# Introduction to QFT

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# General remarks

- I have to make assumptions about what you know
- Please ask questions!
- Email: yg73@cornell.edu
- The plan:
  - Intro to QFT
  - Intro to the SM
  - Flavor (quarks and leptons)

# What is HEP?

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**Find the basic laws of Nature**

More formally

$$\mathcal{L} = ?$$

- We have quite a good answer
- It is very elegant, it is based on axioms and symmetries
- The generalized coordinates are fields
- We use particles to answer this question

# What is mechanics?

- Answer the question: what is $x(t)$?
- A system can have many DOFs, and then we seek to find $\vec{x}(t) \equiv x_1(t)$, $x_2(t)$,...
- Once we know $\vec{x}(t)$ we know any observable
- Solving for $q_1 \equiv x_1 + x_2$ and $q_2 \equiv x_1 - x_2$ is the same as solving for $x_1$ and $x_2$
- The idea of generalized coordinates is very important

How do we solve mechanics?

# How do we find $x(t)$?

- $x(t)$ minimizes the action, $S$. This is an axiom
- There is one action for the whole system

$$S = \int_{t_1}^{t_2} L(x, \dot{x}) dt$$

- The solution is given by the E-L equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

- Once we know $L$ we can find $x(t)$ up to initial conditions
- Mechanics is reduced to the question “what is $L$?”

# An example: Newtonian mechanics

We assume a particle with one DOF and

$$L = \frac{mv^2}{2} - V(x)$$

- We use the E-L equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \qquad L = \frac{mv^2}{2} - V(x)$$

- The solution is $-V'(x) = m\dot{v}$, aka $F = ma$
- Here $F = ma$ is the output, not the starting point!
- Input is $L$. We need to measure the parameters
- So how do we find what is $L$?

# What is L?

$L$ is the most general one that is invariant under some symmetries

- We (again!) rephrase the question. Now we ask what are the symmetries of the system that lead to $L$
- What are the symmetries in Newtonian mechanics?

# What is field theory

# What is a field?

- In math: something that has a value in each point. We can denote it as $\phi(x, t)$
  - Temperature (scalar field)
  - Wind (vector field)
  - Mechanical string (?)
  - The density of people (?)
  - Electric and magnetic fields (vector fields)
- How good is the field description of each of these?
- In physics, fields used to be associated with sources, but now we know that fields are fundamental

# A familiar example: the EM field

- Maxwall Eqs. leads to a wave equations

$$\frac{\partial^2 E(x,t)}{\partial t^2} = c^2 \frac{\partial^2 E(x,t)}{\partial x^2}$$

- The solution is ($A$ and $\varphi_0$ depend on IC)

$$E(x,t) = A\cos(\omega t - kx + \varphi_0), \qquad \omega = ck$$

- Some important implications of the result
  - Each mode has its own amplitude, $A(\omega)$
  - The energy in each $\omega$ is conserved
  - The superposition principle
- Are the statements above exact?

# How to deal with generic field theories

- $\phi(x,t)$ has an infinite number of DOF. It can be an approximation for many (but finite) DOF
- To solve mechanics of fields we need to find $\phi(x,t)$
- Here $\phi$ is the generalized coordinate, while $x$ and $t$ are treated the same (nice!)
- In relativity, $x$ and $t$ are also treated the same
- What is better $x_\mu$ or $t_\mu$?

# Solving field theory

Generalization of mechanics to systems with few "times"

- We still need to minimize $S$

$$S = \int \mathcal{L}\, dx\, dt \qquad \mathcal{L}[\phi(x,t), \dot{\phi}(x,t), \phi'(x,t)]$$

- We usually require Lorentz invariant (and use $c = 1$)

$$S = \int \mathcal{L}\, d^4x \qquad \mathcal{L}[\phi(x,t), \partial_\mu \phi(x_\mu)]$$

# E-L for field theory

- We also have an E-L equation for field theories

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) = \frac{\partial L}{\partial \phi}$$

- In relativistic notation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

- We have a way to solve field theory, just like mechanics. Give me $\mathcal{L}$ and I can know everything!
- Just like in Newtonian mechanics, we want to get $\mathcal{L}$ from symmetries!

