# Amplitudes <br> (in the Standard Model and beyond) 

Vittorio Del Duca<br>ETH Zürich \& INFN LNF

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## The study of scattering amplitudes is on some level the study of classes of special functions

incipit<br>Bourjaily He McLeod von Hippel Wilhelm today

Q At the LHC, particles are produced through the head-on collisions of protons, and in particular through the collisions of quarks and gluons within the protons

Q The probability of a collision event is computed through the cross section, which is given as an integral of (squared) scattering amplitudes over the phase space of the produced particles

Q The scattering amplitudes are given as a power series (loop expansion) in the strong and/or electroweak couplings

The scattering amplitudes are fundamental objects of particle physics

## Scattering amplitudes

9
The scattering amplitudes are given as a loop-momentum expansion in the strong and/or electroweak couplings


The more terms we know in the loop expansion, the more precisely we can compute the cross section

As a matter of fact, we know any amplitude of interest at one loop; several ( $2 \rightarrow 2$ ) amplitudes and one $(2 \rightarrow 3)$ amplitude (but only planar) at two loops; a couple ( $2 \rightarrow \mathrm{I}$ ) amplitudes and one $(2 \rightarrow 2)$ amplitude (in $\mathrm{N}=4 \mathrm{SYM}$ ) at three loops, and nothing beyond that


## Computing amplitudes

Integration by parts (IBP) to reduce the Feynman integral to a suitable basis of master integrals (MI)

It used to be the less problematic part of the workflow.
With $(2 \rightarrow 3)$ amplitudes at two loops it has become a challenge.
Now, several groups work on it
ETH ITP organises a workshop dedicated only to that
Taming the Complexity of Multiloop Integrals, ETH, 4-8 June 2018
Differential Equation method to solve the Mls
$f: \mathrm{N}$-vector of MIs, $A_{i}: \mathrm{NxN}$ matrix, $i=I, \ldots, \#$ external parameters

$$
\partial_{i} f\left(x_{n} ; \varepsilon\right)=A_{i}\left(x_{n} ; \varepsilon\right) f\left(x_{n} ; \varepsilon\right)
$$

but in some cases $\varepsilon$-independent form

$$
\partial_{i} f\left(x_{n} ; \varepsilon\right)=\varepsilon A_{i}\left(x_{n}\right) f\left(x_{n} ; \varepsilon\right)
$$

analysing the space of functions of the Feynman integral where most of the progress has been $\rightarrow$ topic of this talk

What outcome do we expect from the loop expansion of an amplitude?

3
From renormalisability and the infrared structure of the amplitude, we expect that the divergent parts are (poly)logarithmic functions of the external momenta (beyond 2 loops)

But, except for unitarity, we have little guidance for the finite parts. Heuristically, we know that:

- at one loop, logarithmic and dilogarithmic functions of the external momenta occur
- beyond one loop, higher polylogarithmic functions appear and elliptic functions may appear (usually associated to several massive propagators)


## Higgs production at LHC

Q In proton collisions, the Higgs boson is produced mostly via gluon fusion The gluons do not couple directly to the Higgs boson The coupling is mediated by a heavy quark loop
The largest contribution comes from the top loop
The production mode is (roughly) proportional to the top Yukawa coupling $y_{t}$
9 QCD NLO corrections




Djouadi Graudenz Spira Zerwas 1993-I995
Q QCD NLO corrections are about 100\% larger than leading order

9 QCD NNLO corrections are not known

## Higgs production in HEFT

9. $m_{H} \ll 2 m_{t}$

all amplitudes are reduced by one loop
Q ... but, beware of quark mass effects

| $\sigma_{E F T}^{L O}$ | 15.05 pb | $\sigma_{E F T}^{N L O}$ | 34.66 pb |
| :---: | :---: | :---: | :---: |
| $R_{L O} \sigma_{E F T}^{L O}$ | 16.00 pb | $R_{L O} \sigma_{E F T}^{N L O}$ | 36.84 pb |
| $\sigma_{e x ; t}^{L O}$ | 16.00 pb | $\sigma_{e x ; t}^{N L O}$ | 36.60 pb |
| $\sigma_{e x ; t+b}^{L O}$ | 14.94 pb | $\sigma_{e x ; t+b}^{N L O}$ | 34.96 pb |
| $\sigma_{e x ; t+b+c}^{L O}$ | 14.83 pb | $\sigma_{e x ; t+b+c}^{N L O}$ | 34.77 pb |

Anastasiou Duhr Dulat Furlan Gehrmann Herzog Lazopoulos Mistlberger 2016
Q $\quad R_{L O}=\frac{\sigma_{\text {ex:t }}^{L O}}{\sigma_{E F T}^{L O}}=1.063$
rescaled EFT (rEFT) does a good job (<l\%) in approximating the exact (only top) NLO $\sigma$ but misses the $t-b$ interference

## Higgs production in HEFT

Q QCD corrections have been computed at $\mathrm{N}^{3} \mathrm{LO}$
Anastasiou Duhr Dulat Herzog Mistlberger 2015
$\sigma=48.58 \mathrm{pb}_{-3.27 \mathrm{pb}(-6.72 \%)}^{+2.22 \mathrm{pb}(+4.56 \%)}$ (theory) $\pm 1.56 \mathrm{pb}(3.20 \%)\left(\mathrm{PDF}+\alpha_{s}\right)$

Q The breakdown of the cross section

$$
\begin{array}{rlrl}
48.58 \mathrm{pb}= & 16.00 \mathrm{pb} & (+32.9 \%) & \\
& +20.84 \mathrm{pb} & (+42.9 \%) & \\
& (\text { NLO }, \text { rEFT }) \\
& -2.05 \mathrm{pb} & (-4.2 \%) & \\
& +9.56 \mathrm{pb} & (+19.7 \%) & \\
& +0 . c), \text { exact NLO }) \\
& +0.34 \mathrm{pb} & (+0.2 \%) & \\
& +2.40 \mathrm{pb} & (+4.9 \%) & \\
& \left(\mathrm{EW}, \mathrm{rEFT}, 1 / m_{t}\right) \\
& +1.49 \mathrm{pb} & (+3.1 \%) & \\
\left(\mathrm{N}^{3} \mathrm{LO}, \text {, rEFT }\right)
\end{array}
$$

Anastasiou Duhr Dulat Furlan Gehrmann Herzog Lazopoulos Mistlberger 2016
Q Largest uncertainties come from quark mass effects at NNLO and from NLO corrections to QCD-EW interference

## QCD-EW interference



Aglietti Bonciani Degrassi Vicini 2004 (light fermion loop)
Actis Passarino Sturm Uccirati 2008 (heavy fermion loop)

Bonetti Melnikov Tancredi 2017
computed in:
$— m_{w, z} \rightarrow \infty$ limit Anastasiou Boughezal Petriello 2009
— soft approximation Bonetti Melnikov Tancredi 2018

- $m_{w, z} \rightarrow 0$ limit AnastasiouVDD Furlan Mistlberger Moriello Schweitzer Specchia, to appear and found to be about 5.3-5.5\% both at LO and NLO


## QCD NNLO corrections

Q Higgs+2jets amplitudes at one loop


9 Higgs+ljet amplitudes at two loops
Bonciani VDD Frellesvig Henn MorielloV. Smirnov 2016 (only planar diagrams)

multi-scale problem with complicated analytic structure elliptic iterated integrals appear
Q $g g \rightarrow H$ amplitudes at three loops

one-scale problem, but with more elliptic iterated integrals ...

## Higgs $p_{T}$ distribution at LHC

9 leading order


9 high-pt tail of the Higgs pT distribution is sensitive to the structure of the loop-mediated Higgs-gluon coupling New Physics particles circulating in the loop would modify it

Q Full ( $=t+b+c$ ) QCD NLO corrections are not known

9 HEFT $m_{H} \ll 2 m_{t}$ and $p_{T} \ll m_{t}$
QCD corrections are known at NNLO in HEFT, and yield a $15 \%$ increase wrt NLO
Boughezal Caola Melnikov Petriello Schulze 2015
Boughezal Focke Giele Liu Petriello 2015
Chen Cruz-Martinez Gehrmann Glover Jaquier 2016

## Higgs $p_{T}$ distribution at LHC

Q QCD (top) NLO corrections have been computed numerically

$\frac{d \sigma}{d p_{T}^{2}} \propto \frac{1}{p_{T}^{2}} \quad$ in HEFT NLO corrections $\frac{d \sigma}{d p_{T}^{2}} \propto \frac{1}{\left(p_{T}^{2}\right)^{2}} \quad$ in top NLO corrections

No $t-b$ interference

- QCD NLO corrections to $t-b$ interference, using top loop in HEFT and $b$-quark loop in small $m_{b}$ limit


Q In the last few years, a lot of progress has been made in understanding the analytic structure of multi-loop amplitudes, in particular on how the polylogarithmic functions appear at any loop level

9
In particular, a lot of progress has been made:
— in N=4 Super Yang-Mills (SYM)
— in the Regge limit of QCD
— in the Regge limit of $N=4$ SYM
9
This progress has deep implications on how we view scattering amplitudes in the Standard Model

One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane ...

incipit<br>The analytic S-matrix<br>Eden Landshoff Olive Polkinghorne I966

## $N=4$ Super Yang Mills

Q maximal supersymmetric theory (without gravity) conformally invariant, $\beta \mathrm{fn} .=0$

- spin I gluon

4 spin I/2 gluinos
6 spin 0 real scalars
Q ' t Hooft limit: $N_{c} \rightarrow \infty$ with $\lambda=g^{2} N_{c}$ fixed
Q only planar diagrams

Q AdS/CFT duality
Maldacena 97
Q large- $\lambda$ limit of 4dim CFT $\leftrightarrow$ weakly-coupled string theory (aka weak-strong duality)

## N=4 Super Yang Mills

9 amplitudes in planar $\mathrm{N}=4$ SYM are much simpler than in Standard Model processes
use $N=4$ SYM as a computational lab:
Q to learn techniques and tools to be used in Standard Model calculations

Q to learn about the bases of special functions which may occur in realistic scattering processes

