Lie groups

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Chapter 1

Lie Groups

Symmetries play a crucial role in model building. You are already familiar with symmetries and with some of their consequences. For example, nature seems to have the symmetry of the Poincare group, which implies conservation of energy, momentum and angular momentum. In order to understand the interplay between symmetries and interactions, we need a mathematical tool called *Lie groups*. These are the groups that describe all continuous symmetries.

In the following we only give definitions and quote statements without proving them. There are many texts about Lie group where the statements we make below are proven. Three that are very useful for particle physics purposes are the book by Howard Georgi ("Lie Algebras in particle physics"), by Robert Cahn ("Semi-simple Lie algebras and their representations"), by A. Zee ("Group Theory in a Nutshell for Physicists") and the physics report by Richard Slansky ("Group Theory for Unified Model Building", Phys. Rept. 79 (1981) 1).

1.1 Groups

We start by presenting a series of definitions.

Definition: A *group* G is a set $\{x_i\}$ (finite or infinite), with a multiplication law $\cdot$, subject to the following four requirements:

- Closure:
$$x_i \cdot x_j \in G \qquad \forall\, x_i, x_j. \tag{1.1}$$
- Associativity:
$$x_i \cdot (x_j \cdot x_k) = (x_i \cdot x_j) \cdot x_k \qquad \forall\, x_i, x_j, x_k. \tag{1.2}$$
- There is an Identity element I (or e) such that
$$I \cdot x_i = x_i \cdot I = x_i \qquad \forall\, x_i. \tag{1.3}$$

- Each element has an inverse element x_i^{-1}:

$$x_i \cdot x_i^{-1} = x_i^{-1} \cdot x_i = I \qquad \forall\, x_i. \tag{1.4}$$

A group is specified by its multiplication table.

Definition: A group is *Abelian* if all its elements commute:

$$x_i \cdot x_j = x_j \cdot x_i \qquad \forall\, x_i, x_j. \tag{1.5}$$

A *non-Abelian* group is a group that is not Abelian, that is, at least one pair of elements does not commute.

Let us give a few examples:

- Z_2, also known as parity, is a group with two elements, I and P, such that I is the identity and $P^{-1} = P$. This completely specifies the multiplication table. This group is finite and Abelian.

- Z_N, with N a positive integer, is a generalization of Z_2. It contains N elements labeled from zero through $N-1$. The multiplication law is the same as addition modulo N: $x_i \cdot x_j = x_{(i+j)\text{mod } N}$. The identity element is x_0, and the inverse element is given by $x_i^{-1} = x_{N-i}$. This group is also finite and Abelian.

- Multiplication of positive numbers. It is an infinite Abelian group. The identity is the number one and the multiplication law is just a standard multiplication.

- S_3, the group that describes the permutations of three elements. It contains 6 elements. This group is non-Abelian. In your homework you will find for yourself the 6 elements and their multiplication table.

1.2 Representations

One of the most important aspects of group theory that is relevant to physics is related to representation theory, and that is what we discuss next.

Definition: A *representation* is a realization of the multiplication law among matrices.

Definition: Two representations are *equivalent* if they are related by a similarity transformation (that is by a unitary rotation).

Definition: A representation is *reducible* if it is equivalent to a representation that is block diagonal.

Definition: An *irreducible* representation (irrep) is a representation that is not reducible.

Definition: An irrep that contains matrices of size $n \times n$ is said to be of *dimension* n.

Statement: Any reducible representation can be written as a direct sum of irreps

Statement: The dimension of all irreps of an Abelian group is one. For non-Abelian groups there is at least one irrep that has dimension larger than one.

Statement: Any finite group has a finite number of irreps R_i. If N is the number of elements in the group, the irreps satisfy

$$\sum_i [\dim(R_i)]^2 = N. \tag{1.6}$$

Infinite groups have infinite number of irreps.

Statement: For any group there exists a *trivial* representation such that all the matrices are just the number 1. This representation is also called the *singlet* representation. As we see later, it is of particular importance for us.

Let us give some examples for the statements that we made here.

- Z_2: Its trivial irrep is $I = 1$, $P = 1$. The other irrep is $I = 1$, $P = -1$. Clearly these two irreps satisfy Eq. (1.6).
- Z_N: An example of a non-trivial irrep is $x_k = \exp(i2\pi k/N)$.
- S_3: In your homework you will work out its properties.

The groups that we are interested in are *transformation groups of physical systems*. Such transformations are associated with *unitary operators* in the Hilbert space. We often describe the elements of the group by the way that they transform physical states. When we refer to representations of the group, we mean either the appropriate set of unitary operators, or, equivalently, by the matrices that operate on the vector states of the Hilbert space.

1.3 Lie groups and Lie Algebras

While finite groups are very important, the ones that are most relevant to particle physics and, in particular, to the Standard Model, are infinite groups, in particular continuous groups, that is of cardinality $\aleph_1$. These groups are called Lie groups. They give us formal ways to talk about rotations is any real or abstract space. The different groups corresponds to rotations in different spaces.

Definition: A *Lie group* is an infinite group whose elements are labeled by a finite set of N continuous real parameters α_ℓ, and whose multiplication law depends smoothly on the α_ℓ's. The number N is called the dimension of the group.

Different groups have different N. Yet, the dimension of the group does not uniquely defined it. We discuss below the classifications of groups.

Statement: An Abelian Lie group has $N = 1$. A non-Abelian Lie group has $N > 1$.

The first example of an Abelian Lie group is a group we denote by $U(1)$. It represents addition of real numbers modulo 2π, that is, rotation on a circle. Such a group has an infinite number of elements that are labeled by a single continuous parameter α. We can write the group elements as $M = \exp(i\alpha)$. We can also represent it by $M = \exp(2i\alpha)$ or, more generally, as $M = \exp(iX\alpha)$ with X real. Each X generates an irrep of the group.