# Example: a free field theory

- A free particle $L$ has just a kinetic term
- A free field: The "kinetic term" is promoted

$$T \propto \left(\frac{dx}{dt}\right)^2 \Rightarrow T \propto \left(\frac{d\phi}{dt}\right)^2 - \left(\frac{d\phi}{dx}\right)^2 \equiv (\partial_\mu \phi)^2$$

- Free particles, and thus free fields, only have kinetic terms

$$\mathcal{L} = (\partial_\mu \phi)^2 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2}$$

- An $\mathcal{L}$ of a free field gives a wave equation
- As in Newtonian mechanics, what used to be the starting point, here is the final result

# Harmonic oscillator

# The harmonic oscillator

Why do we care so much about harmonic oscillators?

- Because we really care about springs?
- Because we really care about pendulums?

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- Because we really care about springs?
- Because we really care about pendulums?

Because almost any function around its minimum can be approximated as a harmonic function!

- Indeed, we usually expand the potential around one of its minima
- We identify a small parameter, and keep only a few terms in a Taylor expansion

# Classic harmonic oscillator

$$V = \frac{kx^2}{2}$$

We solve the E-L equation and get

$$x(t) = A\cos(\omega t) \qquad \omega^2 = \frac{k}{m}$$

- The period does not depend on the amplitude
- Energy is conserved

Which of the above two statements is a result of the approximation of keeping only the harmonic term in the expansion?

# Coupled oscillators

A clip of coupled oscillators

# Coupled oscillators

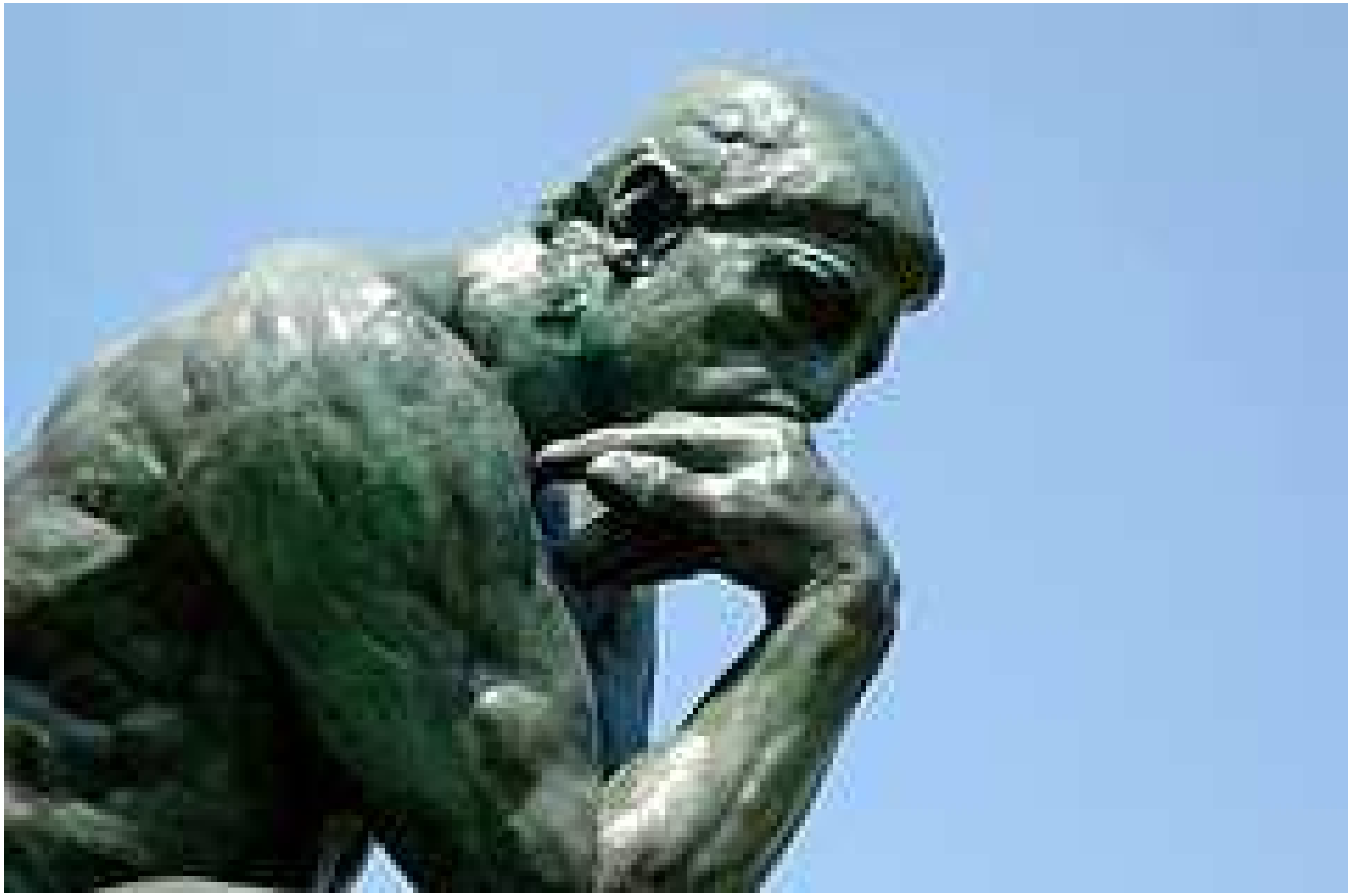
- There are normal modes
- The normal modes are not "local" as in the case of one oscillator
- The energy of each mode is conserved
- This is an approximation!
- Once we keep non-harmonic terms energy moves between modes