## $N=4$ Super Yang Mills

In the last years, a huge progress has been made in understanding the analytic structure of the S-matrix of planar N=4 SYM

Besides the ordinary conformal symmetry, in the planar limit the $S$-matrix exhibits a dual conformal symmetry

Drummond Henn Smirnov Sokatchev 2006
Accordingly, the analytic structure of the scattering amplitudes is highly constrained

4- and 5-point amplitudes are fixed to all loops by the symmetries in terms of the one-loop amplitudes and the cusp anomalous dimension

Anastasiou Bern Dixon Kosower 2003, Bern Dixon Smirnov 2005
Drummond Henn Korchemsky Sokatchev 2007
Beyond 5 points, the finite part of the amplitudes is given in terms of a remainder function $R$. The symmetries only fix the variables of $R$ (some conformally invariant cross ratios) but not the analytic dependence of $R$ on them

## Dual conformal symmetry

9
Dual space $\quad p_{i}=x_{i}-x_{i+1} \equiv x_{i, i+1}$

$$
x_{n+1}=x_{1}
$$

9
one-loop scalar box

$$
\begin{aligned}
& I^{(1)}=\int \frac{d^{4} k}{k^{2}\left(k-p_{1}\right)^{2}\left(k-p_{1}-p_{2}\right)^{2}\left(k+p_{4}\right)^{2}} \\
& p_{1}=x_{12}, \quad p_{2}=x_{23}, \quad p_{3}=x_{34}, \quad p_{4}=x_{41}, \quad k=x_{15}
\end{aligned}
$$

$$
I^{(1)}=\int \frac{d^{4} x_{5}}{x_{15}^{2} x_{25}^{2} x_{35}^{2} x_{45}^{2}}=\frac{1}{x_{13}^{2} x_{24}^{2}} \Phi^{(1)}(s, t)
$$

Q conformal inversion

$$
\begin{aligned}
& x_{i}^{\mu} \rightarrow-\frac{x_{i}^{\mu}}{x_{i}^{2}}, \quad x_{i j}^{2} \rightarrow \frac{x_{i j}^{2}}{x_{i}^{2} x_{j}^{2}, \quad d^{4} x_{5} \rightarrow \frac{d^{4} x_{5}}{x_{5}^{8}}} \\
& \frac{d^{4} x_{5}}{x_{15}^{2} x_{25}^{2} x_{35}^{2} x_{45}^{2}} \rightarrow x_{1}^{2} x_{2}^{2} x_{3}^{2} x_{4}^{2} \frac{d^{4} x_{5}}{x_{15}^{2} x_{25}^{2} x_{35}^{2} x_{45}^{2}} \quad \quad \text { conformally covariant }
\end{aligned}
$$

Q drawing dual graphs is only possible for planar diagrams

## MHV amplitudes in planar N=4 SYM

- at any order in the coupling, colour-ordered maximally helicity violating MHV (- - ++...+) amplitudes in planar N=4 SYM can be written as the tree-level amplitude times a momentum dependent loop coefficient $M_{n}^{(L)}=M_{n}^{(0)} m_{n}^{(L)}$

9 at I loop
Bern Dixon Dunbar Kosower 94
$m_{n}^{(1)}=\sum_{p q} F^{2 \mathrm{me}}(p, q, P, Q) \quad n \geq 6$


Q at 2 loops, iteration formula for the $n-p t$ amplitude
remainder

$$
m_{n}^{(2)}(\epsilon)=\frac{1}{2}\left[m_{n}^{(1)}(\epsilon)\right]^{2}+f^{(2)}(\epsilon) m_{n}^{(1)}(2 \epsilon)+\text { Const }^{(2)}+R
$$

Q at all loops, ansatz for a resummed exponent

$$
m_{n}^{(L)}=\exp \left[\sum_{l=1}^{\infty} a^{l}\left(f^{(l)}(\epsilon) m_{n}^{(1)}(l \epsilon)+\text { Const }^{(l)}+E_{n}^{(l)}(\epsilon)\right)\right]+R
$$

## ABDK/BDS ansatz

9. ABDK/BDS ansatz is valid at all loops for 4-pt and 5-pt amplitudes

$$
\begin{aligned}
M_{n} & =M_{n}^{(0)}\left[1+\sum_{L=1}^{\infty} a^{L} m_{n}^{(L)}(\epsilon)\right] \quad \text { Bern Dixon Smirnov } 05 \\
& =M_{n}^{(0)} \exp \left[\sum_{l=1}^{\infty} a^{l}\left(f^{(l)}(\epsilon) m_{n}^{(1)}(l \epsilon)+\text { Const }^{(l)}+E_{n}^{(l)}(\epsilon)\right)\right]
\end{aligned}
$$

coupling $\quad a=\frac{\lambda}{8 \pi^{2}}\left(4 \pi e^{-\gamma}\right)^{\epsilon} \quad \lambda=g^{2} N \quad$ 't Hooft parameter
$f^{(l)}(\epsilon)=\frac{\hat{\gamma}_{K}^{(l)}}{4}+\epsilon \frac{l}{2} \hat{G}^{(l)}+\epsilon^{2} f_{2}^{(l)} \quad E_{n}^{(l)}(\epsilon)=O(\epsilon)$
$\hat{\gamma}_{K}^{(l)}$ cusp anomalous dimension, known to all orders of $a$
$\hat{G}^{(l)}$ collinear anomalous dimension, known through $\mathrm{O}\left(a^{4}\right)$

Korchemsky Radyuskin 86
Beisert Eden Staudacher 06
Bern Dixon Smirnov 05
Cachazo Spradlin Volovich 07

## Conformally invariant cross ratios

0
for $n=6$, the conformally invariant cross ratios are
$u_{1}=\frac{x_{13}^{2} x_{46}^{2}}{x_{14}^{2} x_{36}^{2}} \quad u_{2}=\frac{x_{24}^{2} x_{15}^{2}}{x_{25}^{2} x_{14}^{2}} \quad u_{3}=\frac{x_{35}^{2} x_{26}^{2}}{x_{36}^{2} x_{25}^{2}}$
$x_{i}$ are variables in a dual space s.t. $\quad p_{i}=x_{i}-x_{i+1}$

thus $\quad x_{k, k+r}^{2}=\left(p_{k}+\ldots+p_{k+r-1}\right)^{2}$
for $n$ points, dual conformal invariance implies dependence on $3 n$ - 15 independent cross ratios
$u_{1 i}=\frac{x_{i+1, i+5}^{2} x_{i+2, i+4}^{2}}{x_{i+1, i+4}^{2} x_{i+2, i+5}^{2}}, \quad u_{2 i}=\frac{x_{N, i+3}^{2} x_{1, i+2}^{2}}{x_{N, i+2}^{2} x_{1, i+3}^{2}}, \quad u_{3 i}=\frac{x_{1 i+4}^{2} x_{2, i+3}^{2}}{x_{1, i+3}^{2} x_{2, i+4}^{2}}$


## Amplitudes in planar $N=4$ SYM

The progress in understanding the analytic structure of the $S$-matrix in planar $N=4$ SYM is also due to an improved understanding of the mathematical structures underlying the scattering amplitudes
n-point amplitudes are expected to be written in terms of iterated integrals (on the space of configurations of points in 3-dim projective space $\operatorname{Conf}_{n}\left(\mathrm{P}^{3}\right)$ )

The simplest case of iterated integrals are the iterated integrals over rational functions, i.e. the multiple polylogarithms (MPL)

$$
G\left(a_{1}, \ldots, a_{n} ; z\right)=\int_{0}^{z} \frac{d t}{t-a_{1}} G\left(a_{2}, \ldots, a_{n} ; t\right) \quad G(a ; z)=\log \left(1-\frac{z}{a}\right) \quad\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{C}
$$

Goncharov 2001
It is thought that maximally helicity violating (MHV) and next-to-MHV (NMHV) amplitudes can be expressed in terms of multiple polylogarithms of uniform transcendental weight

## Amplitudes in planar N=4 SYM

MHV and NMHV amplitudes feature maximal transcendentality, i.e. L-loop amplitudes are expressed in terms of multiple polylogarithms of weight $2 L$ only

MHV amplitudes are pure, i.e. the coefficients of the multiple polylogarithms are (rational) numbers

2-loop 10-pt N3MHV amplitude features elliptic iterated integrals


## Amplitudes in planar $N=4$ SYM

$\odot$
6-pt (N)MHV amplitudes are known analytically up to 5(4) loops
Duhr Smirnov VDD 2009
Goncharov Spradlin Vergu Volovich 2010
Dixon Drummond Henn 201I
Dixon Drummond von Hippel Pennington 2013
Dixon Drummond Duhr Pennington 2014
Caron-Huot Dixon von Hippel McLeod 2016

Dixon Drummond Henn 2011
Dixon von Hippel 2014
Dixon von Hippel McLeod 2015

7-pt MHV amplitudes are known analytically at two loops

$$
\text { Golden Spradlin } 2014
$$

No analytic result is known beyond 7 points (the algebra of the iterated integrals is infinite starting from 8 points)

Golden Goncharov Spradlin VerguVolovich 2013

## Taxonomy of logarithmic functions

## Polylogarithms

9 classical polylogarithms

$$
\operatorname{Li}_{m}(z)=\int_{0}^{z} d t \frac{\mathrm{Li}_{m-1}(t)}{t}=\sum_{n=1}^{\infty} \frac{z^{n}}{n^{m}}
$$

$$
\operatorname{Li}_{1}(z)=\sum_{n=1}^{\infty} \frac{z^{n}}{n}=-\ln (1-z)
$$

9 harmonic polylogarithms (HPLs)

Euler 1768
Spence 1809

$$
H(a, \vec{w} ; z)=\int_{0}^{z} \mathrm{~d} t f(a ; t) H(\vec{w} ; t) \quad f(-1 ; t)=\frac{1}{1+t}, \quad f(0 ; t)=\frac{1}{t}, \quad f(1 ; t)=\frac{1}{1-t}
$$

with $\quad\{a, \vec{w}\} \in\{-1,0,1\}$
Q classical polylogarithms are multiple polylogarithms with specific roots

$$
G\left(\overrightarrow{0}_{n} ; x\right)=\frac{1}{n!} \ln ^{n} x \quad G\left(\vec{a}_{n} ; x\right)=\frac{1}{n!} \ln ^{n}\left(1-\frac{x}{a}\right) \quad G\left(\overrightarrow{0}_{n-1}, a ; x\right)=-\operatorname{Li}_{n}\left(\frac{x}{a}\right)
$$