We are mainly interested in *compact* Lie groups. We do not define this term formally here, but we can use the $U(1)$ example to give an intuitive explanation of what it means. A group of adding with a modulo is compact, while just adding (without the modulo) would be non-compact. In the first, if you repeat the same addition a number of times, you may return to your starting point, while in the latter this would never happen. In other words, in a compact Lie group, the parameters have a finite range, while in a non-compact group, their range is infinite. (Do not confuse that with the number of elements, which is infinite in either case.) Another example is rotations and boosts: Rotations represent a compact group while boosts do not.

Statement: The elements of any compact Lie group can be written as

$$M = \exp(i\alpha_\ell X_\ell) \tag{1.7}$$

such that X_ℓ are specific Hermitian matrices and α_ℓ, as mentioned before, are real numbers. (We use the standard summation convention, that is $\alpha_\ell X_\ell \equiv \sum_\ell \alpha_\ell X_\ell$.)

Definition: The X_ℓ are called the *generators* of the group.

Let us perform some algebra before we turn to our next definition. Consider two elements of a group, A and B, such that in A only $\alpha_a \neq 0$, and in B only $\alpha_b \neq 0$ and, furthermore, $\alpha_a = \alpha_b = \lambda$:

$$A \equiv \exp(i\lambda X_a), \qquad B \equiv \exp(i\lambda X_b). \tag{1.8}$$

Since A and B are in the group, each of them has an inverse. Thus also

$$C = BAB^{-1}A^{-1} \equiv \exp(i\beta_c X_c) \tag{1.9}$$

is in the group. Let us take λ to be a small parameter and expand around the identity. Clearly, if λ is small, also all the β_c are small. Keeping the leading order terms, we get

$$C = \exp(i\beta_c X_c) \approx I + i\beta_c X_c, \qquad C = BAB^{-1}A^{-1} \approx I + \lambda^2 [X_a, X_b]. \tag{1.10}$$

In the $\lambda \to 0$ limit, we have

$$[X_a, X_b] = i\frac{\beta_c}{\lambda^2} X_c. \tag{1.11}$$

The combinations

$$f_{abc} \equiv \lambda^{-2}\beta_c \tag{1.12}$$

are independent of λ. Furthermore, while λ and β_c are infinitesimal, the f_{abc}-constants do not diverge. This brings us to a new set of definitions.

Definition: f_{abc} are called the *structure constants* of the group.

Definition: The commutation relations [see Eq. (1.11)]

$$[X_a, X_b] = i f_{abc} X_c, \tag{1.13}$$

constitute the *algebra* of the Lie group.

Note the following points regarding the Lie Algebra:

- The algebra defines the local properties of the group but not its global properties. Usually, this is all we care about.

- The Algebra is closed under the commutation operator.

- Similar to our discussion of groups, one can define representations of the algebra, that is, matrix representations of X_ℓ. In particular, each representation has its own dimension. (Do not confuse the dimension of the representation with the dimension of the group.)

- The generators satisfy the Jacobi identity

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0. \tag{1.14}$$

- For each algebra there is the trivial (singlet) representation which is $X_\ell = 0$ for all ℓ. The trivial representation of the algebra generates the trivial representation of the group.

- Since an Abelian Lie group has only one generator, its algebra is always trivial. Thus, the algebra of $U(1)$ is the only Abelian Lie algebra.

- Non-Abelian Lie groups have non-trivial algebras.

- The generators of the Non-Abelian Lie groups are traceless.

The example of $SU(2)$ algebra is well-known from Quantum Mechanics:

$$[X_a, X_b] = i\varepsilon_{abc} X_c. \tag{1.15}$$

Here ε_{abc} are the structure constants of the $SU(2)$ group. Usually, in QM, X is called L, S, or J. The three matrices S_x, S_y and S_z for a given spin S corresponds to a a given irrep of $SU(2)$. The $SU(2)$ group represents non-trivial rotations in a two-dimensional complex space. Its algebra is the same as the algebra of the $SO(3)$ group, which represents rotations in the three-dimensional real space.

We should explain what we mean when we say that "the group represents rotations in a space." The QM example makes it clear. Consider a finite Hilbert space of, say, a particle with spin S. The matrices that rotate the direction of the spin are written in terms of exponentials of the S_i operators. For a spin-half particle, the S_i operators are written in terms of the Pauli matrices. For particles with spin different from 1/2, the S_i operators will be written in terms of different matrices. We learn that the group represents rotations in some space, while the various representations correspond to different objects that can "live" in that space.

There are three important irreps that have special names. The first one is the trivial – or *singlet* – representation that we already mentioned. Its importance stems from the fact that it corresponds to something that is symmetric under rotations. While that might sound confusing it is really trivial. Rotation of a singlet does not change its representation. Rotation of a spin half spinor does change the components in the spinor.

The second important irrep is the *fundamental* representation. This is the smallest non-trivial irrep. For $SU(2)$, this is the spinor, or spin half, representation. An important property of the fundamental representation is that it can be used to get all other representations. We return to this point later. Here we just remind you that this statement is well familiar from QM. One can get spin-1 by combining two spin-1/2, and you can get spin-3/2 by combining three spin-1/2. Any non-Abelian Lie group has a fundamental irrep.