$$V(x,y) = \frac{k_1 x^2}{2} + \frac{k_2 y^2}{2} + \alpha x^2 y$$

- What determines the rate of energy transfer?

# Things to think about

- Relations between harmonic oscillators and free fields



# The quantum SHO

# What is QM?

- Many ways to formulate QM
- For example, we promote $x \to \hat{x}$
- We solve QM when we know the wave function $\psi(x, t)$
- How many wave functions describe a system?
- The wave function is mathematically a field

# The quantum SHO

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \qquad E_n = (n + 1/2)\hbar\omega$$

- We also like to use

$$H = (a^\dagger a + 1/2)\hbar\omega \qquad a, a^\dagger \sim x \pm ip \qquad x \sim a + a^\dagger$$

- We call $a^\dagger$ and $a$ creation and annihilation operators

$$E = a|n\rangle \propto |n-1\rangle \qquad a^\dagger|n\rangle \propto |n+1\rangle$$

- So far this is abstract. What can we do with it?

# Couple oscillators

Consider a system with 2 DOFs and same mass with

$$V(x,y) = \frac{kx^2}{2} + \frac{ky^2}{2} + \alpha xy$$

The normal modes are

$$q_\pm = \frac{1}{\sqrt{2}}(x \pm y) \qquad \omega_\pm^2 = \frac{k \pm \alpha}{m}$$

What is the QM spectrum of this system?

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What is the QM spectrum of this system?

$$(n_+ + 1/2)\hbar\omega_+ + (n_- + 1/2)\hbar\omega_- \qquad |n_+, n_-\rangle$$

# Couple oscillators and Fields

- With many DOFs, $a \to a_i \to a(k)$
- And the states

$$|n\rangle \to |n_i\rangle \to |n(k)\rangle$$

- And the energy

$$(n+1/2)\hbar\omega \to \sum(n_i+1/2)\hbar\omega_i \to \int[n(k)+1/2]\hbar\omega(k)dk$$

- Just like in mechanics, we expand around the minimum of the fields, and to leading order we have SHOs
- In QFT fields are operators while $x$ and $t$ are not

# SHO and photons

I have two questions:

- What is the energy that it takes to excite an harmonic oscillator by one level?
- What is the energy of the photon?