Q when the root equals $+1,-1,0$ multiple polylogarithms become HPLs

## Multiple polylogarithms

Q MPLs form a shuffle algebra

$$
G_{\omega_{1}}(z) G_{\omega_{2}}(z)=\sum_{\omega} G_{\omega}(z) \quad \text { with } \omega \text { the shuffle of } \omega_{1} \text { and } \omega_{2}
$$

example

$$
\begin{aligned}
G(a ; z) G(b ; z) & =\int_{0}^{z} \frac{d t_{1}}{t_{1}-a} \int_{0}^{z} \frac{d t_{2}}{t_{2}-b} \\
& =\int_{0}^{z} \frac{d t_{1}}{t_{1}-a} \int_{0}^{t_{1}} \frac{d t_{2}}{t_{2}-b}+\int_{0}^{z} \frac{d t_{2}}{t_{2}-a} \int_{0}^{t_{2}} \frac{d t_{1}}{t_{1}-b} \\
& =G(a, b ; z)+G(b, a ; z)
\end{aligned}
$$

Q $\lim _{z \rightarrow 0} G\left(a_{1}, \ldots, a_{n} ; z\right)=0 \quad$ unless $\quad \vec{a}=\overrightarrow{0}$

- $\frac{\partial}{\partial z} G\left(a_{1}, \ldots, a_{k} ; z\right)=\frac{1}{z-a_{1}} G\left(a_{2}, \ldots, a_{k} ; z\right)$

Q MPLs can be represented as nested harmonic sums

$$
\sum_{n_{1}=1}^{\infty} \frac{u_{1}^{n_{1}}}{n_{1}^{m_{1}}} \sum_{n_{2}=1}^{n_{1}-1} \cdots \sum_{n_{k}=1}^{n_{k-1}-1} \frac{u_{k}^{n_{k}}}{n_{k}^{m_{k}}}=(-1)^{k} G(\underbrace{0, \ldots, 0}_{m_{1}-1}, \frac{1}{u_{1}}, \ldots, \underbrace{0, \ldots, 0}_{m_{k}-1}, \frac{1}{u_{1} \ldots u_{k}} ; 1)
$$

## Hopf algebra and the coproduct

Q multiple polylogarithms form a Hopf algebra with a coproduct

Q algebra is a vector space with a product $\mu: A \otimes A \rightarrow A \quad \mu(a \otimes b)=a \circ b$ that is associative $A \otimes A \otimes A \rightarrow A \otimes A \rightarrow A \quad(a \cdot b) \cdot c=a^{\circ}(b \cdot c)$

Q coalgebra is a vector space with a coproduct $\Delta: B \rightarrow B \otimes B$ that is coassociative $B \rightarrow B \otimes B \rightarrow B \otimes B \otimes B$

$$
\Delta(a)=\sum_{i} a_{i}^{(1)} \otimes a_{i}^{(2)}
$$

$\mu$ puts together; $\Delta$ decomposes
a Hopf algebra is an algebra and a coalgebra, such that product and coproduct are compatible $\Delta(a \cdot b)=\Delta(a) \cdot \Delta(b)$

Q take a word, sum over ways to split it into two: deconcatenation
$T=w x y z$
$\Delta(T)=w x y z \otimes 1+w x y \otimes z+w x \otimes y z+w \otimes x y z+1 \otimes w x y z$
iterate: sum over ways to split it into three

$$
\begin{array}{ll}
w x \otimes y z \rightarrow(w \otimes x) \otimes y z & \text { if sum over all possibilities, } \\
w x \otimes y z \rightarrow w x \otimes(y \otimes z) & \text { get to the same result }
\end{array}
$$


symbols lie within the maximal iteration of a coproduct

## Coproduct on polylogarithms

Q coproduct on classical polylogarithms

$$
\Delta(\ln z)=1 \otimes \ln z+\ln z \otimes 1
$$

$\Delta(\ln y \ln z)=\Delta(\ln y) \cdot \Delta(\ln z)$
$=(1 \otimes \ln y+\ln y \otimes 1) \cdot(1 \otimes \ln z+\ln z \otimes 1)$
$=1 \otimes \ln y \ln z+\ln y \otimes \ln z+\ln z \otimes \ln y+\ln y \ln z \otimes 1$
$\Delta\left(\operatorname{Li}_{2}(z)\right)=1 \otimes \operatorname{Li}_{2}(z)+\operatorname{Li}_{2}(z) \otimes 1-\ln (1-z) \otimes \ln z \operatorname{Sym}[\ln y \ln z]=y \otimes z+z \otimes y$
$\Delta\left(\operatorname{Li}_{n}(z)\right)=1 \otimes \operatorname{Li}_{n}(z)+\operatorname{Li}_{n}(z) \otimes 1+\sum_{k=1}^{n-1} \operatorname{Li}_{n-k}(z) \otimes \frac{\ln ^{k} z}{k!}$
Q ( $n-I, I$ ) component of the coproduct $\Delta_{n-1,1}\left(\operatorname{Li}_{n}(z)\right)=\operatorname{Li}_{n-1}(z) \otimes \ln z$
iterating $\quad \Delta_{1, \ldots, 1}\left(\operatorname{Li}_{n}(z)\right)=-\ln (1-z) \otimes \underbrace{\ln z \otimes \cdots \otimes \ln z}_{n-1}$
$\operatorname{Sym}\left[\mathrm{Li}_{n}(z)\right]=-(1-z) \otimes \overbrace{z \otimes \cdots \otimes z}$
the symbol is the $(I, \ldots, I)$ component of the coproduct
for the constants, define

$$
\begin{array}{ll}
\Delta\left(\zeta_{2 n}\right)=\zeta_{2 n} \otimes 1 & \text { Brown I I } \\
\Delta(\pi)=\pi \otimes 1 & \text { Duhr I2 }
\end{array}
$$

## Coproducts and functional identities

Q weight I $\operatorname{Li}_{1}\left(\frac{1}{z}\right)=-\ln \left(1-\frac{1}{z}\right)=-\ln (1-z)+\ln (-z)=-\ln (1-z)+\ln z-i \pi$
Q weight $2 \quad \Delta_{1,1}\left(\operatorname{Li}_{2}\left(\frac{1}{z}\right)\right)=-\ln \left(1-\frac{1}{z}\right) \otimes \ln \left(\frac{1}{z}\right)$

$$
\begin{aligned}
& =\ln (1-z) \otimes \ln z-\ln z \otimes \ln z+i \pi \otimes \ln z \\
& =\Delta_{1,1}\left(-\operatorname{Li}_{2}(z)-\frac{1}{2} \ln ^{2} z+i \pi \ln z\right)
\end{aligned}
$$

$i \pi$ more than the symbol
so $\quad \operatorname{Li}_{2}\left(\frac{1}{z}\right)=-\operatorname{Li}_{2}(z)-\frac{1}{2} \ln ^{2} z+i \pi \ln z+c \pi^{2}$

$$
z=1 \rightarrow c=\frac{1}{3}
$$

weight 3

$$
\begin{aligned}
\Delta_{1,1,1}\left(\operatorname{Li}_{3}\left(\frac{1}{z}\right)\right) & =-\ln \left(1-\frac{1}{z}\right) \otimes \ln \left(\frac{1}{z}\right) \otimes \ln \left(\frac{1}{z}\right) \\
& =-\ln (1-z) \otimes \ln z \otimes \ln z+\ln z \otimes \ln z \otimes \ln z-i \pi \otimes \ln z \otimes \ln z \\
& =\Delta_{1,1,1}\left(\operatorname{Li}_{3}(z)+\frac{1}{6} \ln ^{3} z-\frac{i \pi}{2} \ln ^{2} z\right)
\end{aligned}
$$

one can do better

$$
\begin{aligned}
\Delta_{2,1}\left(\operatorname{Li}_{3}\left(\frac{1}{z}\right)-\left(\operatorname{Li}_{3}(z)+\frac{1}{6} \ln ^{3} z-\frac{i \pi}{2} \ln ^{2} z\right)\right) & =-\frac{\pi^{2}}{3} \otimes \ln z \\
& =\Delta_{2,1}\left(-\frac{\pi^{2}}{3} \ln z\right)
\end{aligned}
$$

$$
\operatorname{Li}_{3}\left(\frac{1}{z}\right)=\operatorname{Li}_{3}(z)+\frac{1}{6} \ln ^{3} z-\frac{i \pi}{2} \ln ^{2} z-\frac{\pi^{2}}{3} \ln z+c_{1} \zeta_{3}+c_{2} i \pi^{3} \quad z=1 \rightarrow c_{1}=c_{2}=0
$$

## Symbols

Q take a function defined as an iterated integral of logarithms of rational functions $R_{i}$

$$
T^{(k)}=\int_{a}^{b} \mathrm{~d} \ln R_{1} \circ \cdots \circ \mathrm{~d} \ln R_{k}=\int_{a}^{b}\left(\int_{a}^{t} \mathrm{~d} \ln R_{1} \circ \cdots \circ \mathrm{~d} \ln R_{k-1}\right) \mathrm{d} \ln R_{k}(t)
$$

then the total differential can be written as

$$
d T^{(k)}=\sum_{i} T_{i}^{(k-1)} d \ln R_{i}
$$

Q the symbol is defined recursively as $\quad \operatorname{Sym}\left[T^{(k)}\right]=\sum_{i} \operatorname{Sym}\left[T_{i}^{(k-1)}\right] \otimes R_{i}$
as such, the symbol is defined on the tensor product of the group of rational functions, modulo constants

$$
\begin{aligned}
& \cdots \otimes R_{1} R_{2} \otimes \cdots=\cdots \otimes R_{1} \otimes \cdots+\cdots \otimes R_{2} \otimes \cdots \\
& \cdots \otimes\left(c R_{1}\right) \otimes \cdots=\cdots \otimes R_{1} \otimes \cdots
\end{aligned}
$$

