Elliptic non-abelian DT invariants of $\mathbb{C}^3$

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ma per seguir virtute e canoscenza

SISSA

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Partition function of D1/D7 brane system

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- Free field realization of Z_N.

Brane Construction

- Setup (type IIB string):

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
D1	—	—	•	•	•	•	•	•	•	•
D7	—	—	—	—	—	—	—	—	•	•

$\underbrace{x_1 \; x_2}_{T^2} \quad \underbrace{x_3 \; x_4 \; x_5 \; x_6 \; x_7 \; x_8}_{\mathbb{C}^3}$

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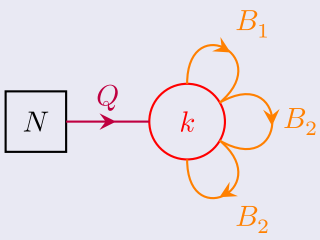

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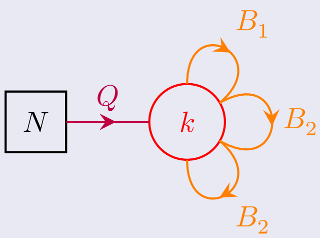

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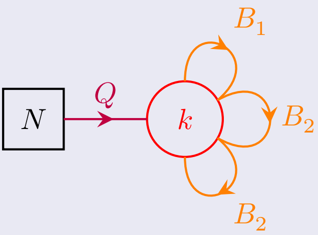

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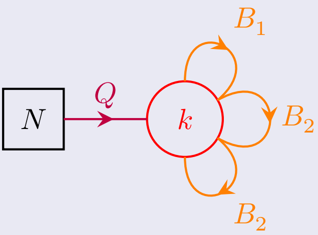

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Interactions

$$W = \operatorname{tr} B_1[B_2, B_3]$$

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Study the Elliptic Genus of this moduli space

Elliptic Genus

The quantity we want to compute is:

$$Z_{N,k}(\vec{\xi},\vec{\epsilon}) = \mathrm{tr}_{\mathrm{RR}}(-1)^F e^{-2\pi(\tau_2 \mathsf{H}+\tau_1 \mathsf{P})} e^{2\pi \mathrm{i} \xi_\alpha \mathsf{F}_\alpha} e^{2\pi \mathrm{i} \epsilon_i \mathsf{R}_i} \ ,$$

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Effect in Path Integral computation

Turn on a flat background gauge field $A^{(\mathsf{F})}$

$$\xi_\alpha = \int_t A_\alpha^{(\mathsf{F})} - \tau \int_s A_\alpha^{(\mathsf{F})}$$

Supersymmetric Localization

Path Integral $\Rightarrow$ Finite Dimensional Integral

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Path Integral ⇒ Finite Dimensional Integral

Given the quantum numbers of various multiplets

	Q	B_1	B_2	B_3	fugacity
$\mathrm{U}(k)$	$\mathbf{k}$	$\mathbf{Adj}$	$\mathbf{Adj}$	$\mathbf{Adj}$	$e^{2\pi \mathrm{i} u_l}$
$\mathrm{SU}(N)$	$\mathbf{N}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$e^{2\pi \mathrm{i} \xi_\alpha}$
$\mathrm{U}(1)_1$	0	1	0	0	$e^{2\pi \mathrm{i} \epsilon_1}$
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there is a recipe [Benini-Eager-Hori-Tachikawa] to get

$$Z_{N,k}(\vec{\xi},\vec{\epsilon}) = \frac{1}{k!}\int_{\mathrm{JK-contour}} \mathrm{d}u_1 \ldots \mathrm{d}u_k \text{Rational function of Jacobi } \theta_1(\tau|\text{fugacities})$$

Computation of $Z_{N,k}$

Anomalies

- $\theta_1(\tau|u+a+b\tau) = (-1)^{a+b} e^{-2\pi \mathrm{i} b u} e^{-\mathrm{i}\pi b^2 \tau} \theta_1(\tau|u)$ for $a, b \in \mathbb{Z}$;

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Pole of the integrand JK allowed $\Leftrightarrow$ N Colored Plane Partition of k

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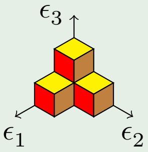

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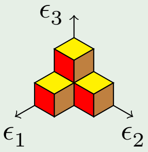

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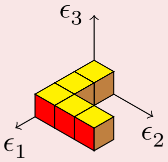

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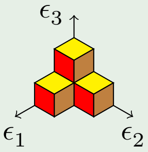

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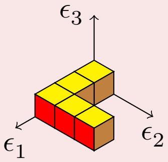

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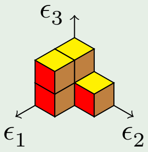

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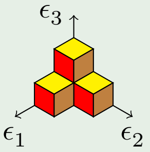

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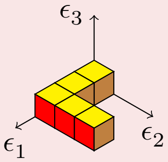

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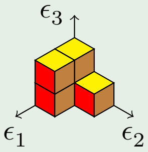

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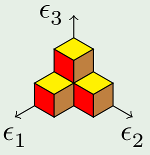

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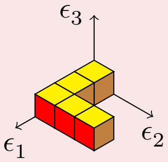

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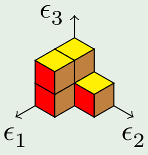

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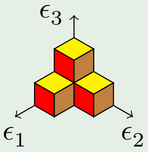

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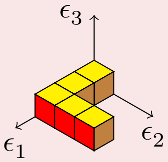

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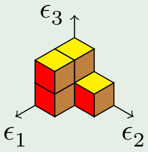

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- The generating function of CPP is $\Phi(v)^N$.

