
Introduction to QFT

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Yesterday

- Particles are excitations of SHOs
- We like to expand around the harmonic terms
- Perturbation theory for the unharmonic terms

PT for 2 SHOs

$$V(x, y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y$$

- We assume that α is small
- Classically α moves energy between the two modes
- How it goes in QM?
- Recall the Fermi golden rule

$$P \propto |\mathcal{A}|^2 \times \text{P.S.} \quad \mathcal{A} \sim \langle f | \alpha x^2 y | i \rangle$$

- The relevant thing to calculate is the transition amplitude, \mathcal{A}

Transitions

$$V(x, y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y$$

- Recall

$$x \sim a_x + a_x^\dagger \quad y \sim a_y + a_y^\dagger$$

- For a given i , when $\mathcal{A} \neq 0$?

$$\mathcal{A} \sim \langle f | \alpha x^2 y | i \rangle$$

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- For a given i , when $\mathcal{A} \neq 0$?

$$\mathcal{A} \sim \langle f | \alpha x^2 y | i \rangle$$

- Since $H_1 \sim x^2 y$ we see that $\Delta n_y = \pm 1$ and $\Delta n_x = 0, \pm 2$
- What could you say if the perturbation was $x^2 y^3$?

Two SHOs with small α

$$V(x, y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y \quad \omega_y = 2\omega_x$$

- Consider $|i\rangle = |0, 1\rangle$
- Since $\omega_y = 2\omega_x$ only $f = |2, 0\rangle$ is allowed by energy conservation and by the perturbation

$$\mathcal{A} \sim \langle 2, 0 | \alpha x^2 y | 0, 1 \rangle \sim \alpha \langle 2, 0 | (a_x + a_x^\dagger)(a_x + a_x^\dagger)(a_y + a_y^\dagger) | 0, 1 \rangle$$

- a_y in y annihilates the y “particle” and $(a_x^\dagger)^2$ in x^2 creates two x “particles”
- It is a decay of a particle y into two x particles with width $\Gamma \propto \alpha^2$ and $\tau = 1/\Gamma$

Even More PT

$$H_1 = \alpha x^2 z + \beta xyz \quad \omega_z = 10, \omega_y = 3, \omega_x = 1$$

- Calculate $y \rightarrow 3x$ using 2nd order PT

$$\mathcal{A} \sim \langle 3, 0, 0 | \mathcal{O} | 0, 1, 0 \rangle \quad \mathcal{O} \sim \sum \frac{\langle 3, 0, 0 | V' | n \rangle \langle n | V' | 0, 1, 0 \rangle}{E_n - E_{0,1,0}}$$

- Which intermediate states? $|1, 0, 1\rangle$ and $|2, 1, 1\rangle$

- $\mathcal{A}_1 = |0, 1, 0\rangle \xrightarrow{\beta} |1, 0, 1\rangle \xrightarrow{\alpha} |3, 0, 0\rangle$

- $\mathcal{A}_2 = |0, 1, 0\rangle \xrightarrow{\alpha} |2, 1, 1\rangle \xrightarrow{\beta} |3, 0, 0\rangle$

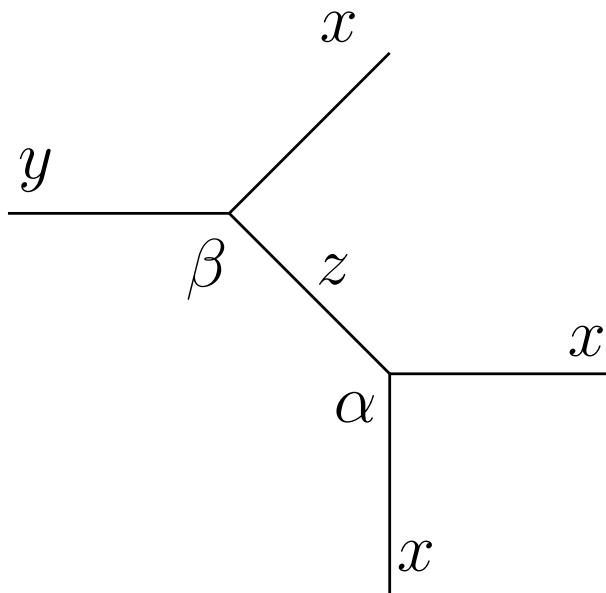
- The total amplitude is then

$$\mathcal{A} \propto \alpha\beta \left(\frac{\#}{\Delta E_1} + \frac{\#}{\Delta E_2} \right) \propto \alpha\beta \left(\frac{\#}{\Delta E^2} \right)$$

