Integrable CFTs in any-D

and non-compact spin chains

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arXiv: 1801.09844 [hep-th] (V.Kazakov and E.O.)

Conformal Quantum Field Theory

- Rotation invariance $SO(D), SO(D-1,1)$
- Scale invariance $x \to \lambda x$
- SCT: $x \to I\,(t + I\,x)$

Conformal Symmetry of the Action $\longrightarrow$ $\boldsymbol{m = 0}$ (no dimensionful parameters)

$$L = \partial_\mu \phi\, \partial^\mu \phi + g\, \phi^4$$

Quantization **breaks** scale invariance

- **No running coupling:** $\beta(g) = \frac{\partial\, g\,(\mu)}{\partial\, \mu} = 0$ $\longrightarrow$ $-\,x^\mu\, \partial_\mu\, G(x) = 2\,\big(1 + \gamma(g)\big)G(x)$

Callan-Symanzik equation ≡ ***Euler***'s formula

Conformal correlators

$$G_2(x,y) = \frac{A}{(x-y)^{2\Delta(g)}} \qquad G_3(x_1, x_2, x_3) = C(g)\ \prod_{i\neq j}(x_i - x_j)^{-2\Delta_{ij}(g)}$$

Solvability & *Integrability*

- Spectrum of *anomalous dimensions*

$$\Delta(g) = \Delta_0 + \gamma(g) \qquad \Delta_0 = [\phi(x)] \qquad \gamma(0) = 0 \qquad G_0(x,y) \propto \frac{1}{(x-y)^{2\Delta_0}}$$

- *Perturbative* or *Exact* solution

$$\gamma(g) = g\,\gamma_1 + g^2\,\gamma_2 + O(g^3) \qquad G(x,y) \propto \frac{1}{(x-y)^{2\Delta_0}}\left[1 + 2\,g\,\gamma_1 \log|x-y| + O(g^2)\right]$$

1-st order diagrams

- *Integrability:* auxiliary integrable **quantum mechanical models** allowing to obtain the set $\{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ in a simple way, relying on the symmetries of the model. [J.Minahan, K.Zarembo '02]

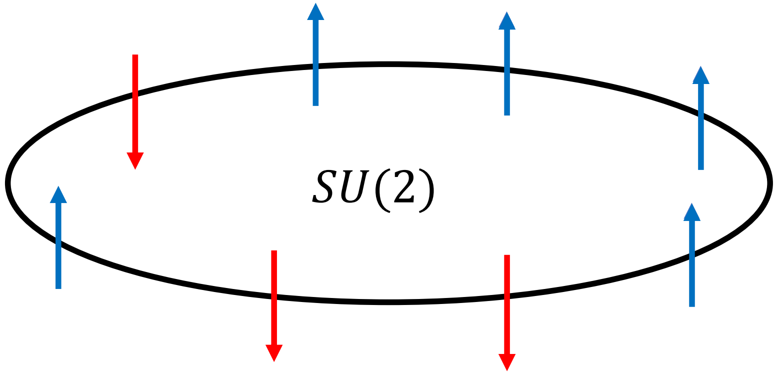
Computing Feynman graphs contributions without *really* compute graphs

Planar *Integrable CFT*

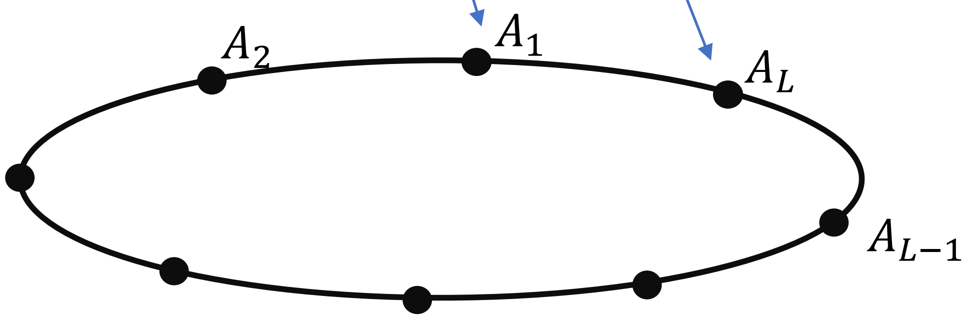
- $\mathcal{N} = 4$ SYM, $\mathcal{N} = 6$ ABJM : **Super**conformal **Gauge** Theories $SU(N_c)$

[J.Minahan, K.Zarembo '02]
[J.sssssasvdvafvafdfbafd'02]

- $SU(N_c)$ gauge group → *t'Hooft* limit: $N_c \to \infty, \;\; g \to 0, \;\; g\sqrt{N_c} = \lambda$ → planar graphs
  → Cyclic operators $Tr[A_1(x) \ldots A_L(y)]$
- Symmetry of the model $PSU(2,2|4)$



$$\frac{1}{2} \otimes \frac{1}{2} \otimes \ldots \otimes \frac{1}{2}$$



$$V_{A_1} \otimes V_{A_2} \otimes \cdots \otimes V_{A_L}$$

Computation of planar Feynmann graphs in $\langle \, Tr[\varphi(x) \ldots \psi(x)] \;\; Tr[\varphi(y) \ldots \psi(y)]^* \rangle$ is mapped to the *diagonalization* of a quantum mechanical chain of **"spins"** → *Bethe Ansatz, Baxter equations, etc.*

