Planar $\mathcal{N} = 4$ Wilson loops/Amplitudes and Integrability

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$\mathcal{N}=4$ Super Yang-Mills

▷ Supersymmetric gauge theory $SU(N)$, $\mathcal{N}=4 \Rightarrow$ Maximal amount of supersymmetries in 4d.

$$\boxed{\mathcal{L} = Tr\left[-\frac{1}{2g_{YM}^2}F_{\mu\nu}F^{\mu\nu} - D_\mu\Phi_A D^\mu\Phi_A - i\bar{\Psi}^A\sigma^\mu D_\mu\Psi_A + \text{interactions}\right]}$$

$\Longrightarrow$ *Fields*: Gluon A_μ, 4 Fermions Ψ^B_α, $\bar{\Psi}^B_{\dot{\alpha}}$, 6 Scalars $\Phi^A \Rightarrow$ adjoint of $SU(N)$

▷ Properties:

- Conformal symmetry at the quantum level ($\beta = 0$)

  $\Longrightarrow$ Full symmetry group $PSU(2,2|4)$
- $SU(4)$ R-symmetry representations, $(A_\mu, \Psi^B_\alpha, \bar{\Psi}^B_{\dot{\alpha}}, \Phi^A) \Leftrightarrow (\mathbf{1}, \mathbf{4}, \bar{\mathbf{4}}, \mathbf{6})$
- Duality with IIB string in $AdS_5 \times S^5$ (AdS/CFT) $\Rightarrow$ Strong/weak
- Planar limit $N \to \infty$, $\lambda \equiv g^2_{YM}N \Rightarrow$ Integrability!

Integrability in $\mathcal{N} = 4$: the spectral problem

▷ $\mathcal{N} = 4$: Operators $\mathcal{O}(x) = Tr[A_1(x) \dots A_n(x)] \Rightarrow$ CFT fixes the 2-point function:

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{x^{2\Delta^{\mathcal{O}}}}, \quad \Delta^{\mathcal{O}}(g) = \Delta_0^{\mathcal{O}} + \gamma^{\mathcal{O}}(g)$$

▷ $\gamma^{\mathcal{O}}(g)$ from Bethe Ansatz, spin chain description

Example: $SU(2)$ sector $\Longrightarrow$ $\text{Tr}[ZZXZ...XZX] \leftrightarrow |\uparrow\uparrow\downarrow\uparrow \dots\dots \downarrow\uparrow\downarrow\rangle$

▷ *Asymptotic* (large L) Bethe Ansatz equations for the full theory $PSU(2,2|4)$, *any loop* [Minahan, Zarembo, Beisert, Staudacher]

▷ Thermodynamic Bethe Ansatz and Quantum Spectral Curve $\Rightarrow$ Finite L

$\Longrightarrow$ Complete solution of the spectral problem for $\mathcal{N} = 4$ SYM

[Bombardelli, Fioravanti, Tateo, Arutyunov, Frolov, Gromov, Kazakov, Vieira]

$\mathcal{N} = 4$ Wilson loops/amplitudes duality

▷ Non local operators: Wilson loops $W(C) = \langle 0 | \mathrm{Tr}\mathcal{P} e^{i \oint_C ds \left(A^\mu \dot{x}_\mu + |\dot{x}| \Phi_i n^i \right)} | 0 \rangle$

Null polygonal contour ⇒ *4d gluon scattering amplitudes*, when $\boxed{p_i \equiv x_{i+1} - x_i}$

- MHV amplitudes ⇔ Bosonic Wilson loops

▷ CFT ⇒ $n > 5$, $3(n-5)$ conformal ratios τ_i,σ_i,ϕ_i (functions of x_i)

▷ *Hint*: Strong coupling from classical string (+ checks at weak coupling)

$$\boxed{W_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_n}} \qquad A_n \Rightarrow \text{Minimal area in } AdS_5$$

[Alday, Maldacena, Korchemsky, Drummond, Sokatchev]

▷ A_n is the free energy of a TBA-like system [Alday, Gaiotto, Maldacena, Sever, Vieira] → functional equations: Y-system with $3(n-5)$ nodes, masses $m = \sqrt{2}, 2$

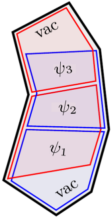
$\Longrightarrow$ Integrable description!