The third important irrep is the *adjoint* representation. It is made out of the structure constants themselves. Think of a matrix representation of the generators. Each entry, T^c_{ij} is labelled by three indices. One is the c index of the generator itself, that runs from 1 to N, such that N depends on the group. The other two indices, i and j, are the matrix indices that run from 1 to the dimension of the representation. One can show that each Lie group has one representation where the dimension of the representation is the same as the dimension of the group. This representation is obtained by defining

$$(X_c)_{ab} \equiv -if_{abc}. \tag{1.16}$$

In other words, the structure constants themselves satisfy the algebra of their own group. (See the homework for more details.) In $SU(2)$, the adjoint representation is that of spin-1. In your homework you will check for yourself that the ε_{ijk} are just the set of the three 3×3 representations of spin 1.

Before closing this section we make several remarks about subalgebras and simple groups.

Definition: A *subalgebra* M is a set of generators that are closed under commutation.

Definition: Consider an algebra L with two subalgebras L_1 and L_2 such that for any $X \in L_1$ and $Y \in L_2$, $[X, Y] = 0$. The algebra L is *not simple* and it can be written as a direct product: $L = L_1 \times L_2$.

Definition: A *simple* Lie algebra is an algebra that cannot be written as a direct product.

Since any algebra can be written as a direct product of simple Lie algebras, we can think about each of the simple algebras separately. A useful example is that of the $U(2)$ group. A $U(2)$ transformation corresponds to a rotation in two-dimensional complex space. This group is not simple:

$$U(2) = SU(2) \times U(1). \tag{1.17}$$

Think, for example, about the rotation of a spinor. It can be separated into two: The trivial rotation is just a $U(1)$ transformation, that is, a phase multiplication of the spinor. The non-trivial rotation is the $SU(2)$ transformation, that is, an internal rotation between the two spin components.

1.4 Roots and Weights

Here we move to discuss properties of the algebra and the representations. From this point on we only consider irreps, and thus we do not distinguish anymore between a representation and an irrep.

Definition: The *Cartan subalgebra* is the maximal commuiting subalgebra. In practice, we find it by first finding a basis where the number of the diagonal generators is maximal. In this basis, the Cartan subalgebra is defined to be the subset of all the diagonal generators. Obviously, these generators all commute with each other and thus they constitute a subalgebra.

Definition: The number of generators in the Cartan subalgebra is called the *rank* of the algebra. At times we aslo refer to it as the rank of the group.

Let us consider a few examples. Since the $U(1)$ algebra has only a single generator, it is of rank one. $SU(2)$ is also rank one. You can make one of its three generators, say S_z, diagonal, but not two of them simultaneously. $SU(3)$ is rank two. We later elaborate on $SU(3)$ in much more detail. (We have to, because the Standard Model has an $SU(3)$ symmetry.)

Our next step is to introduce the terms roots and weights. We do that via an example. Consider the $SU(2)$ algebra. It has three generators. We usually choose S_3 to be in the Cartan subalgebra, and we can combine the two other generators, S_1 and S_2, to a raising and a lowering operator, $S^{\pm} = S_1 \pm iS_2$. Any representation can be defined by the eigenvalues under the operation of the generators in the Cartan subalgebra, in this case S_3. For example, for the spin-1/2 representation, the eigenvalues are $-1/2$ and $+1/2$; For the spin-1 representation, the eigenvalues are -1, 0, and $+1$. Under the operation of the raising (S^+) and lowering (S^-) generators, we "move" from one eigenstate of S_3 to another. For example, for a spin-1 representation, we have $S^-|1\rangle \propto |0\rangle$.

Let us now consider a general Lie group of rank n. Any representation is characterized by the possible eigenvalues of its eigenstates under the operation of the Cartan subalgebra: $|e_1, e_2..., e_n\rangle$.

Statement: We can assemble all the operators that are not in the Cartan subalgebra into "lowering" and "raising" operators. That is, when they act on an eigenstate they either move it to another eigenstate or annihilate it.

Definition: The *weight vectors* (or simply *weights*) of a representation are the possible eigenvalues of the generators in the Cartan subalgebra.

Definition: The *roots* of the algebra are the various ways in which the generators move a state between the possible weights.

Statement: The weights completely describe the representation.

Statement: The roots completely describe the algebra.

Statement: The weights of the adjoint representation are the roots of the Lie algebra.

Note that both roots and weights live in an n-dimensional vector space, where n is the rank of the group. The number of roots is the dimension of the group. The number of weights is the dimension of the representation.

Let us return to our $SU(2)$ example. The vector space of roots and weights is one-dimensional. The three roots are $-1, 0, +1$. The trivial representation has only one weight, zero; The fundamental has two, $\pm 1/2$; The adjoint has three, $0, \pm 1$; and so on. You can also see that the weights of the adjoint irrep are the roots of the algebra.

1.5 $SU(3)$

In this section we discuss the $SU(3)$ group. It is more complicated than $SU(2)$ and it allows us to demonstrate few aspects of Lie groups that cannot be demonstrated with $SU(2)$. Of course, it is also important since it is relevant to physics.