Closer look

$$V' = \alpha x^2 z + \beta xyz \quad \omega_z = 10, \omega_y = 3, \omega_x = 1$$

- We look at $\mathcal{A} = |0, 1, 0\rangle \xrightarrow{\beta} |1, 0, 1\rangle \xrightarrow{\alpha} |3, 0, 0\rangle$



$$\mathcal{A} \propto \frac{\alpha\beta}{\Delta E}$$

What about the other amplitude?

Feynman diagrams

Using PT for fields

- For SHOs we have $x_i \sim a_i + a_i^\dagger$
- For fields we then have

$$\phi \sim \int [a(k) + a^\dagger(k)] dk$$

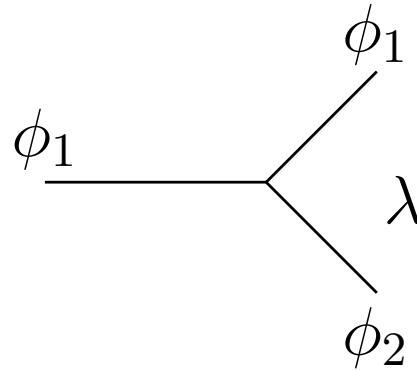
Quantum field = creation and annihilation operators

Feynman diagrams

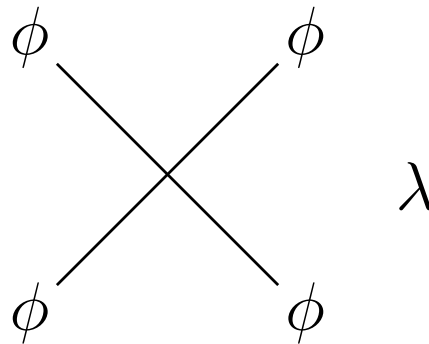
- A graphical way to do perturbation theory with fields
- Unlike SHOs before, a particle can have any energy as long as $E \geq m$
- Operators with 3 or more fields generate transitions between states. They give decays and scatterings
- Decay rates and scattering cross sections are calculated using the Golden Rule
- Amplitudes are calculated from \mathcal{L}
- We generate graphs where lines are particles and vertices are interactions

Examples of vertices

$$\mathcal{L} = \lambda \phi_1^2 \phi_2 :$$



$$\mathcal{L} = \lambda \phi^4 :$$



Calculations

- We usually care about $1 \rightarrow n$ or $2 \rightarrow n$ processes
- We need to make sure we have energy conservation
- External (Internal) particles are called on(off)-shell
 - On-shell: $E^2 = p^2 + m^2$
 - Off-shell: $E^2 \neq p^2 + m^2$
- \mathcal{A} = the product of all the vertices and internal lines
- Each internal line with q^μ gives suppression

$$\frac{1}{m^2 - q^2}$$

- There are many more rules to get all the factors right

Examples of amplitudes

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\Gamma(Z \rightarrow XY)$$

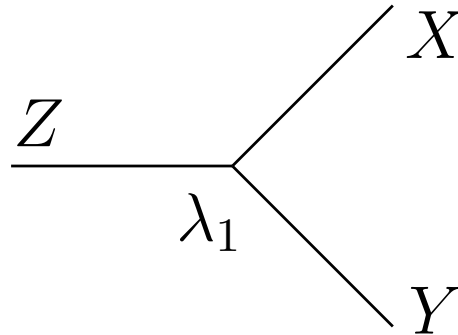
- Energy conservation condition
- Draw the diagram and estimate the amplitude