OPE and the Pentagonal approach

▷ CFT: Variant of the Operator Product Expansion $\Longrightarrow$ Expansions around the collinear limit $\tau_i \to \infty \Rightarrow$ Insertion of (pentagon) $\hat{P}$

[Alday,Basso,Gaiotto,Maldacena,Sever,Vieira]

▷ $2D$ flux-tube: free evolutions and transitions



- $n > 5$: pentagonal decomposition
- Phases $e^{-E_{\psi_i}\tau_i + ip_{\psi_i}\sigma_i + im_{\psi_i}\phi_i} \Rightarrow$ Geometry of the loop
- Cusp $\Rightarrow$ Transition $P(\psi_i|\psi_j) = \langle\psi_j|\hat{P}|\psi_i\rangle \Rightarrow$ Flux-tube dynamics
- Operator $\hat{P} \Longrightarrow$ Twist field (pentagonal monodromy)

▷ Summing over ψ, OPE series $\Rightarrow$ *Non-perturbative in* λ

$$\mathcal{W}_n = \sum_{\{\psi\}} \prod_{i=1}^{n-5} e^{-E_{\psi_i}\tau_i + ip_{\psi_i}\sigma_i + im_{\psi_i}\phi_i} P(0|\psi_1)P(\psi_1|\psi_2)\cdots P(\psi_{n-5}|0)$$

2D Flux-tube: Integrability

▷ GKP string solution: spinning string in AdS_5 [Gubser, Klebanov, Polyakov]

▷ Large spin operator $TrZD_+^s Z$ ($S \to \infty$): operator insertion $\Longrightarrow$ Excitation of the flux-tube

$$TrZD_+^{s-s_1}\hat{O}D_+^{s_1}Z, \qquad \hat{O} = F, \bar{F}, \Psi^a, \bar{\Psi}^{\bar{a}}, \Phi^i$$

Vacuum is $SU(4)$ invariant $\Longrightarrow$ Representations of the $SU(4)$ R-symmetry **1**,**4**,$\bar{\mathbf{4}}$,**6**

▷ Asymptotic Bethe Ansatz for $Tr[ZZ...ZZ]$ (BMN vacuum)

$\Longrightarrow$ Excitation of the GKP flux-tube [Basso, Fioravanti, Piscaglia, Rossi]

▷ ψ are multiparticle Bethe states (gluons and bound states, fermions, scalars)

$\Longrightarrow$ Scattering $S_{a,b}(u,v)$, dispersion $E_a(u)$, $p_a(u)$, transitions $P_{a,b}(u|v) \Rightarrow 2d$ physics

▷ Integration over the rapidities u_i: for the hexagon

$$W_6 = \sum_n \frac{1}{n!} \sum_{part.} \int \prod_{i=1}^{n} \frac{du_i}{2\pi} \mu_i(u_i) e^{-E(u_i)\tau + ip(u_i)\sigma + im_i\phi} P(0|u_1,...,u_n)P(u_1,..,u_n|0)$$

Strong coupling ($\lambda \to \infty$) regime

▷ Two different contributions, of the *same order*:

- Perturbative: classical string in AdS_5, fermions and gluons in the OPE: $\Rightarrow$ Exponential in the coupling $W_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi}A_n}$, minimal area from the TBA-like equations
- Non perturbative: string dynamics in S^5, from the scalars in the OPE [Basso, Sever, Vieira] $\Rightarrow$ Dominant in the collinear limit $\tau \to \infty$

$$\boxed{\mathcal{W} = C_{AdS}(\tau,\sigma,\phi)C_{S^5}(\tau,\sigma)\lambda^B e^{\sqrt{\lambda}A - \frac{\sqrt{\lambda}}{2\pi}A_n(\tau,\sigma,\phi)}\left[1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right]}$$

▷ *Tasks* for the OPE method:

- Reproduce the minimal area for $\lambda \to \infty$
- Analyse the correction from the scalars

AdS_5 string: OPE resummation

▷ Gluons ($m = \sqrt{2}$) and fermions ($m = 1$) $\Longrightarrow SU(4)$ index a for the latter, $\sum_{a=1}^{4}$ in the OPE

▷ Strong coupling, hypothesis: bound state (meson) $\psi\bar{\psi}$ with $m = 2$:

- $SU(4)$ singlet $\Rightarrow$ Simplification, no sum over a
- Not in the spectrum of the Bethe equations, but E_M, p_M, S_{MM}, P_{MM} are still well-defined
- Bound states of n mesons, measure $1/n^2$ (+gluons): $\sum_n \Rightarrow Li_2$ (TBA node)

▷ Singlet $\Rightarrow P(u_1, ..., u_n|0) = \prod_{i<j} P(u_i, u_j|0)$ very simple!