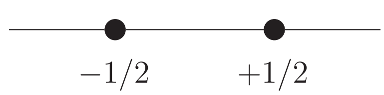
We start by recalling $SU(2)$. The fundamental irrep can be explicitly be written in terms of the Pauli matrices, $X_a = \sigma_a/2$, with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.18}$$

The Pauli matrices are traceless, and only one of them is diagonal, as we expected given that $SU(2)$ has rank one. Given that, we can get the weight diagram of the fundamental representation. We take the two vectors,

$$(1,0)^T, \qquad (0,1)^T \tag{1.19}$$

and apply to them the generator in the Cartan subalgebra, X_3 and find the two weights, $\pm 1/2$, which can be drawn has

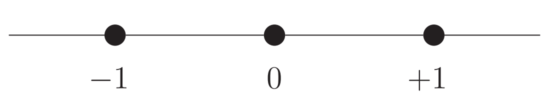


(1.20)

The roots are the combination of generators that move us between the weights. Here they are the known raising and lowering generators

$$\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2). \tag{1.21}$$

They change the weight by ± 1, so the root diagram has one zero root (correspond to σ_3) and ± 1. This root diagram is also the weight diagram of the adjoint representation, also known as spin one, and we draw it below



(1.22)

We now move to $SU(3)$,which is a generalization of $SU(2)$. It may be useful to think about it as rotations in three-dimensional complex space. Similar to $SU(2)$, the full symmetry of these rotations is called $U(3)$, and it can be written as a direct product of simple groups, $U(3) = SU(3) \times U(1)$. The $SU(3)$ algebra has eight generators. (You can see it by recalling that rotation in a complex space is done by unitary matrices, and any unitary matrix can be written with Hermitian matrix in the exponent. There are nine independent Hermitian 3×3 matrices. They can be separated to a unit matrix, which corresponds to the $U(1)$ part, and eight traceless matrices, which correspond to the $SU(3)$ part.) We go on and study $SU(3)$ without proving that it is related to the above intuitive picture of rotation in three dimensional complex space.

Similar to the use of the Pauli matrices for the fundamental representation of $SU(2)$, the fundamental representation of $SU(3)$ is usually written in terms of the Gell-Mann matrices,

$$X_a = \lambda_a/2, \tag{1.23}$$

with

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{1.24}$$

We would like to emphasize the following points:

1. The Gell-Mann matrices are traceless, as they should be.

2. There are many $SU(2)$ subalgebras. One of them is manifest and it is given by λ_1, λ_2 and λ_3.

3. It is manifest that $SU(3)$ is of rank two: λ_3 and λ_8 are in the Cartan subalgebra.

Having explicit expressions of fundamental representation in our disposal, we can draw the weight diagram. In order to do so, let us recall how we do it for the fundamental (spinor) representation of $SU(2)$. We have two basis vectors (spin-up and spin-down); we apply S_z on them and obtain the two weights, $+1/2$ and $-1/2$. Here we follow the same steps. We take the three vectors,

$$(1,0,0)^T, \qquad (0,1,0)^T, \qquad (0,0,1)^T, \tag{1.25}$$

and apply to them the two generators in the Cartan subalgebra, X_3 and X_8. We find the three weights

$$\left(+\frac{1}{2}, +\frac{1}{2\sqrt{3}}\right), \qquad \left(-\frac{1}{2}, +\frac{1}{2\sqrt{3}}\right), \qquad \left(0, -\frac{1}{\sqrt{3}}\right). \tag{1.26}$$

We can plot this in a weight diagram in the $X_3 - X_8$ plane. You will do it in your homework.

Once we have the weights we can get the roots. They are just the combination of generators that move us between the weights. Since $SU(3)$ is rank two, the two roots that are in the Cartan are at the origin. The other six are those that move us between the three weights. We find that they are

$$\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right), \qquad (\pm 1, 0). \tag{1.27}$$

You will plot them in your homework. This root diagram is also the weight diagram of the adjoint representation. In terms of the Gell-Mann matrices, we can see that the raising and lowering generators are proportional to

$$I_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2) \qquad V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5) \qquad U_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7). \tag{1.28}$$

The names I, U, and V are, at this point, just names, but they are useful in some physics cases that you may encounter later in your study.

1.6 Classification and Dynkin diagrams

The $SU(3)$ example allows us to obtain more formal results. In the case of $SU(2)$, it is clear what the raising and lowering operators are. The generalization to groups with higher rank is as follows.

Definition: A *positive (negative) root* is a root whose first non-zero component is positive (negative). A raising (lowering) operator corresponds to a positive (negative) root.

Definition: A *simple root* is a positive root that is not the sum of other positive roots.

Statement: Every rank-k algebra has k simple roots. Which ones they are is a matter of convention, but their relative lengths and angles are fixed.

In fact, it can be shown that the simple roots fully describe the algebra. It can be further be shown that there are only four possible angles and corresponding relative lengths between simple roots:

angle	90°	120°	135°	150°
relative length	N/A	$1:1$	$1:\sqrt{2}$	$1:\sqrt{3}$.

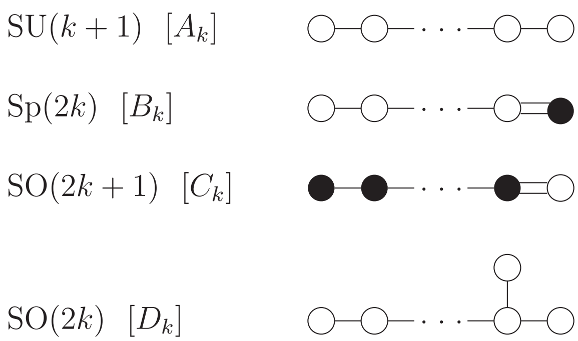
(1.29)

The above rules can be visualized using Dynkin diagrams. Each simple root is described by a circle. The angle between two roots is described by the number of lines connecting the circles:

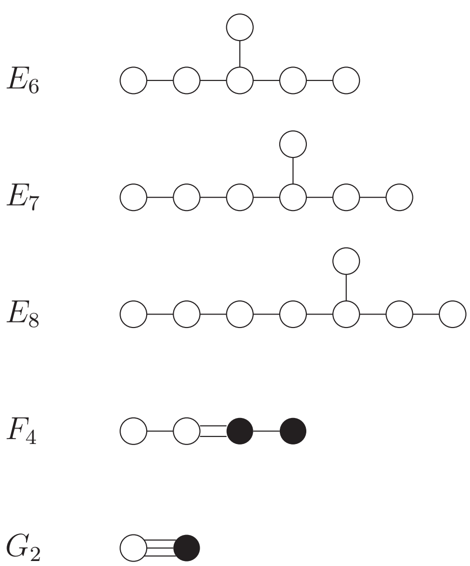
(1.30)

where the solid circle in a link represent the largest root.