Non-gauge integrable CFTs

- Twisting $N = 4$ *SYM:* $SU(2,2) \times SU(4) \longrightarrow SU(2,2) \times U(1)^3$

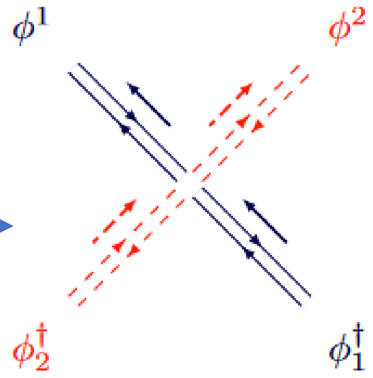
$$\mathcal{L}_{\rm int} = N_c g \,\mathrm{Tr}\Big[\frac{g}{4}\{\phi_i^\dagger, \phi^i\}\{\phi_j^\dagger, \phi^j\} - g\, e^{-i\epsilon^{ijk}\gamma_k} \phi_i^\dagger \phi_j^\dagger \phi^i \phi^j$$
$$- e^{-\frac{i}{2}\gamma_j^-} \bar\psi_j \phi^j \bar\psi_4 + e^{+\frac{i}{2}\gamma_j^-} \bar\psi_4 \phi^j \bar\psi_j + i\epsilon_{ijk} e^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+} \psi^k \phi^i \psi^j$$
$$- e^{+\frac{i}{2}\gamma_j^-} \psi_4 \phi_j^\dagger \psi_j + e^{-\frac{i}{2}\gamma_j^-} \psi_j \phi_j^\dagger \psi_4 + i\epsilon^{ijk} e^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+} \bar\psi_k \phi_i^\dagger \bar\psi_j\Big]$$

$$\mathrm{D}_\mu = \partial_\mu + i\frac{g_{\rm YM}}{\sqrt{2}}[A_\mu\,,\ \cdot\,]$$

1. Double-scaling limit: $\gamma_k \to i\infty$, $g \to 0$, $g^2\, e^{-i\gamma_k} = \xi_k^2$ $\longrightarrow$ *Gauge* fields and ψ_4 decouple
2. $\xi_1 = \xi_2 = 0$ $\longrightarrow$ *Fermionic* fields ψ_k decouple

$$\mathcal{L}_\phi = \frac{N_c}{2}\mathrm{Tr}\left(\partial^\mu\phi_1^\dagger\partial_\mu\phi^1 + \partial^\mu\phi_2^\dagger\partial_\mu\phi^2 + 2\xi^2\,\phi_1^\dagger\phi_2^\dagger\phi^1\phi^2\right)$$

[O.Gurdogan, V.Kazakov '15]



non-unitary theory

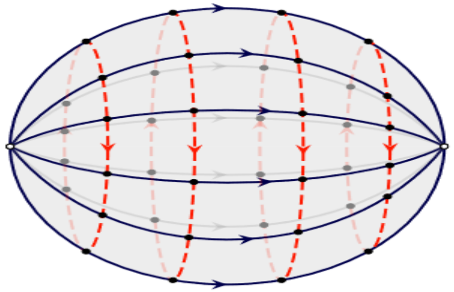
↓

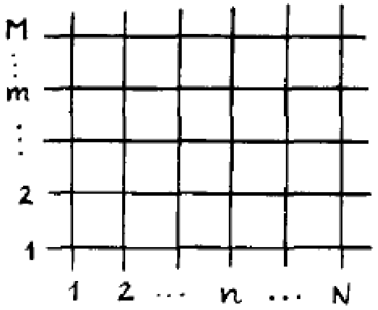
Crucial to planar Integrability

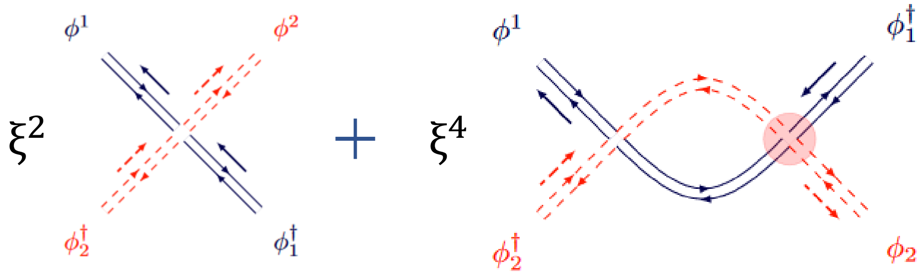
Bi-scalar theory

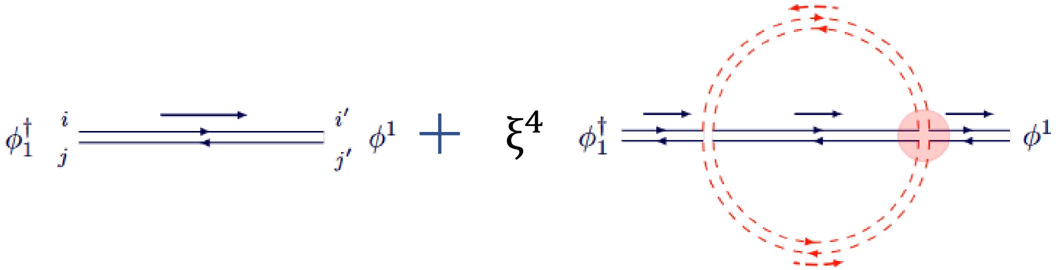
Planar graphs: $Tr[\varphi_1^L(x)]$ → Fishnets with M wrappings → Integrable lattice topology [Zamolodchikov '80]