$\Rightarrow$ Hubbard-Stratonovich, two fields X_g, X_M (gluon, meson)

$$\frac{1}{P(u|v)P(v|u)} \equiv e^{\langle X(u)X(v)\rangle} \Rightarrow \boxed{W = \int DX_g DX_M e^{-S[X_g, X_M]}} \qquad S \sim \sqrt{\lambda}$$

Saddle point $\Rightarrow$ TBA and minimal area $W_6 \sim e^{-\frac{\sqrt{\lambda}}{2\pi}A_6}$ [Fioravanti,Piscaglia,Rossi]

$\Longrightarrow$ Extension to $n > 6$, TBA and Y-system for amplitudes, agreement with the string computations

Bound states $\psi\bar{\psi}$ from the OPE

▷ OPE: n couples $\psi\bar{\psi}$ reads

$$W^{(n)} = \frac{1}{n!n!}\int_{\mathcal{C}}\prod_{k=1}^{n}\left[\frac{du_k}{2\pi}\frac{dv_k}{2\pi}\mu_f(u_k)\mu_f(v_k)e^{-\tau E_f(u_k)+i\sigma p_f(u_k)}\times e^{-\tau E_f(v_k)+i\sigma p_f(v_k)}\right]\Pi^{(n)}_{dyn}\Pi^{(n)}_{mat}$$

▷ Polar structure of the matrix part

$$\Pi^{(n)}_{mat}(\{u_i\},\{v_j\}) = \frac{P^{(n)}(u_1,\ldots,u_n,v_1,\ldots,v_n)}{\prod\limits_{i<j}^{n}[(u_i-u_j)^2+1]\prod\limits_{i<j}^{n}[(v_i-v_j)^2+1]\prod\limits_{i,j=1}^{n}[(u_i-v_j)^2+4]}$$

▷ Strong coupling $\Rightarrow$ Residues in $v_i = u_j - 2$ and properties of $P^{(n)}$:

$$\Longrightarrow\quad W^{(M)} = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{C_S}\prod_{i=1}^{n}\frac{du_i}{2\pi}\hat{\mu}_M(u_i)\prod_{i<j}^{n}\frac{1}{P^{MM}_{reg}(u_i|u_j)P^{MM}_{reg}(u_j|u_i)}\prod_{i<j}^{n}\frac{u_{ij}^2}{u_{ij}^2+1}$$

▷ Analogy with the Nekrasov function $\mathcal{Z}$: $\lambda\to\infty \Leftrightarrow \epsilon\to 0$ (NS limit)

Meson bound states and resummation

▷ Resummation of the series $W^{(M)}$: works also for $\mathcal{Z}$

- Polar part $\Rightarrow$ $\boxed{\prod_{i<j}^{n} \frac{u_{ij}^2}{u_{ij}^2+1} = \frac{1}{i^n} \det\left(\frac{1}{u_i - u_j - i}\right)}$ $\Rightarrow$ Fredholm determinant
- Hubbard-Stratonovich for the regular part $P_{reg}^{MM}(u|v)$

$\Longrightarrow$ $\boxed{W^{(M)} \simeq \langle \det(1 - M[X]) \rangle}$, average over the field $X(u)$

▷ Strong coupling $\Rightarrow$ $\boxed{\frac{1}{n} TrM^n = -\frac{1}{n^2} \int \frac{du}{2\pi i} \mu_M(u) e^{-n\tau E_M(u)+\dots} e^{nX(u)} + O(1)}$

$\Longrightarrow$ Sum over bound states $\sum_n x^n/n^2 \Rightarrow$ Dilogarithm $Li_2(x)$ (TBA node)

$$\boxed{W^{(M)} \simeq \left\langle \exp\left[-\int du \mu_M(u) Li_2\left[e^{-\tau E_M(u)+\dots}\right]\right]\right\rangle}$$ $\Rightarrow$ The same for $\mathcal{Z}$

▷ Including gluons + saddle point

$\Longrightarrow$ TBA equations directly from the OPE, no additional assumption

Non-perturbative regime, scalars contribution

▷ Mass $m \sim e^{-\sqrt{\lambda}/4}$, 2d non-linear (relativistic) $O(6)$ σ-model $\Leftrightarrow$ String on S^5
W_N is a $(N-4)$ point function in the $O(6)$, for the hexagon:

$$\boxed{W_6(z) = \langle \hat{P}(z)\hat{P}(0)\rangle_{O(6)} \simeq C z^{-J} (\log 1/z)^s, \quad z \to 0}, \qquad z = m\sqrt{\tau^2 + \sigma^2}$$

▷ Scaling dimension of $\hat{P} \Rightarrow J = 1/24, s = -1/36$

$\Longrightarrow$ *non-perturbative enhancement* $\boxed{W_6 \simeq f_6 \lambda^{-\frac{7}{288}} e^{\frac{\sqrt{\lambda}}{144}}}$ [Basso, Sever, Vieira]