Examples of amplitudes

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\Gamma(Z \rightarrow XY)$$

- Energy conservation condition $m_Z > m_X + m_Y$
- Draw the diagram and estimate the amplitude



$$\mathcal{A} \propto \lambda_1$$

Examples of amplitudes (2)

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\Gamma(Y \rightarrow 3X)$$

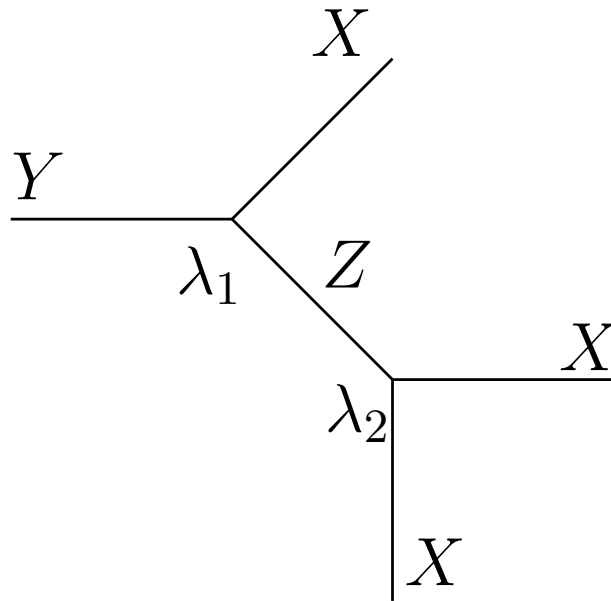
- Energy conservation condition
- Draw the diagram and estimate the amplitude

Examples of amplitudes (2)

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\Gamma(Y \rightarrow 3X)$$

- Energy conservation condition $m_Y > 3m_X$
- Draw the diagram and estimate the amplitude



$$\begin{aligned} \mathcal{A} &\propto \lambda_1 \lambda_2 \times \frac{1}{\Delta E_Z^2} \\ &= \lambda_1 \lambda_2 \times \frac{1}{m_Z^2 - q^2} \end{aligned}$$

Examples of amplitudes (HW)

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\sigma(XX \rightarrow XY)$$

- Energy conservation condition
- Draw the diagram and estimate the amplitude

Some summary

- Quadratic terms describe free fields. Free particles cannot be created nor decay
- We use perturbation theory where terms with 3 or more fields in \mathcal{L} are considered small
- These terms can generate and destroy particles and give dynamics
- Feynman diagrams are a tool to calculate transition amplitudes
- Many more details are needed to get calculation done
- Once calculations and experiments to check them are done, we can test our theory

Symmetries

How to “built” Lagrangians

- \mathcal{L} is:
 - The most general one that is invariant under some symmetries
 - We work up to some order (usually 4)
- We need the following input:
 - What are the symmetries we impose
 - What DOFs we have and how they transform under the symmetry
- The output is
 - A Lagrangian with N parameters
 - We need to measure its parameters and test it

Symmetries and representations

Example: 3d real space in classical mechanics

- We require that \mathcal{L} is invariant under rotation
- All our DOFs are assigned into vector representations ($\vec{r}_1, \vec{r}_2, \dots$)
- We construct invariants from these DOFs. They are called singlets or scalars

$$C_{ij} \equiv \vec{r}_i \cdot \vec{r}_j$$

- We then require that V is a function of the C_{ij} s

Generalizations

- In mechanics, \vec{r} lives in 3d real space and is a vector
- Fields do not live in real space. They live in some mathematical space
- They also do not have to be vectors, but can be scalars or tensors (representation)
- The idea is similar to what we did in mechanics
 - We require \mathcal{L} to be invariant under rotation in that mathematical space
 - Thus \mathcal{L} depends only on combinations of fields that form singlets
- All this is related to a subject called Lie groups
- We usually care about $SO(N)$, $SU(N)$ and $U(1)$

Combining representations

- It is all about generating singlets
- We all know that we can combine vectors in real space to generate singlets
- We also know how to make a spin zero from 2 spin half spinors (spin zero is a singlet!)
- There is a generalization of this procedure to any mathematical space
- As of now, all we need to know are $SU(3)$, $SU(2)$ and $U(1)$