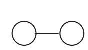
There are seven classes of Lie groups. Four classes are infinite, and three classes, called the exceptional groups, each have only a finite number of Lie groups. Below you can find all the sets. The number of circles is the rank of the group. Note that different names for the infinite groups are used in the physics and mathematics communities. Below we give both names, but we use only the physics names from now on. In the diagrams below k is the rank of the group.



(1.31)



(1.32)

Consider, for example, $SU(3)$. The two simple roots are equal in length and have an angle of 120° between them. Thus, the Dynkin diagram is just  .

Dynkin diagrams provide a very good tool to tell us also about what the subalgebras of a given algebra are. We do not describe the procedure in detail here, and you are encouraged to read it for yourself in one of the books. One simple point to make is that removing a simple root always corresponds to a subalgebra. For example, removing simple roots you can see the following breaking pattern:

$$E_6 \to SO(10) \to SU(5) \to SU(3) \times SU(2). \tag{1.33}$$

You may find such a breaking pattern in the context of Grand Unified Theories (GUTs).

Finally, we would like to mention that the algebras of some small groups are identical

$$SU(2) \simeq SO(3) \simeq Sp(2), \quad SU(4) \simeq SO(6), \quad SO(4) \simeq SU(2) \times SU(2), \quad SO(5) \simeq Sp(4). \tag{1.34}$$

1.7 Naming representations

We are now back to discussing representations. How do we name an irrep? In the context of $SU(2)$, which is rank one, there are three different ways to do so.

(i) We denote an irrep by its highest weight. For example, spin-0 denotes the singlet representation, spin-1/2 refers to the fundamental representation, where the highest weight is 1/2, and spin-1 refers to the adjoint representation, where the highest weight is 1.

(ii) We can define the irrep according to the dimension of the representation-matrices, which is also the number of weights. Then the singlet representation is denoted by 1, the fundamental by 2, and the adjoint by 3.

(iii) We can name the representation by the number of times we can apply S_- to the highest weight without annihilating it. In this notation, the singlet is denoted as (0), the fundamental as (1), and the adjoint as (2).

Before we proceed, let us explain in more detail what we mean by "annihilating the state". Let us examine the weight diagram. In $SU(2)$, which is rank-one, this is a one dimensional diagram. For example, for the fundamental representation, it has two entries, at $+1/2$ and $-1/2$. We now take the highest weight (in our example, $+1/2$), and move away from it by applying the root that corresponds to the lowering operator, -1. When we apply it once, we move to the lowest weight, $-1/2$. When we apply it once more, we move out of the weight diagram, and thus "annihilate the state". Thus, for the spin-1/2 representation, we can apply the root corresponding to S_- once to the highest weight before moving out of the weight diagram, and — in the naming scheme (iii) — we call the representation (1).

We are now ready to generalize this to general Lie algebras. Either of the methods (ii) and (iii) are used. Method (ii) is straightforward, but somewhat problematic as there could be several different representations with the same dimension. We give an example of such a situation later.

In scheme (iii), the notation used for a specific state is unambiguous. To use it, we must order the simple roots in a well-defined (even if arbitrary) order. Then we have a unique highest weight. We denote a representation of a rank-k algebra as a k-tuple, such that the first entry is the maximal number of times that we can apply the first simple root on the highest weight before the state is annihilated, the second entry refers to the maximal number of times that we can apply the second simple root on the highest weight before annihilation, and so on. Take again $SU(3)$ as an example. We order the Cartan subalgebra as X_3, X_8 and the two simple roots as

$$S_1 = \left(+\frac{1}{2}, +\frac{\sqrt{3}}{2}\right), \qquad S_2 = \left(+\frac{1}{2}, -\frac{\sqrt{3}}{2}\right). \tag{1.35}$$

Consider the fundamental representation where we chose the highest weight to be $\left(1/2, 1/(2\sqrt{3})\right)$. Subtracting S_1 twice or subtracting S_2 once from the highest weight would annihilate it. Thus the fundamental representation is denoted by $(1, 0)$. You can work out the case of the adjoint representation and find that it should be denoted as $(1, 1)$. In fact, it can be shown that any pair of non-negative integers forms a different irrep. (For $SU(2)$ with the naming scheme (iii), any non-negative integer defines a different irrep.)

From now on we limit our discussion mostly to $SU(N)$.

Statement: For any group the singlet irrep is $(0, 0, ..., 0)$.

Statement: For any $SU(N)$ algebra, the fundamental representation is $(1, 0, 0, ..., 0)$.

Statement: For any $SU(N \geq 3)$ algebra, the adjoint representation is $(1, 0, 0, ..., 1)$.

Definition: For any $SU(N)$, the *conjugate representation* is the one where the order of the k-tuple is reversed.

For example, $(0, 1)$ is the conjugate of the fundamental representation, which is usually called the anti-fundamental representation. An irrep and its conjugate have the same dimension. In the naming scheme (ii), they are called m and $\bar{m}$. Note that some representations are self–conjugate, *e.g.*, the adjoint representation, such representation are also called real representations. Representations that are not self conjugate are also called complex representations. Also note that all irreps of $SU(2)$ are real.