Result for $Z_{N,k}$

We have for $\epsilon = \frac{n}{N}$ with $n \in \mathbb{Z}$ that

$$Z_{N,k}(\vec{\xi}, \vec{\epsilon}) = \sum \text{Residues at pole corresponding to } N \text{ CPP of } k$$
$$= \begin{cases} (-1)^{nk} \Phi^{(\gcd(n,N))}_{\frac{k}{N}\gcd(n,N)} & \text{if } \frac{N}{\gcd(n,N)} | k \\ 0 & \text{otherwise} \end{cases}$$

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Comments

- The result is a number!

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$$Z_{N,k}(\vec{\xi}, \vec{\epsilon}) = \sum \text{Residues at pole corresponding to } N \text{ CPP of } k$$
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- Interpretation on terms of F theory.

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- The result can be written in a compact form resumming it as before $\widetilde{Z}_N(\vec{\xi},\vec{\epsilon};v) = \sum_{k=0}^{\infty} \widetilde{Z}_{N,k}(\vec{\xi},\vec{\epsilon})v^k$, it is

$$\widetilde{Z}_N(\vec{\epsilon};v) = \mathtt{PE}_{v,\vec{p}}\left[-\frac{(1-p_1p_2)(1-p_1p_3)(1-p_2p_3)}{(1-p_1)(1-p_2)(1-p_3)}\times \right.$$
$$\left. \times\, p^{-\frac{N}{2}}\frac{1-p^N}{1-p}\frac{v}{(1-vp^{-\frac{N}{2}})(1-vp^{\frac{N}{2}})}\right]$$

($p_i \equiv e^{2\pi \mathrm{i}\epsilon_u}$ and $p \equiv e^{2\pi\mathrm{i}\epsilon}$).

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- Computation of 11d-SUGRA index in the Ω-background gives exactly the argument of $\mathtt{PE}$.

Further reduction: D(-1)/D5

Study the equivariant volume $\overline{Z}_{N,k}$ SUSY Matrix Model

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- This is an example of *trivial factorization* [Nekrasov].

Free field realization of Z_N

The propagator of free a free boson on torus is

$$\langle \phi(u,\bar{u})\phi(w,\bar{w})\rangle_{T^2} = \log G(u,\bar{u};w,\bar{w})\ ,$$

where

$$G(u,\bar{u};w,\bar{w}) = e^{-\frac{2\pi}{\tau_2}[\Im(u-w)]^2} \left|\frac{\theta_1(\tau|u-w)}{2\pi\eta^3(\tau)}\right|^2 \ .$$

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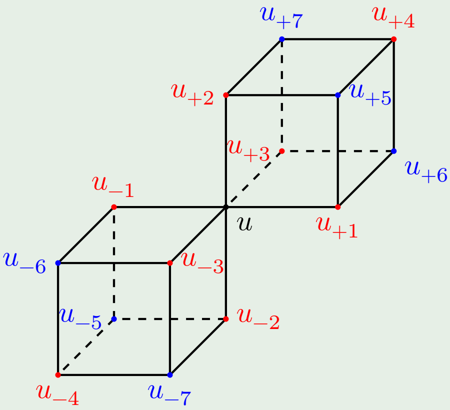
- It contains Jacobi θ_1 function;
- Differently from the planar case, it is *not* a sum of a holomorphic and anti-holomorphic part because of the $\Im^2$.

Multi-local Vertex Operator

Consider the following vertex operator

$$\mathcal{V}_{\vec{\epsilon}}(u) = \prod_{i=1}^{7} :e^{\lambda_i \phi_i(u_{+i})} :: e^{-\lambda_i \phi_i(u_{-i})}: ,$$



with

$$u_{\pm 1} = u \pm \frac{\epsilon_1}{2}, \qquad u_{\pm 2} = u \pm \frac{\epsilon_2}{2},$$
$$u_{\pm 3} = u \pm \frac{\epsilon_3}{2}, \qquad u_{\pm 4} = u \pm \frac{\epsilon}{2},$$
$$u_{\pm 5} = u \pm \frac{\epsilon_1+\epsilon_2}{2}, \quad u_{\pm 6} = u \pm \frac{\epsilon_1+\epsilon_3}{2},$$
$$u_{\pm 7} = u \pm \frac{\epsilon_2+\epsilon_3}{2};$$

and

$$\vec{\lambda} = (1, 1, 1, 1, \mathrm{i}, \mathrm{i}, \mathrm{i}).$$

Source ("Hamiltonian") operator

Let us consider the following operator:

$$H = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \partial \phi_4(u) \omega(w) \mathrm{d}w \,,$$

where the contour Γ is around $u = 0$ and encloses all $u_{\pm i}$, and

$$\omega(w) = -\sum_{\alpha=1}^{N} \log \theta_1 \left(\tau \middle| w + \xi_\alpha - \frac{\epsilon}{2} \right)$$

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Combining these results it is possible to recover

$$Z_N(v) = \left\langle e^H e^{v \oint_{\text{JK}} \mathcal{V}_{\vec{\epsilon}}(u) \mathrm{d}u} \right\rangle_{\text{hol.}} .$$

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It can be a starting point for a generalization of *Hirota Bilinear Equations*.

Conclusion and Perspective

What we did

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- Understand the possible link between the free field realization point of view and integrable hierarchies.

Thank you for your attention!