virtual π^0 transition form factor



isospin decomposition:

$$F_{\pi^0\gamma^*\gamma^*}\left(q_1^2, q_2^2\right) = F_{vs}\left(q_1^2, q_2^2\right) + \left(q_1^2 \leftrightarrow q_2^2\right)$$

once subtracted dispersion relation in isovector virtuality:

$$F_{vs}\left(s_1, s_2\right) = F_{vs}\left(0, s_2\right) + \frac{e s_1}{12\pi^2} \int_{4M_\pi^2}^{\infty} \frac{q_\pi^3\left(x\right) F_\pi^{V*}\left(x\right) f_1\left(x, s_2\right)}{x^{3/2}\left(x - s_1\right)} \, \mathrm{d}\, x$$

$$q_\pi\left(x\right) = \sqrt{\frac{x}{4} - M_\pi^2}$$

[Hoferichter et al., EPJ C **74**(2014)]

Kinematics

Mandelstam variables:

$$\begin{aligned}
t_{\gamma\pi}\left(s, z, q^2, M_\pi^2\right) &= a\left(s, q^2, M_\pi^2\right) + b\left(s, q^2, M_\pi^2\right) z \\
a\left(s, q^2, M_\pi^2\right) &= \frac{3M_\pi^2 + q^2 - s}{2} \\
b\left(s, q^2, M_\pi^2\right) &= \frac{\sigma\left(s, M_\pi^2\right)}{2}\sqrt{\lambda(s, q^2, M_\pi^2)} \\
\sigma\left(s, M_\pi^2\right) &= \sqrt{1 - \frac{4M_\pi^2}{s}} \\
u_{\gamma\pi}\left(s, z, q^2, M_\pi^2\right) &= t_{\gamma\pi}\left(s, -z, q^2, M_\pi^2\right)
\end{aligned}$$

threshold:

$$s^{\text{thr}} = \begin{cases} \left(\sqrt{q^2} + M_\pi\right)^2, & q^2 > M_\pi^2 \\ 4M_\pi^2, & q^2 \leq M_\pi^2 \end{cases}.$$