$$\langle Tr(\phi_1^L)(x)\, Tr(\phi_1^L)^*(y)\rangle = \; x \quad \quad y$$









- No **mass** generation.
- ξ^2 is **not** renormalized in the planar limit

→ Fishnets are **quantum** conformal graphs.

$$\langle Tr(\phi_1^L)(x)\, Tr(\phi_1^L)^*(y)\rangle = c\,(x-y)^{-2\Delta(\xi)}$$

General *Fishnet Field Theory* [V.Kazakov, E.O.'18]

- Field Theory described by the same conformal planar graphs in dimension $D \in \mathbb{N}$:

$$[\phi_1] + [\phi_2] = \frac{D}{2} \quad \Leftrightarrow \quad [\phi_1^* \phi_2^* \phi_1 \phi_2] = D$$

- Isotropic lattice $[\phi_1] = [\phi_2]$

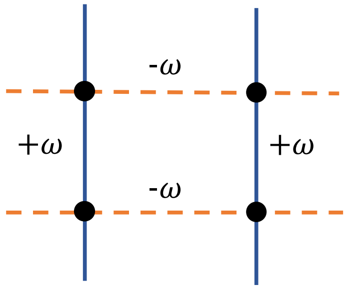
$$G_0(x-y) = |x-y|^{-\frac{D}{2}} \quad \Leftrightarrow \quad L_{kin}(x) = -\, \phi_i^* \left(\partial_\mu \partial^\mu\right)^{\frac{D}{4}} \phi_i$$

→ $\frac{D}{4} = n$

→ $non-locality$

- Non-isotropic lattice $[\phi_1] \neq [\phi_2]$

$$G_{0,1}(x-y) = |x-y|^{-\frac{D}{2}+\omega} \qquad G_{0,2}(x-y) = |x-y|^{-\frac{D}{2}+\omega}$$

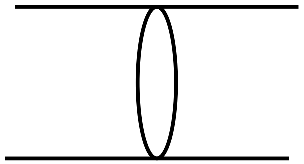
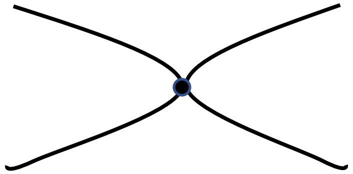


- (D,ω)
  - → $D = 1, \omega = 0$ → "scalar version" of **cSYK** model [D.Gross, V.Rosenhaus '17]
  - → $D = 2, \omega \to 1$ → **BFKL** spectrum (Pomeron, Odderon, etc.) [Balitski,Kuraeev,Fadin,Lipatov]
  - → $D = \infty, \omega = 0$

Conformal RG points

$L=2$ $\quad \langle Tr[\varphi_1^2(x)]Tr[\varphi_1^2(y)]^*\rangle$ has extra UV divergencies $\longrightarrow$ $\alpha^2\, Tr[\varphi_1^2(x)]^*\, Tr[\varphi_1^2(x)]$

Planar counter-term

Primitive divergencies $\left\{ \right.$  ξ^4 $+$  α^2 $\left. \right\}$

Coupling α^2 runs $\longrightarrow$ the *complete* theory is **not** quantum conformal. [J.Fokken, G. Sieg, M.Wilhelm '15]

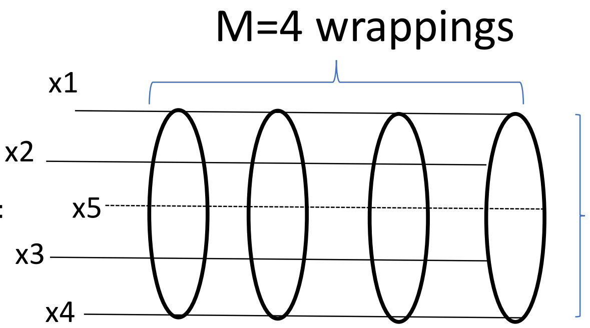
$\exists$ **critical points** (perturbatively) $\beta(\alpha_1^{\pm})=0$

$$\alpha_1^2(\xi)=\mp\frac{i\,\xi^2}{2\,\Gamma\left(\frac{D}{4}\right)^2}\pm\frac{\psi^{(0)}\left(\frac{D}{2}\right)-\psi^{(0)}\left(\frac{D}{4}\right)}{2\,\Gamma\left(\frac{D}{4}\right)^2\Gamma\left(\frac{D}{2}\right)}\,\xi^4+O(\xi^6)$$

[Grabner, Gromov, Korchemsky, Kazakov '17]
[Kazakov, E.O. '18]