Invariant of complex numbers

- $U(1)$ is rotation in 1d complex space
- Each complex number comes with a q that tells us how much it rotates
- When we rotate the space by an angle θ , the number rotate as

$$X \rightarrow e^{iq\theta} X$$

- Consider $q_X = 1$, $q_Y = 2$, $q_Z = 3$ and write 3rd and 4th order invariants

$$XX^*YY^* \quad X^2Y^*$$

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$$XX^*YY^* \quad X^2Y^* \quad XYZ^* \quad X^3Z^* \quad Y^2X^*Z^*$$

$SU(2)$

- $U(2)$ is rotation in 2d complex space. We have $U(2) = SU(2) \times U(1)$
- $SU(2)$ is locally the same as rotation in 3d real space
- Rotations in this space are non-Abelian (non-commutative)
- It depends on the representation: scalar, spinor, vector
- Spin in QM is described by $SU(2)$ rotations, so we use the same language to describe it
- For the SM all we care is that $1/2 \times 1/2 \ni 0$ so we know how to generate singlets
- How can we generate invariants from spin $1/2$ and spin $3/2$?

$SU(3)$

- $U(3)$ is rotation in 3d complex space. We have $U(3) = SU(3) \times U(1)$
- The representations we care about are singlets, triplets and octets
- Unlike $SU(2)$, in $SU(3)$ we have complex representations, 3 and $\bar{3}$
- The three quarks form a triplet (the three colors)
- To form a singlet we need to know that

$$3 \times \bar{3} \ni 1 \quad 3 \times 3 \times 3 \ni 1$$

- This is why we have baryons and mesons

A game

A game calls “building invariants”

- Symmetry is $SU(3) \times SU(2) \times U(1)$
 - $U(1)$: Add the numbers (\bar{X} has charge $-q$)
 - $SU(2)$: $2 \times 2 \ni 1$ and recall that 1 is a singlet
 - $SU(3)$: we need $3 \times \bar{3} \ni 1$ and $3 \times 3 \times 3 \ni 1$

● Fields are

$$Q(3, 2)_1 \quad U(3, 1)_4 \quad D(3, 1)_{-2} \quad H(1, 2)_3$$

● What 3rd and 4th order invariants can we built?

$$(HH^*)^2 \quad H^3 \quad UDD \quad QUD \quad HQU^*$$

● HW: Find more invariants

Lorentz invariants

Lorentz invariants

- The representations we care about are
 - Singlet: Spin zero (scalars, denote by ϕ)
 - LH and RH fields: Spin half (fermions, ψ_L, ψ_R)
 - Vector: Spin one (gauge boson, denote by A_μ)
- Fermions are more complicated

$$\mathcal{L} \sim \bar{\psi} \partial_\mu \gamma^\mu \psi$$

- Since \mathcal{L} has dimension 4, ψ is dimension 3/2
- For fermions when we expand up to 4th order we can have at most two fermion fields
- Under Lorentz, the basic fields are left-handed and right-handed. A mass term must involve both $m\bar{\psi}_L\psi_R$

Local symmetries

Local symmetry

Basic idea: rotations depend on x and t

$$\phi(x_\mu) \rightarrow e^{iq\theta} \phi(x_\mu) \xrightarrow{\text{local}} \phi(x_\mu) \rightarrow e^{iq\theta(x_\mu)} \phi(x_\mu)$$

- It is kind of logical and we think that all imposed symmetries in Nature are local
- The kinetic term $|\partial_\mu \phi|^2$ is not invariant
- We want a kinetic term (why?)
- We can save the kinetic term if we add a field that is
 - Massless
 - Spin 1
 - Adjoint representation: $q = 0$ for $U(1)$, triplet for $SU(2)$, and octet for $SU(3)$

Gauge symmetry

- Fermions are called matter fields. What they are and their representation is an input
- Gauge fields are known as force fields

Local symmetries \Rightarrow force fields