Q if $T$ is a multiple polylogarithm $G$, then

$$
d G\left(a_{n-1}, \ldots, a_{1} ; a_{n}\right)=\sum_{i=1}^{n-1} G\left(a_{n-1}, \ldots, \hat{a}_{i}, \ldots, a_{1} ; a_{n}\right) d \ln \left(\frac{a_{i}-a_{i+1}}{a_{i}-a_{i-1}}\right)
$$

the symbol is

$$
\operatorname{Sym}\left(G\left(a_{n-1}, \ldots, a_{1} ; a_{n}\right)\right)=\sum_{i=1}^{n-1} \operatorname{Sym}\left(G\left(a_{n-1}, \ldots, \hat{a}_{i}, \ldots, a_{1} ; a_{n}\right)\right) \otimes\left(\frac{a_{i}-a_{i+1}}{a_{i}-a_{i-1}}\right)
$$

Q the symbol knows about the discontinuities of $T$; if

$$
\operatorname{Sym}\left[T^{(k)}\right]=R_{1} \otimes \cdots \otimes R_{k}
$$

then $T$ has a branch cut at $R_{I}=0$, and the symbol of the discontinuity is

$$
\operatorname{Sym}\left[\operatorname{Disc}_{R_{1}}\left(T^{(k)}\right)\right]=R_{2} \otimes \cdots \otimes R_{k}
$$

Q $\operatorname{Disc}(\ln x \ln y)= \begin{cases}2 \pi i \ln x & \text { along the } y \operatorname{cut}[-\infty, 0] \\ 2 \pi i \ln y & \text { along the } x \operatorname{cut}[-\infty, 0]\end{cases}$

$$
\operatorname{Sym}[\ln x \ln y]=x \otimes y+y \otimes x
$$

Q in general, if $\operatorname{Disc}(f g)=\operatorname{Disc}(f) g+f \operatorname{Disc}(g)$

$$
\begin{array}{ll}
\text { and } & \operatorname{Sym}[f]=\otimes_{i=1}^{n} R_{i} \quad \operatorname{Sym}[g]=\otimes_{i=n+1}^{m} R_{i} \\
\text { then } & \operatorname{Sym}[f g]=\sum_{\sigma} \otimes_{i=1}^{n} R_{\sigma(i)}
\end{array}
$$

where $\sigma$ denotes the set of all shuffles of $n+(m-n)$ elements

$$
\begin{aligned}
& \text { e.g. } \quad \operatorname{Sym}[f]=R_{1} \otimes R_{2} \quad \operatorname{Sym}[g]=R_{3} \otimes R_{4} \\
& \begin{aligned}
\operatorname{Sym}[f g] & =R_{1} \otimes R_{2} \otimes R_{3} \otimes R_{4}+R_{1} \otimes R_{3} \otimes R_{2} \otimes R_{4}+R_{1} \otimes R_{3} \otimes R_{4} \otimes R_{2} \\
& +R_{3} \otimes R_{1} \otimes R_{2} \otimes R_{4}+R_{3} \otimes R_{1} \otimes R_{4} \otimes R_{2}+R_{3} \otimes R_{4} \otimes R_{1} \otimes R_{2}
\end{aligned}
\end{aligned}
$$

Q symbols form a shuffle algebra, i.e. a vector space with a shuffle product (also iterated integrals and multiple polylogarithms form shuffle algebras)

## MPLs, coproduct and unitarity

Q multiple polylogarithms form a Hopf algebra with a coproduct

$$
\Delta\left(L_{w}\right)=\sum_{k=0}^{w} \Delta_{k, w-k}\left(L_{w}\right)=\sum_{k=0}^{w} L_{k} \otimes L_{w-k}
$$

- the coproduct steers the functional identities among MPLs, thus it allows us to reduce a given set of MPLs of weight $n$ to a (minimal) basis of MPLs of weight $\leq n$, which we are then to analytically continue from Euclidean to Minkowski space, and to evaluate numerically

Q the analytic structure of amplitudes is constrained by unitarity and the optical theorem $\operatorname{Disc}(M)=i M M^{\dagger}$
Q discontinuity(derivative) acts in the first(last) entry of the coproduct

$$
\Delta \mathrm{Disc}=(\mathrm{Disc} \otimes \mathrm{id}) \Delta \quad \Delta \partial=(\mathrm{id} \otimes \partial) \Delta
$$

then the coproduct of an amplitude is related to unitarity
in particular, for massless amplitudes
$\Delta(M)=\ln \left(s_{i j}\right) \otimes \ldots$
massless amplitudes may have branch points when Mandelstam invariants vanish $s_{i j} \rightarrow 0$ or become infinite $s_{i j} \rightarrow \infty$

## Regge limit

## Regge limit of QCD

- In perturbative QCD, in the Regge limit $s \geqslant t$, any scattering process is dominated by gluon exchange in the $t$ channel

Q For a 4-gluon tree amplitude, we obtain

$$
\begin{aligned}
& \mathcal{M}_{a a^{\prime} b b^{\prime}}^{g g \rightarrow g}(s, t)=2 g_{s}^{2}\left[\left(T^{c}\right)_{a a^{\prime}} C_{\nu_{a} \nu_{a^{\prime}}}\left(p_{a}, p_{a^{\prime}}\right)\right] \frac{s}{t}\left[\left(T_{c}\right)_{b b^{\prime}} C_{\nu_{b} \nu_{b^{\prime}}}\left(p_{b}, p_{b^{\prime}}\right)\right] \\
& C_{\nu_{a} \nu_{a^{\prime}}}\left(p_{a}, p_{a^{\prime}}\right) \text { are called impact factors }
\end{aligned}
$$

Q leading logarithms of $s / t$ are obtained by the substitution

$$
\frac{1}{t} \rightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}
$$

Q $\alpha(t)$ is the Regge gluon trajectory, with infrared coefficients

$$
\begin{array}{lll}
\alpha(t)=\frac{\alpha_{s}(-t, \epsilon)}{4 \pi} \alpha^{(1)}+\left(\frac{\alpha_{s}(-t, \epsilon)}{4 \pi}\right)^{2} \alpha^{(2)}+\mathcal{O}\left(\alpha_{s}^{3}\right) & \alpha_{s}(-t, \epsilon)=\left(\frac{\mu^{2}}{-t}\right)^{\epsilon} \alpha_{s}\left(\mu^{2}\right) \\
\alpha^{(1)}=C_{A} \frac{\widehat{\gamma}_{K}^{(1)}}{\epsilon}=C_{A} \frac{2}{\epsilon} & \alpha^{(2)}=C_{A}\left[-\frac{b_{0}}{\epsilon^{2}}+\widehat{\gamma}_{K}^{(2)} \frac{2}{\epsilon}+C_{A}\left(\frac{404}{27}-2 \zeta_{3}\right)+n_{f}\left(-\frac{56}{27}\right)\right]
\end{array}
$$

Q in the Regge limit, the amplitude is invariant under $s \leftrightarrow u$ exchange.
To NLL accuracy, the amplitude is given by

$$
\mathcal{M}_{a a^{\prime} b b^{\prime}}^{g g \rightarrow g}(s, t)=2 g_{s}^{2} \frac{s}{t}\left[\left(T^{c}\right)_{a a^{\prime}} C_{\nu_{a} \nu_{a^{\prime}}}\left(p_{a}, p_{a^{\prime}}\right)\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)}+\left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[\left(T_{c}\right)_{b b^{\prime}} C_{\nu_{b} \nu_{b^{\prime}}}\left(p_{b}, p_{b^{\prime}}\right)\right]
$$

## Balitski Fadin Kuraev Lipatov

BFKL is a resummation of multiple gluon radiation out of the gluon exchanged in the $t$ channel

the resummation yields an integral (BFKL) equation for the evolution of the gluon propagator in 2-dim transverse momentum space the BFKL equation is obtained in the limit of strong rapidity ordering of the emitted gluons, with no ordering in transverse momentum -multi-Regge kinematics (MRK)
the solution is a Green's function of the momenta flowing in and out of the gluon ladder exchanged in the $t$ channel

## Multi-Regge kinematics in N=4 SYM

9
In the Euclidean region (where all Mandelstam invariants are negative), amplitudes in MRK factorise completely in terms of building blocks which are expressed in terms of Regge poles and can be determined to all orders through the 4-pt and 5-pt amplitudes. Thus the remainder functions $R$ vanish at all points

After analytic continuation to some regions of the Minkowski space, the amplitude develops cuts. The discontinuity of the amplitude is described by a dispersion relation for octet exchange, which is similar to the singlet BFKL equation in QCD

Bartels Lipatov Sabio-Vera 2008
9
Accordingly, 6-pt amplitudes have been thoroughly examined, both at weak and at strong coupling

Basso Caron-Huot Sever 2014

9
In particular, 6-pt amplitudes at weak coupling can be expressed in terms of single-valued harmonic polylogarithms

## Regge factorisation of the n-pt amplitude

$$
\begin{aligned}
& m_{n}(1,2, \ldots, n)=s\left[g C\left(p_{2}, p_{3}\right)\right] \frac{1}{t_{n-3}}\left(\frac{-s_{n-3}}{\tau}\right)^{\alpha\left(t_{n-3}\right)}\left[g V\left(q_{n-3}, q_{n-4}, \kappa_{n-4}\right)\right] \\
& \cdots \times \frac{1}{t_{2}}\left(\frac{-s_{2}}{\tau}\right)^{\alpha\left(t_{2}\right)}\left[g V\left(q_{2}, q_{1}, \kappa_{1}\right)\right] \frac{1}{t_{1}}\left(\frac{-s_{1}}{\tau}\right)^{\alpha\left(t_{1}\right)}\left[g C\left(p_{1}, p_{n}\right)\right]
\end{aligned}
$$