Tayloring fishnets

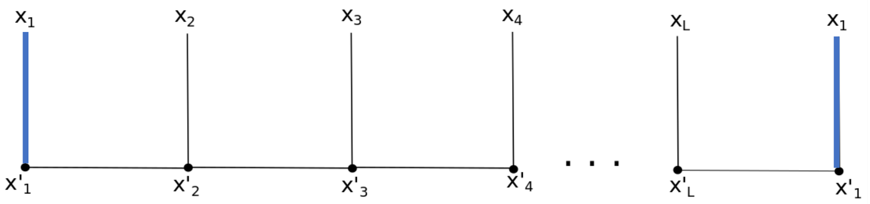
$$(H_L)^M =$$



L=5 lines →

Integral taylor

$$H_L(x_1 \dots x_L \mid x'_1 \dots x'_L) =$$



$SO(D+1,1)$ *homogeneous, scalar* spin chain →

Yang-Baxter equation

$$R_{12}(u)\, R_{23}(v)\, R_{31}(u-v) = R_{31}(u) \mathrm{R}_{23}(v)\, \mathrm{R}_{12}(u-v)$$

Quantum Integrals of motion

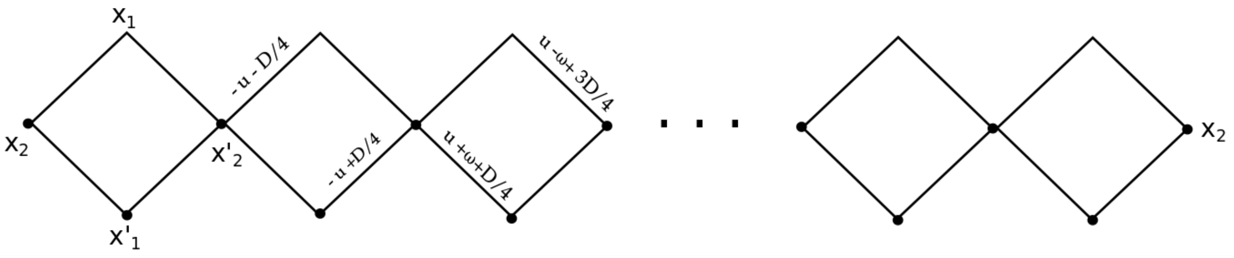
$$\{Q_1, \dots, Q_L\} \qquad [Q_h, Q_k] = 0$$

$$rep\ SO(D+1,1) = \left(\frac{D-2\,\omega}{4}, 0, 0\right) = (\Delta, l, \dot{l})$$

Scaling dimension

Rotations group labels

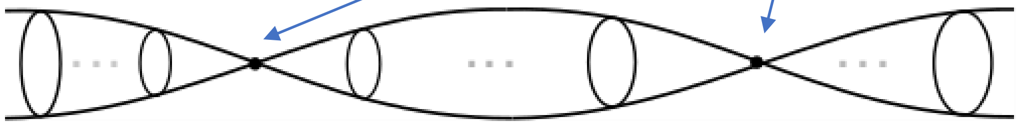
$$T(u) =$$



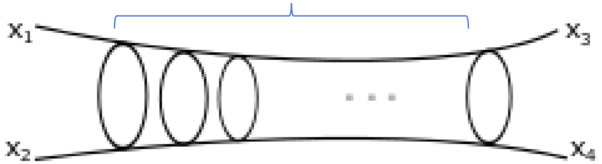
$$u = u_0(D, \omega)$$

$$\rightarrow H_L(x_1 \dots x_L \mid x'_1 \dots x'_L)$$

$$\langle Tr[\varphi_1(x_1)\varphi_1(x_2)]^* \, Tr[\varphi_1(x_3)\varphi_1(x_4)]\rangle = \sum_{L=0}^{\infty} \alpha^{2L} \left(\sum_{M=0}^{\infty} \xi^{4M} \, G_{\{M\}} \right)^L$$



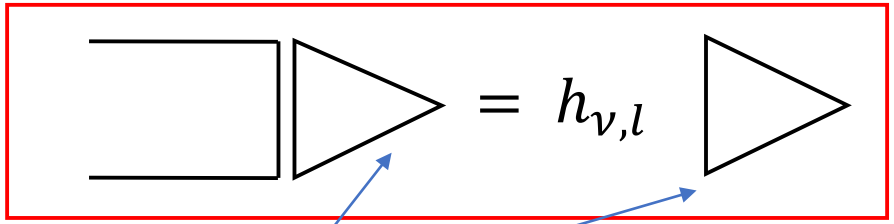
M wrappings



$$G_{\{M\}} = (H_2)^M \, G_0(x_3' - x_3) G_0(x_4' - x_4)$$ → Graph computation ≡ spin chain diagonalization

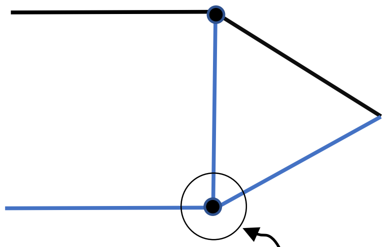
$$\left(\tfrac{D}{4},0,0\right) \otimes \left(\tfrac{D}{4},0,0\right) = \oplus_{\nu,l} \left(\tfrac{D}{2} + i\,\nu, l, l\right)$$

Clebsch-Gordan decomposition over **principal series** [Todorov, et al.]