We now return to the notion that the groups that we are dealing with are transformation groups of physical states. These physical states are often just particles. For example, when we talk about the $SU(2)$ group that is related to the spin transformations, the physical system that is being transformed is often that of a single particle with well-defined spin. In this context we often abuse the language by saying that the particle is, for example, in the spin-1/2 representation of $SU(2)$. What we mean is that, as a state in the Hilbert space, it transforms by the spin operator in the 1/2 representation of $SU(2)$. Similarly, when we say that the proton and the neutron form a doublet of isospin-$SU(2)$, we mean that we represent p by the vector-state $(1, 0)^T$ and n by the vector-state $(0, 1)^T$, so that the appropriate representation of the isospin generators is the 2×2 Pauli matrices. In other words, we loosely speak of "particles in a representation" when we mean "the representation of the group generators acting on the vector states that describe these particles."

How many particles there are in a given irrep? Here, again, we consider only $SU(N)$ and state the results, and we use naming convention (iii).

- Consider an (α) representation of $SU(2)$. It has

$$N = \alpha + 1, \tag{1.36}$$

particles. The singlet (0), fundamental (1) and adjoint (2) representations have, respectively, 1, 2, and 3 particles.

- Consider an (α, β) representation of $SU(3)$. It has

$$N = (\alpha + 1)(\beta + 1)\,\frac{\alpha + \beta + 2}{2} \tag{1.37}$$

particles. The singlet $(0, 0)$, fundamental $(1, 0)$ and adjoint $(1, 1)$ representations have, respectively, 1, 3, and 8 particles.

- Consider an (α, β, γ) representation of $SU(4)$. It has

$$N = (\alpha + 1)(\beta + 1)(\gamma + 1)\,\frac{\alpha + \beta + 2}{2}\,\frac{\beta + \gamma + 2}{2}\,\frac{\alpha + \beta + \gamma + 3}{3} \tag{1.38}$$

particles. The singlet $(0, 0, 0)$, fundamental $(1, 0, 0)$ and adjoint $(1, 0, 1)$ representations have, respectively, 1, 4, and 15 particles. Note that there is no $\alpha + \gamma + 2$ factor. Only a consecutive sequence of the label integers appears in any factor.

- The generalization to any $SU(N)$ is straightforward. It is easy to see that the fundamental of $SU(N)$ has N particles and the adjoint has $N^2 - 1$ particles.

In $SU(2)$, the number of particles in a representation is unique. In a general Lie group, however, the case may be different. Yet, it is often used to identify irreps. For example, in $SU(3)$ we usually call the fundamental 3, and the adjoint 8. For the anti-fundamental we use $\bar{3}$. In cases where there are several irreps with the same number of particles we often use a prime to distinguish them. For example, both $(4, 0)$ and $(2, 1)$ contain 15 particles. We denote them by 15 and $15'$ respectively.

Last, we remark on how the count goes in terms of subgroup irreps. For example, any $SU(3)$ irreps can be written in terms of their $SU(2)$ subgroup ireps. Two useful decompositions are $3 = 2 + 1$ and $8 = 3 + 2 + 2 + 1$, where the irreps on the left are those of $SU(3)$ and on the right of $SU(2)$. You can decude them from the weight diagrams of the $SU(3)$ irreps simply by moving only in one of the $SU(2)$ subgroup directions on the diagrams, for example, only moving in the X_3 direction. While we do not elaborate further here, we only mention that there is a way to decompose any irrep of a bigger group as a sum of irreps of its subgroups.

1.8 Combining representations

When we study spin, we learn how to combine $SU(2)$ representations. The canonical example is to combine two spin-1/2 to generate a singlet (spin-0) and a triplet (spin-1). We often write it as $1/2 \times 1/2 = 0 + 1$. There is a similar method to combine representations for any Lie group. The basic idea is, just like in $SU(2)$, that we need to find all the possible ways to combine the indices and then assign them to the various irreps. That way we know what irreps are in the product representation and the corresponding Clebsch-Gordan (GB) coefficients.

Here, however, we do not explain how to construct the product representation. The reason is that often all we want to know is what irreps appear in the product representation, without the need to get all the CG coefficients. (In particular, many times all we care about is how to generate the singlet.) There is a simple way to do just this for a general $SU(N)$. This method is called *Young Tableaux*, or Young Diagrams. The details of the method are well explained in several places, for example, in the PDG. In the homework you are asked to learn how to use it.

Here we give few examples for combining irreps in $SU(3)$ that we will use when we discuss the SM. Using naming scheme (ii) we have

$$3 \times \bar{3} = 1 + 8, \qquad 3 \times 3 = \bar{3} + 6, \qquad 3 \times 6 = 10 + 8. \tag{1.39}$$

From this we can conclude that

$$3 \times 3 \times 3 = 10 + 8 + 8 + 1. \tag{1.40}$$

Note that the number of particles on both sides are equal, as they should. Of particular interest to us is that $3 \times \bar{3}$ and $3 \times 3 \times 3$ contain the singlet irrep.

When we combine identical irreps, the results have well defined symmetry properties under exchange of the two irreps, that is, they are even or odd under an exchange. For example, when combining two spin halves, the singlet is odd while the triplet is even under the exchange. For $SU(2)$ the rule is simple, the highest irrep is symmetric and then going down it alternates. For other groups it is more complicated and we do not discuss it here. When it is important we may add a subscript to denote the symmetry property. For example, in $SU(2)$ we may write $2 \times 2 = 1_a + 3_s$ and $3 \times 3 = 5_s + 3_a + 1_s$. In the example above for $SU(3)$ we may write

$$3 \times 3 = \bar{3}_a + 6_s, \qquad 3 \times 3 \times 3 = 10_s + 8_m + 8_m + 1_a, \tag{1.41}$$

where 8_m refer to a mixed symmetry.