$n$-pt amplitude in the multi-Regge limit

$$
\begin{aligned}
y_{3} & \gg y y_{4} \gg \cdots>y_{n} ; \quad\left|p_{3 \perp}\right| \simeq\left|p_{4 \perp}\right| \ldots \simeq\left|p_{n \perp}\right| \\
s & \gg s_{1}, s_{2}, \ldots, s_{n-3} \gg-t_{1},-t_{2} \ldots,-t_{n-3}
\end{aligned}
$$

the l-loop n-pt amplitude can be assembled using the l-loop trajectories, vertices and coefficient functions, determined through the --loop 4-pt and 5-pt amplitudes
in Euclidean space,
no violation of the BDS ansatz can be found in the multi-Regge limit


## Discontinuity of the amplitude in MRK

6-pt amplitude

continue to a Minkowski region
$s_{34}, s_{56}<0 \quad s, s_{45}>0$
one cross ratio picks up a phase
$u_{1}=\frac{s_{12} s_{45}}{s_{345} s_{456}} \rightarrow\left|u_{1}\right| e^{-2 \pi i}$
compute $\left.\quad \operatorname{Disc}(\mathcal{M})\right|_{s_{45}}$
n-pt amplitude


## Moduli space of Riemann spheres

in MRK, there is no ordering in transverse momentum, i.e. only the $n-2$ transverse momenta are non-trivial
dual conformal invariance in transverse momentum space implies dependence on $n-5$ cross ratios of the transverse momenta


$$
z_{i}=\frac{\left(\mathbf{x}_{1}-\mathbf{x}_{i+3}\right)\left(\mathbf{x}_{i+2}-\mathbf{x}_{i+1}\right)}{\left(\mathbf{x}_{1}-\mathbf{x}_{i+1}\right)\left(\mathbf{x}_{i+2}-\mathbf{x}_{i+3}\right)}=-\frac{\mathbf{q}_{i+1} \mathbf{k}_{i}}{\mathbf{q}_{i-1} \mathbf{k}_{i+1}} \quad i=1, \ldots, n-5
$$

Q $\mathcal{M}_{0, \mathrm{p}}=$ space of configurations of $p$ points on the Riemann sphere Because we can fix 3 points at $0, I, \infty$, its dimension is $\operatorname{dim}\left(\mathscr{M}_{0, \mathrm{p}}\right)=p-3$
$\mathscr{M}_{0, n-2}$ is the space of the MRK, with $\operatorname{dim}\left(\mathscr{M}_{0, n-2}\right)=n-5$
Its coordinates can be chosen to be the $z$ 's, i.e. the cross ratios of the transverse momenta

VDD Drummond Druc Duhr Dulat Marzucca Papathanasiou Verbeek 2016
on $\mathscr{M}_{0, n-2,}$ the singularities are associated to degenerate configurations
when two points merge $x_{i} \rightarrow x_{i+1}$
i.e. when momentum $p_{i}$ becomes soft $p_{i} \rightarrow 0$

## Iterated integrals on $\mathbb{M}_{0, n-2}$

Q iterated integrals on $\mathscr{M}_{0, \mathrm{p}}$ can be written as multiple polylogarithms
amplitudes in MRK can be written in terms of multiple polylogarithms
unitarity implies that for massless amplitudes
$\Delta(M)=\ln \left(s_{i j}\right) \otimes \ldots$
in particular, for amplitudes in MRK
$\Delta(M)=\ln \left|\mathbf{x}_{i}-\mathbf{x}_{j}\right|^{2} \otimes \ldots$
Q except for the soft limit $p_{i} \rightarrow 0$, in MRK the transverse momenta never vanish $\left|\mathrm{x}_{i}-\mathrm{x}_{j}\right|^{2} \neq 0 \Longrightarrow$ single-valued functions

Q therefore, $n$-point amplitudes in MRK of planar N=4 SYM can be written in terms of single-valued iterated integrals on $\mathscr{N}_{0, n-2}$

VDD Drummond Druc Duhr Dulat Marzucca Papathanasiou Verbeek 2016
Q for $\mathrm{n}=6$, iterated integrals on $\mathscr{M}_{0,4}$ are harmonic polylogarithms thus, 6-point amplitudes in MRK of can be written in terms of single-valued harmonic polylogarithms (SVHPL)

## MRK in $N=4$ SYM

In MRK, 6-pt MHV and NMHV amplitudes are known at any number of loops
Lipatov Prygarin 2010-2011
Dixon Duhr Pennington 2012
Lipatov Prygarin Schnitzer 2012
knowing the space of functions of the n-point amplitudes in MRK, (i.e. that is made of single-valued iterated integrals on $\mathscr{M}_{0, n-2}$ ) allowed us to compute all MHV amplitudes at $\ell$ loops in LLA in terms of amplitudes with up to ( $\ell+4$ ) points, in practice up to 5 loops, and all non-MHV amplitudes in LLA up 8 points and 4 loops

VDD Drummond Druc Duhr Dulat Marzucca Papathanasiou Verbeek 2016
for MHV amplitudes in MRK at LLA at:

- at 2 loop, the $n$-pt remainder function $R_{n}{ }^{(2)}$ can be written as a sum of 2-loop 6-pt remainder functions $R_{6}{ }^{(2)}$

Prygarin Spradlin VerguVolovich 2011
Bartels Kormilitzin Lipatov Prygarin 2011
Bargheer Papathanasiou Schomerus 2015

- at 5 loops, the n-pt remainder function $R_{n}{ }^{(5)}$ can be written as a sum of 5-loop 6-, 7-, 8- and 9-pt amplitudes

VDD Drummond Druc Duhr Dulat Marzucca Papathanasiou Verbeek 2016
... extended to 7-pt (N)MHV amplitudes at 5(4) loops in NLLA

## Single-valued polylogarithms

Q Single-valued functions are real analytic functions on the complex plane

Q Because the discontinuities of the classical polylogarithms are known

$$
\Delta \mathrm{Li}_{n}(z)=2 \pi i \frac{\log ^{n-1} z}{(n-1)!}
$$

one can build combinations of classical polylogarithms such that all branch cuts cancel on the punctured plane $\mathrm{C} /\{0, \mathrm{I}\}$ (Riemann sphere with punctures)

Q An example is the Bloch-Wigner dilogarithm
$S V P_{2}(z)=\operatorname{Im}\left[\operatorname{Li}_{2}(z)\right]-\log |z| \arg (1-z)$
in general

$$
S V P_{n}(z)=R_{n}\left[\sum_{k=0}^{n-1} \frac{2^{k} B_{k}}{k!} \log ^{k}|z| \mathrm{Li}_{n-k}(z)\right] \quad B_{k} \text { Bernouilli numbers } \quad R_{n}= \begin{cases}\operatorname{Re} & \text { odd } n \\ \operatorname{Im} & \text { even } n\end{cases}
$$

## Single-valued harmonic polylogarithms

Q define a function $\mathscr{L}$ that is real-analytic and single-valued on $\mathbb{C} /\{0,1\}$ and that has the same properties as the HPLs
the SVHPLs $\mathscr{L}_{\omega}(\mathrm{z})$ also form a shuffle algebra
$\mathcal{L}_{\omega_{1}}(z) \mathcal{L}_{\omega_{2}}(z)=\sum_{\omega} \mathcal{L}_{\omega}(z) \quad$ with $\omega$ the shuffle of $\omega_{\text {I }}$ and $\omega_{2}$
Q SVHPLs can be explicitly expressed as combinations of HPLs such that all the branch cuts cancel

$$
\begin{aligned}
\mathcal{L}_{0}(z) & =H_{0}(z)+H_{0}(\bar{z})=\ln |z|^{2} \\
\mathcal{L}_{1}(z) & =H_{1}(z)+H_{1}(\bar{z})=-\ln |1+z|^{2} \\
\mathcal{L}_{0,1}(z) & =\frac{1}{4}\left[-2 H_{1,0}+2 \bar{H}_{1,0}+2 H_{0} \bar{H}_{1}-2 \bar{H}_{0} H_{1}+2 H_{0.1}-2 \bar{H}_{0,1}\right] \\
& =\operatorname{Li}_{2}(z)-\operatorname{Li}_{2}(\bar{z})+\frac{1}{2} \ln |z|^{2}(\ln (1-z)-\ln (1-\bar{z}))
\end{aligned}
$$

## Single-valued multiple polylogarithms

Q Single-valued multiple polylogarithms (SVMPL) can be constructed through a map that to each multiple polylogarithm associates its single-valued version

Brown 2004, 2013, 2015
examples of SVMPLs

$$
\begin{aligned}
\mathcal{G}_{a}(z)= & G_{a}(z)+G_{\bar{a}}(\bar{z})=\ln \left|1-\frac{z}{a}\right|^{2} \\
\mathcal{G}_{a, b}(z) & =G_{a, b}(z)+G_{\bar{b}, \bar{a}}(\bar{z})+G_{b}(a) G_{\bar{a}}(\bar{z})+G_{\bar{b}}(\bar{a}) G_{\bar{a}}(\bar{z}) \\
& -G_{a}(b) G_{\bar{b}}(\bar{z})+G_{a}(z) G_{\bar{b}}(\bar{z})-G_{\bar{a}}(\bar{b}) G_{\bar{b}}(\bar{z})
\end{aligned}
$$

Q 3-mass triangles with massless propagators


$$
z \bar{z}=\frac{p_{1}^{2}}{p_{3}^{2}} \quad(1-z)(1-\bar{z})=\frac{p_{2}^{2}}{p_{3}^{2}}
$$

can be written in terms of SVMPLs

Q IR structure of a QCD amplitude with $n$ massless partons

$$
\mathcal{M}_{n}\left(\left\{p_{i}\right\}, \alpha_{s}\right)=Z_{n}\left(\left\{p_{i}\right\}, \alpha_{s}, \mu\right) \mathcal{H}_{n}\left(\left\{p_{i}\right\}, \alpha_{s}, \mu\right)
$$

$Z_{n}$ is solution to the RGE equation

$$
Z_{n}=\mathrm{P} \exp \left\{-\frac{1}{2} \int_{0}^{\mu^{2}} \frac{d \lambda^{2}}{\lambda^{2}} \Gamma_{n}\left(\left\{p_{i}\right\}, \lambda, \alpha_{s}\left(\lambda^{2}\right)\right)\right\}
$$