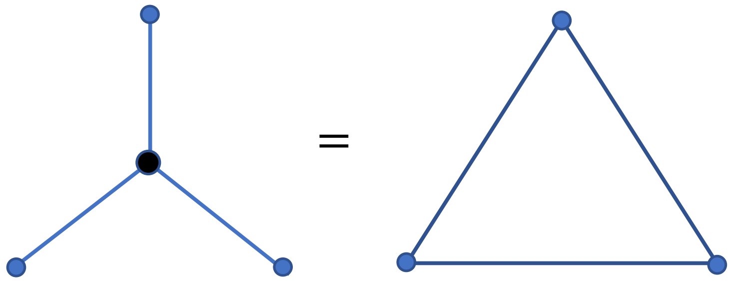


Conformal triangle $\left\{\left(\tfrac{D}{4},0,0\right), \left(\tfrac{D}{4},0,0\right), \left(\tfrac{D}{2} + i\,\nu, l, l\right)\right\}$

[Sotkov, Zaikov '76]



Uniqueness relations



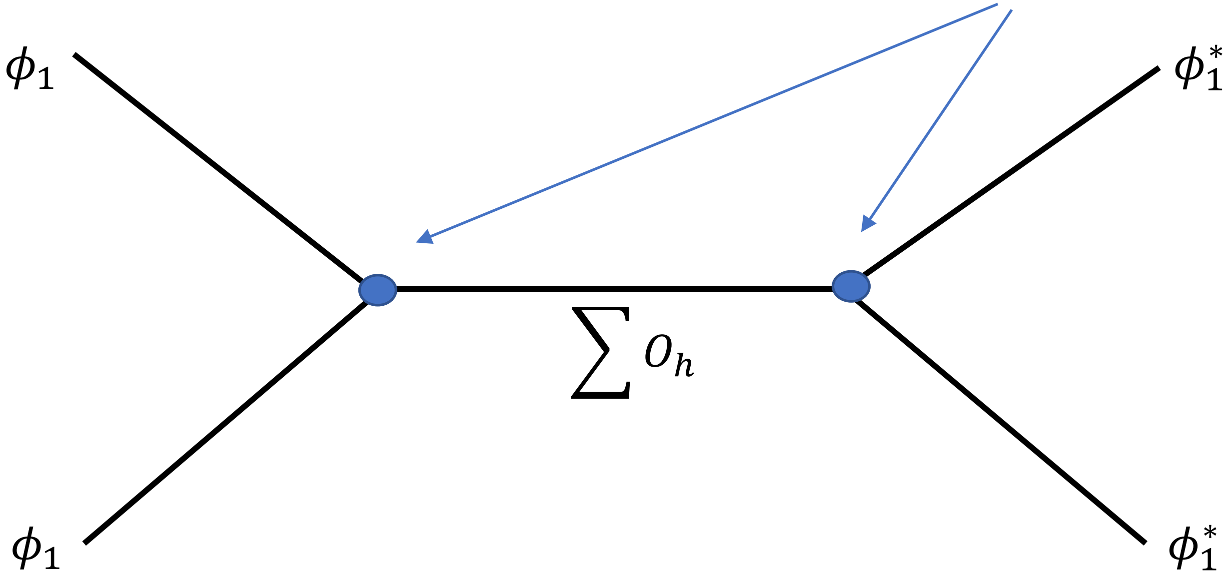
Exact 4-points function

$$G(u,v) = \sum_h C^2_{O_h\phi\phi}\, u^{\Delta_h - S_h} g_h(u,v)$$

$$u = x_{12}^2 x_{34}^2/(x_{13}^2 x_{24}^2)$$
$$v = x_{14}^2 x_{23}^2/(x_{13}^2 x_{24}^2)$$

Conformal cross ratios

$\{O_h(x)\}$ Algebra of Conformal local operators → Completeness allows decomposition



Exact 4-points function

$$\mathcal{G}(u,v) = \sum_{S/2\in\mathbb{Z}_+} \int_{-\infty}^{\infty} d\nu \mu_{\Delta,S} \frac{u^{(\Delta-S)/2} g_{\Delta,S}(u,v)}{1-\xi^4/h_{\Delta,S}}$$

$$\left.\begin{aligned} u &= x_{12}^2 x_{34}^2/(x_{13}^2 x_{24}^2) \\ v &= x_{14}^2 x_{23}^2/(x_{13}^2 x_{24}^2) \end{aligned}\right\}$$

Conformal cross ratios

Exact 4-points function

$$\mathcal{G}(u,v) = \sum_{S/2\in\mathbb{Z}_+} \int_{-\infty}^{\infty} d\nu \mu_{\Delta,S} \frac{u^{(\Delta-S)/2} g_{\Delta,S}(u,v)}{1-\xi^4/h_{\Delta,S}}$$

$$u = x_{12}^2 x_{34}^2/(x_{13}^2 x_{24}^2)$$
$$v = x_{14}^2 x_{23}^2/(x_{13}^2 x_{24}^2)$$

Conformal cross ratios

Plancherel measure SO(D+1,1)