The symmetry properties are important when the two irreps that we are combining are identical. For example, the antisymmetric combination of two vectors in real three dimensional space is the cross product. Being antisymmetric we know it vanishes identically for two identical vectors, $\vec{a} \times \vec{a} = 0$. This result is a general one, and applies to all fully antisymmetric combinations in any group.

Combining representations plays a very important role in physics. In particular, we would like to find a combination of representations that is combined into the singlet one, as such a combination is invariant under rotation in the relevant space. For that, we mention the following properties:

Statement: The combination of an irrep with its conjugates contains the singlet, $R \times \bar{R} \ni 1$.

Statement: Consider two irreps that are not the conjugate of each other, that is $R_1 \neq \bar{R}_2$. The combination of them does not contains the singlet, $R_1 \times R_2 \not\ni 1$.

Homework

Question 1.1: S_3

In this question we study the group S_3. It is the smallest finite non-Abelian group. You can think about it as all possible permutation of three elements. The group has 6 elements. Thinking about the permutations we see that we get the following representation of the group:

$$
\begin{aligned}
() &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad & (12) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(13) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad & (23) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
(123) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad & (321) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\end{aligned}
\tag{1.42}
$$

The names are instructive. For example, (12) represents exchanging the first and second elements. (123) and (321) are cyclic permutation to the right or left.

1. Write explicitly the 6×6 multiplication table for the group.

2. Show that the group is non-Abelian. Hint, it is enough to find one example.

3. Z_3 is a sub group of S_3. Find three generators that correspond to a Z_3.

4. In Eq. (1.6 we mentioned the following theorem for finite groups

$$\sum_i \left[\dim(R_i)\right]^2 = N, \tag{1.43}$$

   where N is the number of elements in the group and R_i runs over all the irreps. Based on this, prove that the representation in Eq. (1.42) is reducible. Then, write it explicitly in a $(1+2)$ block diagonal representation. (Hint: find a vector which is an eigenvector of all the above matrices.)

5. In the last item you found a two dimensional and a one dimensional representations of S_3. Based on Eq. (1.43) you know that there is only one more representation and that it is one dimensional. Find it.

Question 1.2: Lie algebras

Consider two general elements of a Lie groups,

$$A \equiv \exp(i\lambda X_a), \qquad B \equiv \exp(i\lambda X_b). \tag{1.44}$$

where X_i is a generator. We think about λ as a small parameter. Then, consider a third element

$$C = BAB^{-1}A^{-1} \equiv \exp(i\beta_c X_c). \tag{1.45}$$

Expand C in powers of λ and show that at lowest order you get the Lie algebra

$$[X_a, X_b] = i f_{abc} X_c, \qquad f_{abc} \equiv \frac{\beta_c}{\lambda^2}. \tag{1.46}$$

Question 1.3: The adjoint irrep

1. We mentioned that the structure constants are a irrep of their group. To see that explicitly we define
$$(X_c)_{ab} \equiv -i f_{abc}. \tag{1.47}$$
Show that
$$[X_a, X_b] = i f_{abc} X_c. \tag{1.48}$$
2. Show explicitly that you obtain the spin one irrep of $SU(2)$ for the case where $f_{abc} = \epsilon_{abc}$.

Question 1.4: $SU(3)$

1. Draw the weight diagram of the fundamental irrep of $SU(3)$
2. Draw the root diagram of $SU(3)$
3. The three Gell–Mann matrices, $a\lambda_1$, $a\lambda_2$ and $a\lambda_3$ satisfy an SU(2) algebra, also known as Isospin, where a is a constant. What is a?
4. Does this fact mean that $SU(3)$ is not a simple Lie group?

5. There are other independent combinations of Gell–Mann matrices that satisfy $SU(2)$ algebras. Here we discuss two of them, called U-spin and V-spin. Their raising operators are given by

$$V_+ = \frac{1}{2}(\lambda_4 + i\lambda_5)\,, \qquad U_+ = \frac{1}{2}(\lambda_6 + i\lambda_7). \tag{1.49}$$

Find these algebras. Hint: Look at the root diagram.

Question 1.5: $SO(4) \sim SU(2) \times SU(2)$ and more

In this question you show that the symmetry of a $-\alpha/r$ potential is in fact similar to a rotation in $4d$ and not just the real $3d$. This extra symmetry is the reason that the Kepler trajectories are close and that to leading order the hydrogen energy levels depend only on n and not also on ℓ. We then see how this is connected to the Lorentz group.

1. Show that in a Coulomb potential, the so-called Runge-Lenz $\vec{A}$ vector is conserved in classical mechanics, where

$$\vec{A} = \frac{m\alpha\vec{r}}{r} - \vec{p} \times \vec{L}. \tag{1.50}$$

2. The quantum mechanical extension of the Runge-Lenz vector is

$$\vec{A} = \frac{m\alpha\vec{r}}{r} - \frac{1}{2}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}). \tag{1.51}$$

Show that this commutes with the Hamiltonian

$$H = \frac{p^2}{2m} - \frac{\alpha}{r}. \tag{1.52}$$

3. Derive the commutation relations between A_i and L_k:

$$[L_i, A_j] = i\epsilon_{ijk}A_k, \qquad [A_i, A_j] = -iH\epsilon_{ijk}L_k\ . \tag{1.53}$$

4. Consider a particular energy eigenstate at energy level E (since this is a bound state $E < 0$), and instead of A consider the rescaled operator $u_i = A_i/\sqrt{-2E}$. Show that the commutation relations are now

$$[L_i, u_j] = i\epsilon_{ijk}u_k, \qquad [u_i, u_j] = 2i\epsilon_{ijk}L_k\,. \tag{1.54}$$

The interpretation of these relations together with the usual $[L_i, L_j]$ commutators is that theoperators $\vec{L}, \vec{u}$ are the generators of rotations of a 4-dimensional Euclidean space x, y, z, u, where $L_{x,y,z}$ are rotations in the usual yz, xz, yx planes, while $u_{x,y,z}$ are rotations in the xu, yu, zu planes. This is the additional hidden symmetry of the Coulomb problem.