$\Gamma_{n}$ is the soft anomalous dimension

$$
\begin{gathered}
\Gamma_{n}\left(\left\{p_{i}\right\}, \lambda, \alpha_{s}\right)=\Gamma_{n}^{\operatorname{dip}}\left(\left\{p_{i}\right\}, \lambda, \alpha_{s}\right)+\Delta_{n}\left(\left\{\rho_{i j k l}\right\}, \alpha_{s}\right) \\
\Gamma_{n}^{\mathrm{dip}}\left(\left\{p_{i}\right\}, \lambda, \alpha_{s}\right)=-\frac{1}{2} \hat{\gamma}_{K}\left(\alpha_{s}\right) \sum_{i<j} \log \left(\frac{-s_{i j}}{\lambda^{2}}\right) \mathbf{T}_{\mathbf{i}} \cdot \mathbf{T}_{\mathbf{j}}+\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} \gamma_{\mathbf{J}_{\mathbf{i}}}\left(\alpha_{\mathbf{s}}\right) \\
\rho_{i j k l}=\frac{\left(-s_{i j}\right)\left(-s_{k l}\right)}{\left(-s_{i k}\right)\left(-s_{j l}\right)} \quad \text { Becher Neubert; Gardi Magnea } 2009
\end{gathered}
$$

At 2 loops, $\Delta^{(2)}=0, \Gamma_{2}$ : Catani 1998; Aybat Dixon Sterman 2006
At 3 loops,

$$
\begin{aligned}
& \text { lt } 3 \text { loops, } \quad \Delta_{4}^{(3)}\left(\rho_{1234}, \rho_{1432}, \alpha_{s}\right)=16 \mathbf{T}_{1}^{a_{1}} \mathbf{T}_{2}^{a_{2}} \mathbf{T}_{3}^{a_{3}} \mathbf{T}_{4}^{a_{4}}\left\{f^{a_{1} a_{2} b} f^{a_{3} a_{4} b}\left[F\left(1-\frac{1}{z}\right)-F\left(\frac{1}{z}\right)\right]\right. \\
& \left.+f^{a_{1} a_{3} b} f^{a_{4} a_{2} b}[F(z)-F(1-z)]+f^{a_{1} a_{4} b} f^{a_{2} a_{3} b}\left[F\left(\frac{1}{1-z}\right)-F\left(\frac{z}{z-1}\right)\right]\right\} \\
& \rho_{1234}=z \bar{z} \quad \rho_{1432}=(1-z)(1-\bar{z}) \quad F(z)=\mathcal{L}_{10101}(z)+2 \zeta_{2}\left[\mathcal{L}_{001}(z)+\mathcal{L}_{100}(z)\right]+6 \zeta_{4} \mathcal{L}_{1}(z)
\end{aligned}
$$

is given in terms of SVHPLs

## Mueller-Navelet jets



Dijet production cross section with two tagging jets in the forward and backward directions
$p_{a}=x_{a} P_{A} \quad p_{b}=x_{b} P_{B}$ incoming parton momenta
$S$ : hadron centre-of-mass energy
$s=x_{a} x_{b} S$ : parton centre-of-mass energy
$E_{T j}$; jet transverse energies
$\Delta y=\left|y_{j_{1}}-y_{j_{2}}\right| \simeq \log \frac{s}{E_{T j_{1}} E_{T j_{2}}}$
is the rapidity interval between the tagging jets gluon radiation is considered in MRK and resummed through the LL BFKL equation

Mueller-Navelet evaluated the inclusive dijet cross section up to 5 loops

## Mueller-Navelet dijet cross section

9
the cross section for dijet production at large rapidity intervals

$$
\Delta y=y_{1}-y_{2}=\ln \left(\frac{\hat{s}}{-t}\right) \gg 1
$$

$$
\text { with } \quad \hat{s}=x_{a} x_{b} S, \quad t=-\sqrt{p_{1 \perp}^{2} p_{2 \perp}^{2}}
$$

$$
\chi_{\nu, n}
$$

with $\quad \eta \equiv \frac{C_{A} \alpha_{s}}{\pi} \Delta y \quad$ and $\phi$ the angle between $\mathrm{q}_{1}{ }^{2}$ and $\mathrm{q}_{2}{ }^{2}$
and the LL BFKL eigenvalue

$$
\chi_{\nu, n}=-2 \gamma_{E}-\psi\left(\frac{1}{2}+\frac{|n|}{2}+i \nu\right)-\psi\left(\frac{1}{2}+\frac{|n|}{2}-i \nu\right)
$$

## Mueller-Navelet dijet cross section

Q azimuthal angle distribution ( $\phi_{\mathrm{ij}}=\phi-\pi$ )

$$
\begin{aligned}
& \frac{d \hat{\sigma}_{g g}}{d \phi_{j j}}=\frac{\pi\left(C_{A} \alpha_{s}\right)^{2}}{2 E_{\perp}^{2}}\left[\delta\left(\phi_{j j}-\pi\right)+\sum_{k=1}^{\infty}\left(\sum_{n=-\infty}^{\infty} \frac{e^{i n \phi}}{2 \pi} f_{n, k}\right) \eta^{k}\right] \\
& \text { with } \quad f_{n, k}=\frac{1}{2 \pi} \frac{1}{k!} \int_{-\infty}^{\infty} d \nu \frac{\chi_{\nu, n}^{k}}{\nu^{2}+\frac{1}{4}}
\end{aligned}
$$

Q the dijet cross section is $\quad \hat{\sigma}_{g g}=\frac{\pi\left(C_{A} \alpha_{s}\right)^{2}}{2 E_{\perp}^{2}} \sum_{k=0}^{\infty} f_{0, k} \eta^{k}$

$$
\begin{aligned}
f_{0,0} & =1 \\
f_{0,1} & =0 \\
f_{0,2} & =2 \zeta_{2} \\
f_{0,3} & =-3 \zeta_{3} \\
f_{0,4} & =\frac{53}{6} \zeta_{4} \\
f_{0,5} & =-\frac{1}{12}\left(115 \zeta_{5}+48 \zeta_{2} \zeta_{3}\right)
\end{aligned}
$$

## Mueller-Navelet jets and SVHPLs

The singlet LL BFKL ladder in QCD, and thus the dijet cross section in the high-energy limit, can also be expressed in terms of SVHPLs, i.e. in terms of single-valued iterated integrals on $\mathscr{M}_{0,4}$

Dixon Duhr Pennington VDD 2013
Mueller \& Navelet evaluated analytically the inclusive dijet cross section up to 5 loops. We evaluated it analytically up to 13 loops

Also, we could evaluate analytically the dijet cross section differential in the jet transverse energies or the azimuthal angle between the jets (up to 6 loops)

## BFKL Green's function and single-valued functions

use complex transverse momentum

$$
\tilde{q}_{k} \equiv q_{k}^{x}+i q_{k}^{y}
$$

and a complex variable $\quad z \equiv \frac{\tilde{q}_{1}}{\tilde{q}_{2}}$
the BFKL Green's function can be expanded into a power series in $\eta_{\mu}=\bar{\alpha}_{\mu} y$

$$
f^{L L}\left(q_{1}, q_{2}, \eta_{\mu}\right)=\frac{1}{2} \delta^{(2)}\left(q_{1}-q_{2}\right)+\frac{1}{2 \pi \sqrt{q_{1}^{2} q_{2}^{2}}} \sum_{k=1}^{\infty} \frac{\eta_{\mu}^{k}}{k!} f_{k}^{L L}(z)
$$

where the coefficient functions $f_{k}$ are given by the Fourier-Mellin transform

$$
f_{k}^{L L}(z)=\mathcal{F}\left[\chi_{\nu n}^{k}\right]=\sum_{n=-\infty}^{+\infty}\left(\frac{z}{\bar{z}}\right)^{n / 2} \int_{-\infty}^{+\infty} \frac{d \nu}{2 \pi}|z|^{2 i \nu} \chi_{\nu n}^{k}
$$

(3) the $f_{k}$ have a unique, well-defined value for every ratio of the magnitudes of the two jet transverse momenta and angle between them.
So, they are real-analytic functions of $z$

## Azimuthal angle distribution

Q this allows us to write the azimuthal angle distribution as

$$
\frac{d \hat{\sigma}_{g g}}{d \phi_{j j}}=\frac{\pi\left(C_{A} \alpha_{s}\right)^{2}}{2 E_{\perp}^{2}}\left[\delta\left(\phi_{j j}-\pi\right)+\sum_{k=1}^{\infty} \frac{a_{k}\left(\phi_{j j}\right)}{\pi} \eta^{k}\right]
$$

where the contribution of the $k^{\text {th }}$ loop is

$$
a_{k}\left(\phi_{j j}\right)=\int_{0}^{\infty} \frac{d|w|}{|w|} f_{k}\left(w, w^{*}\right)=\frac{\operatorname{Im} A_{k}\left(\phi_{j j}\right)}{\sin \phi_{j j}}
$$

with

$$
\begin{aligned}
A_{1}\left(\phi_{j j}\right)= & -\frac{1}{2} H_{0}, \\
A_{2}\left(\phi_{j j}\right)= & H_{1,0}, \\
A_{3}\left(\phi_{j j}\right)= & \frac{2}{3} H_{0,0,0}-2 H_{1,1,0}+\frac{5}{3} \zeta_{2} H_{0}-i \pi \zeta_{2}, \\
A_{4}\left(\phi_{j j}\right)= & -\frac{4}{3} H_{0,0,1,0}-H_{0,1,0,0}-\frac{4}{3} H_{1,0,0,0}+4 H_{1,1,1,0}-\zeta_{2}\left(2 H_{0,1}+\frac{10}{3} H_{1,0}\right)+\frac{4}{3} \zeta_{3} H_{0}+i \pi\left(2 \zeta_{2} H_{1}-2 \zeta_{3}\right), \\
A_{5}\left(\phi_{j j}\right)= & -\frac{46}{15} H_{0,0,0,0,0}+\frac{8}{3} H_{0,0,1,1,0}+2 H_{0,1,0,1,0}+2 H_{0,1,1,0,0}+\frac{8}{3} H_{1,0,0,1,0}+2 H_{1,0,1,0,0} \\
& +\frac{8}{3} H_{1,1,0,0,0}-8 H_{1,1,1,1,0}-\zeta_{2}\left(\frac{33}{5} H_{0,0,0}-4 H_{0,1,1}-4 H_{1,0,1}-\frac{20}{3} H_{1,1,0}\right) \\
& -\zeta_{3}\left(2 H_{0,1}+\frac{8}{3} H_{1,0}\right)+\frac{217}{15} \zeta_{4} H_{0}+i \pi\left[\zeta_{2}\left(\frac{10}{3} H_{0,0}-4 H_{1,1}\right)+4 \zeta_{3} H_{1}-\frac{10}{3} \zeta_{4}\right]
\end{aligned}
$$

where $\quad H_{i, j, \ldots} \equiv H_{i, j, \ldots}\left(e^{-2 i \phi_{j j}}\right)$