Conformal partial waves

$$J^m = \sum_{i=1}^{4} J^m(x_i)$$

$$(J^m J_m)\ \ g_{\Delta,S}(u,v) = C_2(\Delta,S)\ g_{\Delta,S}(u,v)$$

Pole equation

$$h_{\Delta,S} \equiv \frac{\Gamma\left(\frac{3D}{4}-\frac{\Delta-S}{2}\right)\Gamma\left(\frac{D}{4}+\frac{\Delta+S}{2}\right)}{\Gamma\left(\frac{D}{4}-\frac{\Delta-S}{2}\right)\Gamma\left(-\frac{D}{4}+\frac{\Delta+S}{2}\right)} = \xi^4$$

Even $D = 2m$: factorizes into a polynomial equation

$$\Delta_0 - S = \{m, m+2, \cdots, 3m-2\},$$

$$Tr(\phi_1^2), \qquad Tr(\phi_1 \partial_S^+ \phi_1)$$

m poles → m *exchanged operators* in the *OPE channel* $\Delta(\xi^2)$

m residues → *Structure constants for 3-points correlators* $C_{O\phi\phi}(\xi^2)$

Non-perturbative

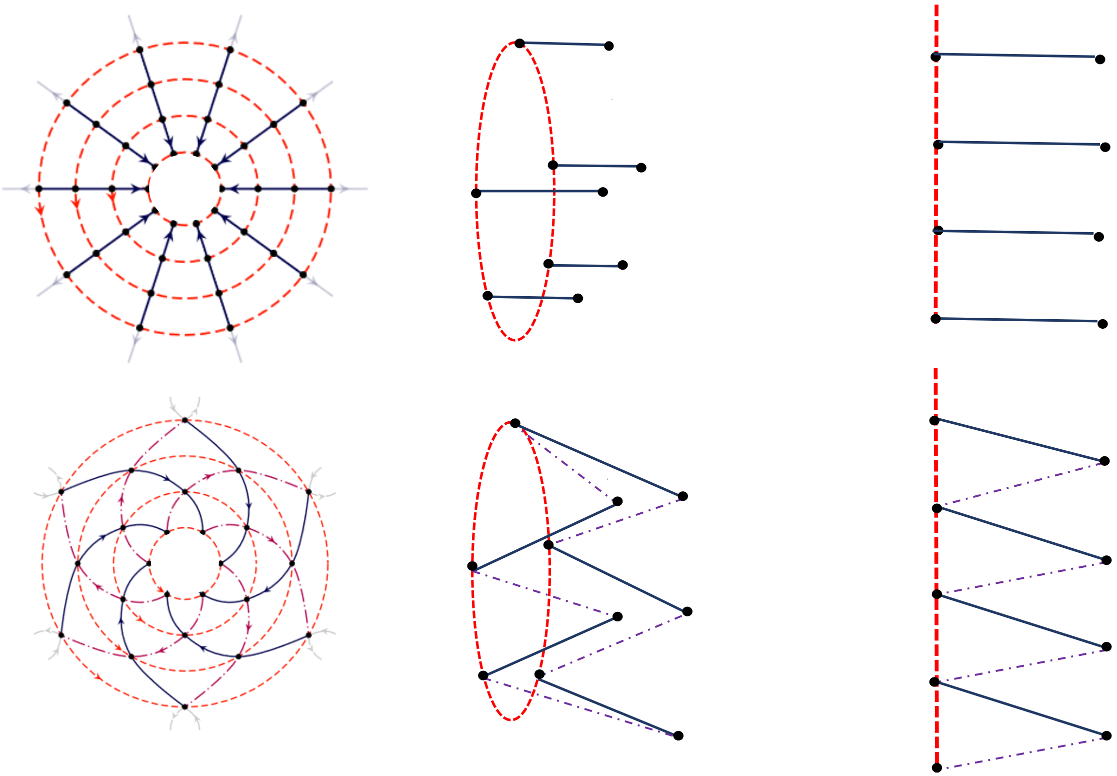
Замолодчиков*’s Field Theories*

- “Honeycomb” Fishnet Field Theory
- Strongly twisted $\mathcal{N}=6$ ABJM $\longrightarrow$ “Triangular” Fishnet Field Theory

[Caetano, Gurdogan, Kazakov’17]

Quantum Conformality

[Mamroud, Torrents ’17]



Homogeneous, *scalar* spin chains

$$[\phi_k] = \frac{D}{4} \longrightarrow rep\ SO(D+1,1) = \left(\frac{D}{4}, 0,0\right)$$

$$[\phi_k] = \frac{D}{3} \longrightarrow rep\ SO(D+1,1) = \left(\frac{D}{3}, 0,0\right)$$