# SHO and photons

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- What is the energy that it takes to excite an harmonic oscillator by one level?
- What is the energy of the photon?

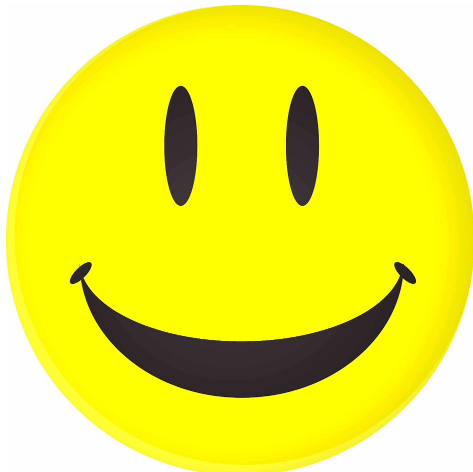
Same answer

$$\hbar\omega$$

- Why is the answer to both question the same? Can we learn anything from it?

# What is a particle?

Excitations of SHOs are particles



# More on QFT

# What about masses?

- A “free” Lagrangian gives massless particle

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 \Rightarrow \omega = k \quad (\text{or } E = P)$$

- We can add “potential” terms (without derivatives)

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2}m^2\phi^2$$

- Here $m$ is the mass of the particle. Still free particle
- (HW) Show that $m$ is a mass of the particle by showing that $\omega^2 = k^2 + m^2$. To do it, use the E-L Eq. and “guess” a solution of the form $\phi = e^{i(kx-\omega t)}$.

# What about other terms?

- How do we choose what terms to add to $\mathcal{L}$?
- Must be invariant under the symmetries
- We keep some leading terms (usually, up to $\phi^4$)
- Lets add $\lambda\phi^4$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4}\lambda\phi^4$$

- We get the non-linear wave equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = m^2 \phi + \lambda \phi^3$$

- We do not know how to solve it

# What about fermions?

- We saw how massless and massive bosons are related to SHO
- The spin of the particle is related to the polarization of the classical field
  - Scalar field: spin zero particles
  - vector field: spin one particles
- Can we construct a fermionic harmonic oscillator?

$$[a, a^\dagger] = 1 \quad \to \quad \{b, b^\dagger\} = 1$$

- No classical analog since $b^2 = 0$
- We can then think of fermionic fields. They can generate only one particle in a given state

# A short summary

- Particles are excitations of fields
- The fundamental Lagrangian is giving in terms of fields
- Our aim is to find $\mathcal{L}$
- We can only solve the linear case, that is, the equivalent of the SHO
- What can we do with higher order terms?

# Perturbation theory

# Perturbation theory

$$H = H_0 + H_1 \qquad H_1 \ll H_0$$

- In many cases, perturbation theory is a mathematical tool
- There are cases, however, that PT is a better way to describe the physics
- Many times we prefer to work with EV of $H$ (why?)
- Yet, at times it is better to work with EV of $H_0$ (why?)

# 1st and 2nd order PT

$$H = H_0 + H_1 \qquad H_1 \ll H_0$$

- In first order we care only about the states with the same energy

$$\langle f|H_1|i\rangle \qquad E_f = E_i$$

- 2nd order pertubation theory probe the whole spectrum

$$\sum_n \frac{\langle f|H_1|n\rangle\langle n|H_1|i\rangle}{E_n - E_f} \qquad E_f = E_i \qquad E_n \neq E_i$$

# PT for 2 SHOs

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y$$

- We assume that $\alpha$ is small
- Classically $\alpha$ moves energy between the two modes
- How it goes in QM?
- Recall the Fermi golden rule

$$P \propto |\mathcal{A}|^2 \times \text{P.S.} \qquad \mathcal{A} \sim \langle f|\alpha x^2 y|i\rangle$$

- The relevant thing to calculate is the transition amplitude, $\mathcal{A}$.