## Mueller-Navelet dijet cross section reloaded

9
the MN dijet cross section is $\quad \hat{\sigma}_{g g}=\frac{\pi\left(C_{A} \alpha_{s}\right)^{2}}{2 E_{\perp}^{2}} \sum_{k=0}^{\infty} f_{0, k} \eta^{k}$
the first 5 loops were computed by Mueller-Navelet.
Dixon Duhr Pennington VDD 2013
We computed it through the 13 loops

$$
\begin{aligned}
f_{0,6} & =\frac{13}{4} \zeta_{3}^{2}+\frac{3737}{120} \zeta_{6}, \\
f_{0,7} & =-\frac{87}{5} \zeta_{3} \zeta_{4}-\frac{116}{9} \zeta_{2} \zeta_{5}-\frac{3983}{144} \zeta_{7}, \\
f_{0,8} & =-\frac{37}{75} \zeta_{5,3}+\frac{64}{15} \zeta_{2} \zeta_{3}^{2}+\frac{369}{20} \zeta_{5} \zeta_{3}+\frac{50606057}{453600} \zeta_{8}, \\
f_{0,9} & =-\frac{139}{60} \zeta_{3}^{3}-\frac{15517}{252} \zeta_{6} \zeta_{3}-\frac{3533}{63} \zeta_{4} \zeta_{5}-\frac{557}{15} \zeta_{2} \zeta_{7}-\frac{5215361}{60480} \zeta_{9}, \\
f_{0,10} & =-\frac{2488}{4725} \zeta_{5,3} \zeta_{2}-\frac{94721}{211680} \zeta_{7,3}+\frac{1948}{105} \zeta_{4} \zeta_{3}^{2}+\frac{2608}{105} \zeta_{2} \zeta_{5} \zeta_{3}+\frac{12099}{224} \zeta_{7} \zeta_{3}+\frac{1335931}{47040} \zeta_{5}^{2}+\frac{25669936301}{63504000} \zeta_{1 c} \\
f_{0,11} & =\frac{62}{315} \zeta_{5,3} \zeta_{3}+\frac{83}{120} \zeta_{5,3,3}-\frac{2872}{945} \zeta_{2} \zeta_{3}^{3}-\frac{13211}{672} \zeta_{5} \zeta_{3}^{2}-\frac{661411}{3024} \zeta_{8} \zeta_{3} \\
& -\frac{242776937}{725760} \zeta_{11}-\frac{605321}{3024} \zeta_{5} \zeta_{6}-\frac{2583643}{16200} \zeta_{4} \zeta_{7}-\frac{28702763}{340200} \zeta_{2} \zeta_{9}, \\
f_{0,12} & =\frac{74711}{162000} \zeta_{5,3} \zeta_{4}-\frac{13793}{7560} \zeta_{6,4,1,1}+\frac{3965011}{793800} \zeta_{7,3} \zeta_{2}-\frac{33356851}{4082400} \zeta_{9,3} \\
& +\frac{252163}{181440} \zeta_{3}^{4}+\frac{620477}{10080} \zeta_{6} \zeta_{3}^{2}+\frac{8101339}{75600} \zeta_{4} \zeta_{5} \zeta_{3}+\frac{342869}{3780} \zeta_{2} \zeta_{7} \zeta_{3} \\
& +\frac{101571047}{680400} \zeta_{9} \zeta_{3}+\frac{71425871}{1587600} \zeta_{2} \zeta_{5}^{2}+\frac{904497401571619}{620606448000} \zeta_{12}+\frac{484414571}{2721600} \zeta_{5} \zeta_{7}, \\
f_{0,13} & =\frac{4513}{1890} \zeta_{5,3} \zeta_{5}+\frac{27248}{23625} \zeta_{5,3,3} \zeta_{2}-\frac{97003}{235200} \zeta_{5,5,3}+\frac{13411}{75600} \zeta_{7,3} \zeta_{3} \\
& +\frac{7997743}{12700800} \zeta_{7,3,3}-\frac{187318}{14175} \zeta_{4} \zeta_{3}^{3}-\frac{125056}{4725} \zeta_{2} \zeta_{5} \zeta_{3}^{2}-\frac{17411413}{302400} \zeta_{7} \zeta_{3}^{2} \\
& -\frac{5724191}{100800} \zeta_{5}^{2} \zeta_{3}-\frac{1874972477}{2376000} \zeta_{10} \zeta_{3}-\frac{2418071698069}{2235340800} \zeta_{13} \\
& -\frac{2379684877}{6048000} \zeta_{11} \zeta_{2}-\frac{297666465053}{523908000} \zeta_{6} \zeta_{7}-\frac{1770762319}{2494800} \zeta_{5} \zeta_{8}-\frac{229717224973}{628689600} \zeta_{4} \zeta_{9}
\end{aligned}
$$

## Regge limit in the next-to-leading logarithmic approximation

## BFKL eigenvalue at NLLA

At NLLA in QCD and in $\mathrm{N}=4 \mathrm{SYM}$, the eigenvalue is

$$
\omega_{\nu n}^{(1)}=\frac{1}{4} \delta_{\nu n}^{(1)}+\frac{1}{4} \delta_{\nu n}^{(2)}+\frac{1}{4} \delta_{\nu n}^{(3)}+\gamma_{K}^{(2)} \chi_{\nu n}-\frac{1}{8} \beta_{0} \chi_{\nu n}^{2}+\frac{3}{2} \zeta_{3}
$$

with one-loop beta function and two-loop cusp anomalous dimension

$$
\beta_{0}=\frac{11}{3}-\frac{2 N_{f}}{3 N_{c}} \quad \gamma_{K}^{(2)}=\frac{1}{4}\left(\frac{64}{9}-\frac{10 N_{f}}{9 N_{c}}\right)-\frac{\zeta_{2}}{2}
$$

and with

$$
\begin{aligned}
\delta_{\nu n}^{(1)}= & \partial_{\nu}^{2} \chi_{\nu n} \\
\delta_{\nu n}^{(2)}= & -2 \Phi(n, \gamma)-2 \Phi(n, 1-\gamma) \quad \chi_{\nu n}=\omega_{\nu n}^{(0)} \\
\delta_{\nu n}^{(3)}= & -\frac{\Gamma\left(\frac{1}{2}+i \nu\right) \Gamma\left(\frac{1}{2}-i \nu\right)}{2 i \nu}\left[\psi\left(\frac{1}{2}+i \nu\right)-\psi\left(\frac{1}{2}-i \nu\right)\right] \\
& \times\left[\delta_{n 0}\left(3+\left(1+\frac{N_{f}}{N_{c}^{3}}\right) \frac{2+3 \gamma(1-\gamma)}{(3-2 \gamma)(1+2 \gamma)}\right)-\delta_{|n| 2}\left(\left(1+\frac{N_{f}}{N_{c}^{3}}\right) \frac{\gamma(1-\gamma)}{2(3-2 \gamma)(1+2 \gamma)}\right)\right]
\end{aligned}
$$

$\Phi(n, \gamma)$ is a sum over linear combinations of $\psi$ functions and $\gamma$ is a shorthand $\gamma=1 / 2+\mathrm{iv}$

In blue we labeled the terms which occur only in QCD, in red the ones which occur in QCD and in N=4 SYM

## Fourier-Mellin transform

At NLLA, the BFKL gluon ladder is

$$
f^{N L L}\left(q_{1}, q_{2}, \eta_{s_{0}}\right)=\frac{1}{2 \pi \sqrt{q_{1}^{2} q_{2}^{2}}} \sum_{k=1}^{\infty} \frac{\eta_{s_{0}}^{k}}{k!} f_{k+1}^{N L L}(z) \quad \eta_{s_{0}}=\bar{\alpha}_{S}\left(s_{0}\right) y
$$

with coefficients given by the Fourier-Mellin transform

$$
f_{k}^{N L L}(z)=\mathcal{F}\left[\omega_{\nu n}^{(1)} \chi_{\nu n}^{k-2}\right]=\sum_{n=-\infty}^{+\infty}\left(\frac{z}{\bar{z}}\right)^{n / 2} \int_{-\infty}^{+\infty} \frac{d \nu}{2 \pi}|z|^{2 i \nu} \omega_{\nu n}^{(1)} \chi_{\nu n}^{k-2} \quad \chi_{\nu n}=\omega_{\nu n}^{(0)}
$$

using the explicit form of the eigenvalue

$$
\omega_{\nu n}^{(1)}=\frac{1}{4} \delta_{\nu n}^{(1)}+\frac{1}{4} \delta_{\nu n}^{(2)}+\frac{1}{4} \delta_{\nu n}^{(3)}+\gamma_{K}^{(2)} \chi_{\nu n}-\frac{1}{8} \beta_{0} \chi_{\nu n}^{2}+\frac{3}{2} \zeta_{3}
$$

the coefficients can be written as

$$
f_{k}^{N L L}(z)=\frac{1}{4} C_{k}^{(1)}(z)+\frac{1}{4} C_{k}^{(2)}(z)+\frac{1}{4} C_{k}^{(3)}(z)+\gamma_{K}^{(2)} f_{k-1}^{L L}(z)-\frac{1}{8} \beta_{0} f_{k}^{L L}(z)+\frac{3}{2} \zeta_{3} f_{k-2}^{L L}(z)
$$

with $\quad C_{k}^{(i)}(z)=\mathcal{F}\left[\delta_{\nu n}^{(i)} \chi_{\nu n}^{k-2}\right]$
the weight of $f_{N L L}^{k}$ is
weight $\left(f{ }^{N L L} L_{k}\right)=k \quad k \quad 0 \leq w \leq k \quad k-2 \leq w \leq k \quad k-1 \quad k$