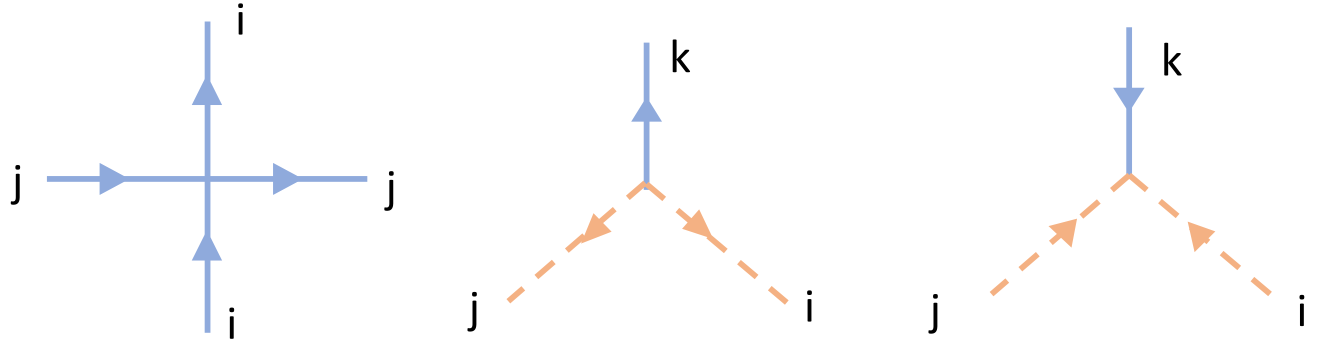
$$[Y_k] = \frac{D}{6} \longrightarrow rep\ SO(D+1,1) = \left(\frac{D}{6}, 0,0\right)$$

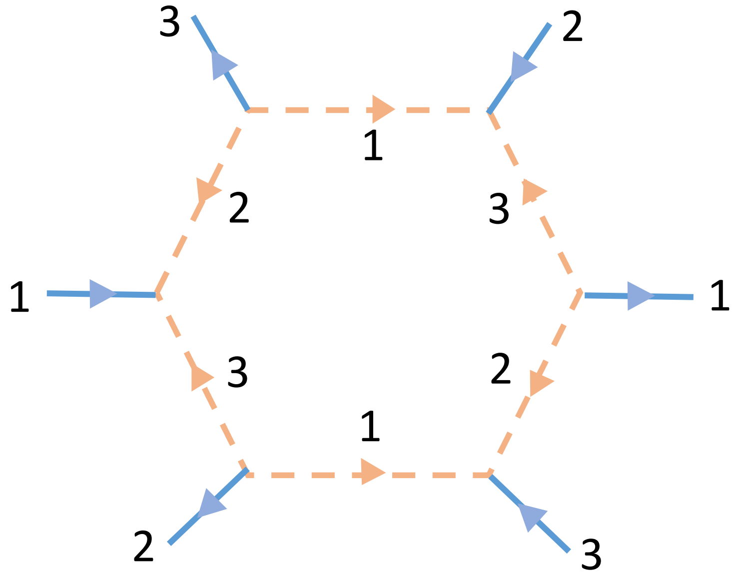
General *DS limit* $N = 4$

Three bi-scalar vertices

Yukawa couplings

$$\mathcal{L}_{\rm int} = N_c \,\mathrm{Tr}\Big(\xi_1^2\, \phi_2^\dagger\phi_3^\dagger\phi^2\phi^3 + \xi_2^2\, \phi_3^\dagger\phi_1^\dagger\phi^3\phi^1 + \xi_3^2\, \phi_1^\dagger\phi_2^\dagger\phi^1\phi^2$$
$$+ i\sqrt{\xi_2\xi_3}(\psi^3\phi^1\psi^2 + \bar\psi_3\phi_1^\dagger\bar\psi_2) + i\sqrt{\xi_1\xi_3}(\psi^1\phi^2\psi^3 + \bar\psi_1\phi_2^\dagger\bar\psi_3) + i\sqrt{\xi_1\xi_2}(\psi^2\phi^3\psi^1 + \bar\psi_2\phi_3^\dagger\bar\psi_1)\Big).$$

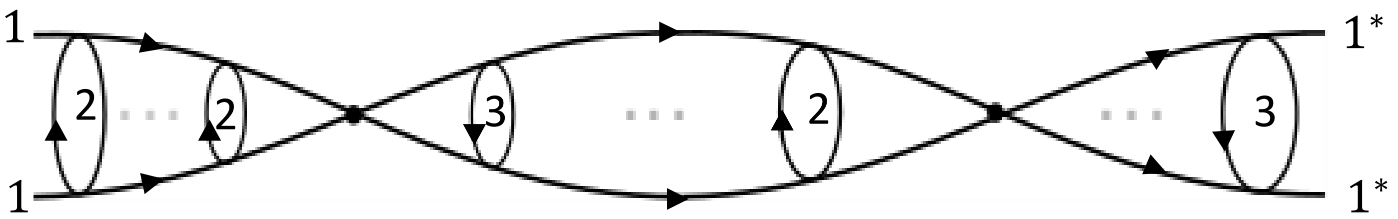




Chirality
$i > j > k \;\in \mathbb{Z}_3$

Imposed by $N = 4$

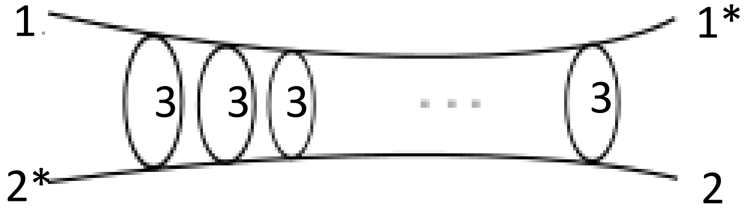
$\langle Tr[\varphi_1(x_1)\varphi_1(x_2)]^*\, Tr[\varphi_1(x_3)\varphi_1(x_4)]\rangle =$