5. Instead of the $\vec{L}, \vec{u}$ operators let us introduce the operators $\vec{j}_{1,2}$ with

$$\vec{j}_1 = \frac{1}{2}(\vec{L} + \vec{u}), \qquad \vec{j}_2 = \frac{1}{2}(\vec{L} - \vec{u})\,. \tag{1.55}$$

Show, that these now satisfy two independent angular momentum algebras

$$[j_{1i}, j_{1j}] = i\epsilon_{ijk}j_{1k}, \qquad [j_{2i}, j_{2j}] = i\epsilon_{ijk}j_{2k}, \qquad [j_{1i}, j_{2j}] = 0\,. \tag{1.56}$$

6. Since you have two independent angular momenta, introduce the usual angular momentum representations $j_1^2 = j_1(j_1+1), j_2^2 = j_2(j_2+1)$, where $j_{1,2}$ can be half-integers. Show using a straight calculation using the original definition of $\vec{u}$ that

$$\vec{u} \cdot \vec{L} = \vec{L} \cdot \vec{u} = 0, \qquad L^2 + u^2 = -1 - \frac{1}{2E}\,. \tag{1.57}$$

Argue that this implies $j_1 = j_2 \equiv j$, and

$$j(j+1) = -\frac{1}{4}\left(1 + \frac{1}{2E}\right)\,. \tag{1.58}$$

7. Introducing the notation $(2j+1) = n, n = 1, 2, 3, ...$ find the expression of the energy levels of the hydrogen atom.

8. Using the properties of the angular momentum representations $j_{1,2}$ calculate the degeneracy level of the energy level n. What are the possible values of the physical angular momentum? Compare these answers to those obtained while solving the hydrogen atom the traditional way.

9. We now relate it all to the Lorentz group. The Lie algebra of the SO(3,1) is given by

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\left(g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma}\right). \tag{1.59}$$

where we defined the "4-dimensional" angular momentum as

$$L_{\mu\nu} \equiv i(x_\mu\partial_\nu - x_\nu\partial_\mu). \tag{1.60}$$

They related to rotations (L_i) and boosts (K_i) as

$$L_i = \frac{1}{2}\epsilon_{ijk}M_{jk}, \qquad K_i = M_{0i} \tag{1.61}$$

such that

$$[L_i, L_j] = i\epsilon ijkL_k, \qquad [K_i, K_j] = -2i\epsilon ijkL_k, \qquad [L_i, K_j] = i\epsilon ijkK_k. \tag{1.62}$$

Defining $u_i = iK_i$ show that you recover Eq. (1.54). That show that SO(3,1) localy is like $SU(2) \times SU(2)$.

Question 1.6: Dynkin diagrams

1. Draw the Dynkin diagram of SO(10).

2. What is the rank of SO(10)?

3. How many generators there are for SO(10)? (We did not proof a general formula for the number of generators for SO(N). It should be simple for you to find such a formula using your understanding of rotations in real N-dimensional spaces.)

4. Based on the Dynkin diagram show that SO(10) has the following subalgebras

$$\mathrm{SO}(8), \qquad \mathrm{SU}(5), \qquad \mathrm{SU}(4) \times \mathrm{SU}(2), \qquad \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2). \tag{1.63}$$

   In each case show which simple root you can remove from the SO(10) Dynkin diagram.

Question 1.7: Representations

Here we practice finding the number of degrees of freedom in a given irrep.

1. In SU(5), how many particles there are in the $(1, 1, 0, 0)$ irrep?

2. In SU(3) how many particles there are in the following irreps

$$(3, 0), \quad (2, 2). \tag{1.64}$$

3. Consider the $(3, 0)$ irrep of $SU(3)$. Draw its weight diagram and from it decompose it into its $SU(2)$ irreps.

Question 1.8: Looking for singlets

1. Conisder an irrep, R. Using the statments that were made in the text, show that $R \times R \ni 1$ if and only if R is a real irrep.

2. All the irreps of $SU(2)$ are real. Show the above result explicitly in $SU(2)$.

3. Cosnider R to be an adjoint. Is $R \times R \ni 1$?

Question 1.9: Combining irreps

Here we are going to practice the use of Young Tableaux. The details of the method can be found in the PDG (there is a link in the website of the course). Study the algorithm and do the following calculations. Make sure you check that the number of particles on both sides is the same. Write your answer both in the k-tuple notation and the number notation. For example, in SU(3) you should write

$$(1,0) \times (0,1) = (0,0) + (1,1), \qquad 3 \times \bar{3} = 1 + 8. \tag{1.65}$$

1. In SU(3) calculate

$$3 \times 3, \qquad 3 \times 8, \qquad \overline{10} \times 8. \tag{1.66}$$

2. As we discuss in the book, the strong interaction is described by an $SU(3)$ gauge group. Observable bound states must be a color singlet. The quarks, q, are $SU(3)$ triplets, anti-quarks $\bar{q}$, are in anti-triplets and the gluons, g are color octets. Given that assignment, which of the following could be an observable bound state?

$$q\bar{q}, \qquad qq, \qquad qg, \qquad gg, \qquad q\bar{q}g, \qquad qqq. \tag{1.67}$$

3. Find what is $\bar{5}$ and 10 in SU(5) in a k-tuple notation.

4. Calculate 10×10 in SU(5).