## generalised SVMPLs

9
$C_{k}^{(1)}(z) \quad$ are SVHPLs of uniform weight k with singularities at $\mathrm{z}=0$ and $\mathrm{z}=1$
$C_{k}^{(3)}(z) \quad$ are MPLs of type $G\left(a_{1}, \ldots, a_{n} ;|z|\right)$ with $a_{k} \in\{-i, 0, i\}$
they are SV functions of $z$ because they have no branch cut on the positive real axis, and have weight $0 \leq w \leq k$

For $C_{k}^{(2)}(z)$ one needs Schnetz' generalised SVMPLs with singularities at

$$
z=\frac{\alpha \bar{z}+\beta}{\gamma \bar{z}+\delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}
$$

then one can show that $C_{k}^{(2)}(z)$ are Schnetz' generalised SVMPLs

$$
\mathcal{G}\left(a_{1}, \ldots, a_{n} ; z\right) \quad \text { with singularities at } \quad a_{i} \in\{-1,0,1,-1 / \bar{z}\}
$$

In moment space, the maximal weight of the BFKL eigenvalue and of the anomalous dimensions of the leading twist operators which control the Bjorken scaling violations in QCD is the same as the corresponding quantities in $\mathrm{N}=4$ SYM (Principle of Maximal Transcendentality)

Interestingly, in transverse momentum space at NLLA, the maximal weight of the BFKL ladder in QCD is not the same as the one of the ladder in $N=4$ SYM

## BFKL ladder in a generic $\mathrm{SU}\left(\mathrm{N}_{\mathrm{c}}\right)$ gauge theory

9
one can consider the BFKL eigenvalue at NLLA in a $\operatorname{SU}\left(\mathrm{N}_{\mathrm{c}}\right)$ gauge theory with scalar or fermionic matter in arbitrary representations

$$
\omega_{\nu n}^{(1)}=\frac{1}{4} \delta_{\nu n}^{(1)}+\frac{1}{4} \delta_{\nu n}^{(2)}+\frac{1}{4} \delta_{\nu n}^{(3)}\left(\tilde{N}_{f}, \tilde{N}_{s}\right)+\frac{3}{2} \zeta_{3}+\gamma^{(2)}\left(\tilde{n}_{f}, \tilde{n}_{s}\right) \chi_{\nu n}-\frac{1}{8} \beta_{0}\left(\tilde{n}_{f}, \tilde{n}_{s}\right) \chi_{\nu n}^{2}
$$

Kotikov Lipatov 2000
with $\beta_{0}\left(\tilde{n}_{f}, \tilde{n}_{s}\right)=\frac{11}{3}-\frac{2 \tilde{n}_{f}}{3 N_{c}}-\frac{\tilde{n}_{s}}{6 N_{c}}$

$$
\gamma^{(2)}\left(\tilde{n}_{f}, \tilde{n}_{s}\right)=\frac{1}{4}\left(\frac{64}{9}-\frac{10 \tilde{n}_{f}}{9 N_{c}}-\frac{4 \tilde{n}_{s}}{9 N_{c}}\right)-\frac{\zeta_{2}}{2}
$$

$$
\tilde{n}_{f}=\sum_{R} n_{f}^{R} T_{R} \quad \tilde{n}_{s}=\sum_{R} n_{s}^{R} T_{R} \quad \operatorname{Tr}\left(T_{R}^{a} T_{R}^{b}\right)=T_{R} \delta^{a b} \quad T_{F}=\frac{1}{2}
$$

$\tilde{n}_{s}\left(\tilde{n}_{f}\right)=$ number of scalars (Weyl fermions) in the representation $R$
$\delta_{\nu n}^{(3)}\left(\tilde{N}_{f}, \tilde{N}_{s}\right)=\delta_{\nu n}^{(3,1)}\left(\tilde{N}_{f}, \tilde{N}_{s}\right)+\delta_{\nu n}^{(3,2)}\left(\tilde{N}_{f}, \tilde{N}_{s}\right)$
with $\quad \tilde{N}_{x}=\frac{1}{2} \sum_{R} n_{x}^{R} T_{R}\left(2 C_{R}-N_{c}\right), \quad x=f, s$
Necessary and sufficient conditions for a $\operatorname{SU}\left(\mathrm{N}_{\mathrm{c}}\right)$ gauge theory to have a BFKL ladder of maximal weight are:

- the one-loop beta function must vanish
- the two-loop cusp AD must be proportional to $\zeta_{2}$
- $\delta_{\nu n}^{(3,2)}$ must vanish $\rightarrow 2 \tilde{N}_{f}=N_{c}^{2}+\tilde{N}_{s}$

There is no theory whose BFKL ladder has uniform maximal weight which agrees with the maximal weight terms of QCD

## Matter in the fundamental and in the adjoint

We solve the conditions above for matter in the fundamental $F$ and in the adjoint $A$ representations.We obtain:

$$
2 n_{f}^{F}=n_{s}^{F} \quad 2 n_{f}^{A}=2+n_{s}^{A}
$$

which describes the spectrum of a gauge theory with $N$ supersymmetries and $n^{F}=n_{f}^{F}$ chiral multiplets in $F$ and $n^{A}=n_{f}^{A}-N$ chiral multiplets in $A$

There are four solutions to those conditions

| $\mathcal{N}$ | 4 | 2 | 1 | 1 |
| :---: | :---: | :---: | :---: | :---: |
| $n_{A}$ | 0 | 0 | 0 | 2 |
| $n_{F}$ | 0 | $1 N_{c}$ | $6 N_{c}$ | $2 N_{c}$ |

— the first is $N=4$ SYM
— the second is $N=2$ superconformal QCD with $N_{f}=2 N_{c}$ hypermultiplets

- the third is $N=I$ superconf. QCD

Caveat:
because the one-loop beta function is fixed by matter loops in gluon self-energies, we are only sensitive to the matter content of a theory, and not to its details (like scalar potential or Yukawa couplings)

## Hic sunt leones ...

## Elliptic iterated integrals

Q 2-loop sunrise graph


Broadhurst I989; ...; Bloch Vanhove 2013; ...

0
2-loop 3-pt functions
electroweak form factor


Aglietti Bonciani Grassi Remiddi 2007
Q 2-loop 4-pt function for Higgs +1 jet


## massless elliptic iterated integrals

2-loop 10-pt N3MHV amplitude in planar N=4 SYM


Bourjaily McLeod Spradlin von Hippel Wilhelm 2017
traintracks:
L-loop Feynman integrals involving ( $2 L+6$ ) massless legs they occur in massless $\varphi^{4}$ and in planar $N=4$ SYM


Q iterated integrals on $\mathscr{N}_{0, \mathrm{p}}$ are multiple polylogarithms
$\mathscr{M}_{0, \mathrm{p}}=$ space of configurations of $p$ points on the Riemann sphere
$G(a, \vec{w} ; z)=\int_{0}^{z} \frac{\mathrm{~d} t}{t-a} G(\vec{w} ; t), \quad G(a ; z)=\ln \left(1-\frac{z}{a}\right)$

$$
a, \vec{w} \in \mathbb{C}
$$

Q iterated integrals on a torus ...
Brown Levin 2011
$\tilde{\Gamma}\left(\begin{array}{c}n_{1} \ldots n_{k} \\ z_{1} \ldots z_{k}\end{array} ; z, \tau\right)=\int_{0}^{z} d t g^{\left(n_{1}\right)}\left(t-z_{1}, \tau\right) \tilde{\Gamma}\left(\begin{array}{c}n_{2} \ldots n_{k} \\ z_{2} \ldots z_{k}\end{array} ; t, \tau\right)$

$$
n_{i} \in \mathbb{N}, \quad z_{i} \in \mathbb{C}
$$

with kernels defined through the Eisenstein-Kronecker series
$F(z, \alpha, \tau)=\frac{1}{\alpha} \sum_{n=0}^{\infty} g^{(n)}(z, \tau) \alpha^{n}=\frac{\theta_{1}^{\prime}(0, \tau) \theta_{1}(z+\alpha, \tau)}{\theta_{1}(z, \tau) \theta_{1}(\alpha, \tau)}$

$\theta_{\text {I }}$ Jacobi theta function; $g^{(n)}$ has at most simple poles at $z=m+n \tau$

$$
m, n \in \mathbb{Z}
$$

... are elliptic multiple polylogarithms (eMPL)
$E_{3}\left(\begin{array}{c}n_{1} \ldots n_{k} \\ z_{1} \ldots z_{k}\end{array} ; z, \vec{a}\right)=\int_{0}^{z} d t \varphi_{n_{1}}\left(z_{1}, t, \vec{a}\right) E_{3}\left(\begin{array}{c}n_{2} \ldots n_{k} \\ z_{2} \ldots z_{k}\end{array} ; t, \vec{a}\right) \quad n_{i} \in \mathbb{Z}, \quad z_{i} \in \mathbb{C} \quad a_{i} \in \mathbb{R}$
with $\quad \vec{a}=\left(a_{1}, a_{2}, a_{3}\right)$ are the zeroes of the elliptic curve $y^{2}=\left(x-a_{1}\right)\left(x-a_{2}\right)\left(x-a_{3}\right)$
and $E_{3}(; z, \vec{a})=1$
Q 2-loop sunrise can be written in terms of eMPLs

## Conclusions

Q
In the last few years, a lot of progress has been made in understanding the analytic structure of multi-loop amplitudes
we understand the analytic and algebraic properties of amplitudes, when they are written in terms of MPLs and/or SVMPLs
an in-depth exploration of how elliptic iterated integrals arise at 2 loops and beyond has just begun