$Tr(\phi_1\ \phi_2^*)$

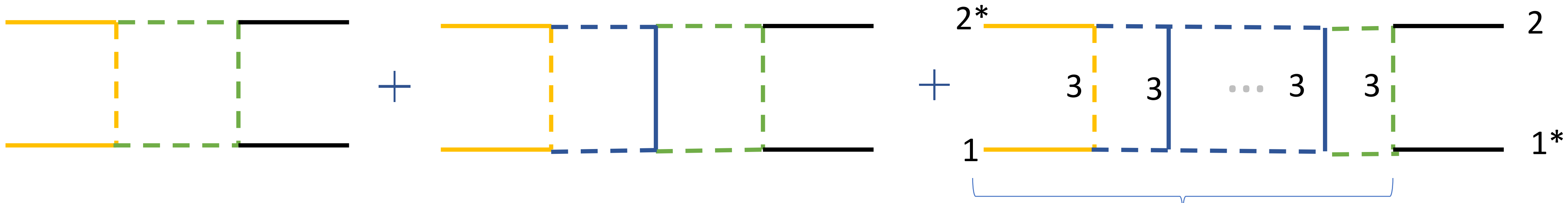
- Two geometric series of planar Fishnets

Scalar

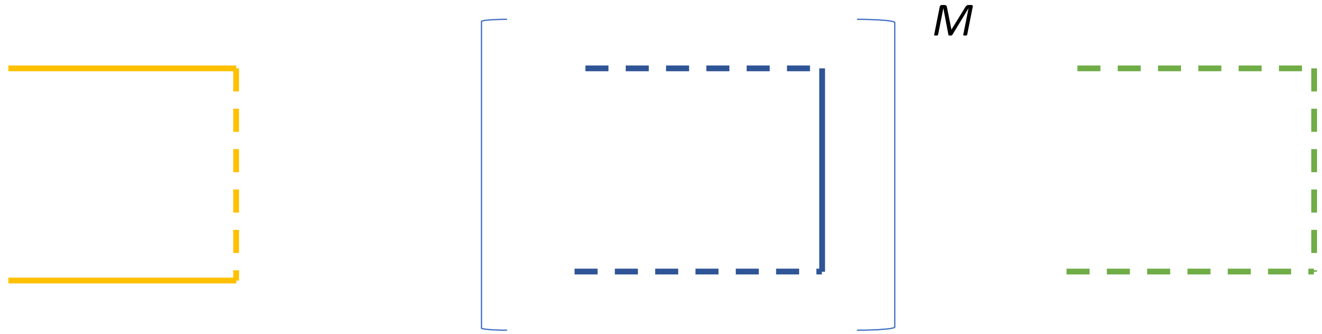


"Scalar taylor"

Fermionic loop enters the net

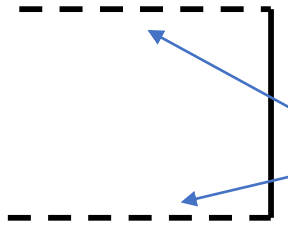


- Three *"Fermionic taylors"*



By symmetry is diagonalized by principal series eigenfunctions

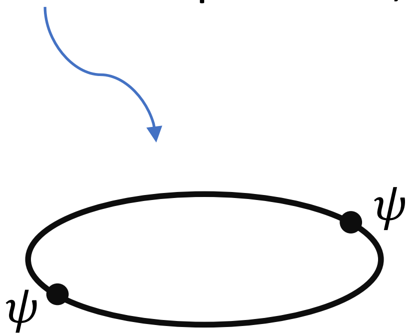
$$(1,0,0) \otimes (1,0,0) = \oplus_{\nu,l}\,(2 + i\,\nu, l, l)$$



scalar: $(1,0,0)$

Massless fermions: $\left(\frac{3}{2},\frac{1}{2},0\right)$ or $\left(\frac{3}{2},0,\frac{1}{2}\right)$ → Left XOR Right chiral fermions

Thus the first and last operator are ***Intertwiners*** between *scalar* and *fermionic* spin chain, while the bulk operator is a charge of the fermionic chain.

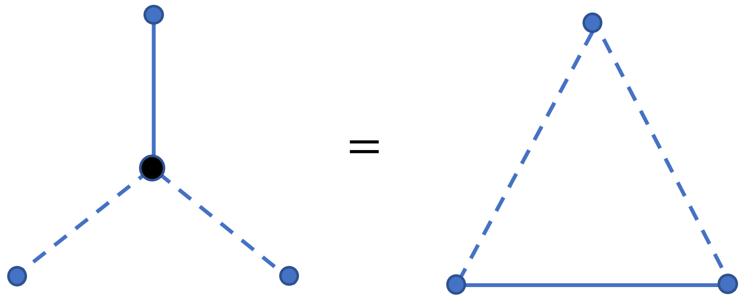


$$\left(\frac{3}{2},\frac{1}{2},0\right)\otimes\left(\frac{3}{2},\frac{1}{2},0\right)$$

$$\left(\frac{3}{2},0,\frac{1}{2}\right)\otimes\left(\frac{3}{2},0,\frac{1}{2}\right)$$

Fermionic Uniqueness relation (Star-Triangle)

[Chicherin, Derkachev, Isaev '12]



Conclusions