▷ <u>*Goal*</u>: Compute J, s, C from the OPE series (Form factor expansion)

$$\Longrightarrow \quad \boxed{W_6 = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int \prod_{i=1}^{2n} d\theta_i G^{(2n)}(\theta_1, \ldots, \theta_{2n}) e^{-z \sum_k \cosh \theta_k}}$$

▷ Asymptotic factorisation of $G^{(2n)} \Longrightarrow g^{(2n)} \to 0$ when the rapidities are sent far away

Connected functions: series represtation for J

▷ The series of the logarithm contains the connected counterparts $g^{(2n)}$

$$\mathcal{F}_6 \equiv \log W_6 = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \prod_{i=1}^{2n} d\theta_i g^{(2n)}(\theta_1, \cdots, \theta_n) e^{-z \sum_k \cosh \theta_k}$$

▷ $g^{(2n)}$ depends on the differences $\alpha_i \equiv \theta_{i+1} - \theta_1$, we integrate over θ_1 to get

$$\mathcal{F}_6 = 2 \sum_{n=1}^{\infty} \int \prod_{i=1}^{2n-1} d\alpha_i g^{(2n)}(\alpha_1, \ldots, \alpha_{2n-1}) K_0(z\xi)$$

▷ Expand for small $z \Rightarrow K_0(x) = -\log x + O(1)$, *Series* for J: (also $s, \log C$)

$$J = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \prod_{i=1}^{2n-1} d\alpha_i g^{(2n)}(\alpha_1, \ldots, \alpha_{2n-1}) = \sum_{n=1}^{\infty} J^{(2n)}$$

▷ *Fast convergence*: $J^{(2)} + J^{(4)} \simeq J = 1/24$ with 99% of accuracy

Scalar Polygons $N > 6$: recursion formula

▷ Polygons $N > 6$, same picture: N-gonal Wilson loop $\Leftrightarrow$ $(N-4)$-point function

$$\boxed{W_N(\tau_i,\sigma_i) = \langle \hat{P}(z_1) \ldots . \hat{P}(z_{N-4}) \rangle , \quad z_{i+i} - z_i = m(\tau_i,\sigma_i)}$$

▷ Logarithm $\mathcal{F}_N = \log W_N$, *multiconnected parts* $G^{(2n_1,\ldots,2n_{N-5})} \Rightarrow g^{(2n_1,\ldots,2n_{N-5})}$

▷ Features of $g^{(2n_1,\ldots,2n_{N-5})} \Rightarrow$ Recursion among the polygons:

$$\boxed{\mathcal{F}_N = \mathcal{F}_{N-1}(1,\ldots,N-6) + \mathcal{F}_{N-1}(2,\ldots,N-5) - \mathcal{F}_{N-2}(2,\ldots,N-6) + \sum_{\{n\}=1}^{\infty} \mathcal{F}_N^{(2n_1,\cdots,2n_{N-5})}}$$

$\Rightarrow$ Picture: two inner $(N-1)$-gons minus the overlapping $(N-2)$-gon, plus corrections ($\rightarrow 0$ for large N) $\Rightarrow$ Same for J_N, s_N

▷ Simplest solution with $J_4 = J_5 = 0$ (square and pentagon are trivial):

$$\boxed{J_N = \frac{(N-4)(N-5)}{12N}} \Rightarrow \text{agreement with the BSV proposal } W \sim e^{\sqrt{\lambda} J_N}$$

$\Longrightarrow$ Does it hold for s_N as well?

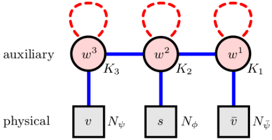
Twist operator $\hat{P}$: form factors

▷ Ψ_B, Φ_A charged under $SU(4)$ ⇒ sum over internal indices, for the hexagon:

$$\boxed{\sum_{\{a\}} |\langle 0|\hat{P}|\Phi_{a_1}(u_1)\cdots\Phi_{a_n}(u_n)\rangle|^2 = \Pi_{mat}(\{u\})\Pi_{dyn}(\{u\})}$$ ⇒ Simpler than usual

- $\Pi^{(2n)}_{dyn}$ ⇒ Two-body factorizable, coupling constant g
- $\Pi^{(2n)}_{mat}$ ⇒ Coupling independent, NOT factorized

▷ Integral representation: auxiliary roots of $SU(4)$ spin chain:



- Different variables, solid blue line
  $f(x) = x^2 + 1/4$
- Self interaction, shaded red line
  $g(x) = x^2(x^2+1)$

Scalars ⇒

$$\boxed{\Pi^{(2n)}_{mat} = \frac{1}{(2n)!(n!)^2}\int_{-\infty}^{+\infty}\prod_{k=1}^{n}\frac{da_k dc_k}{(2\pi)^2}\prod_{k=1}^{2n}\frac{db_k}{2\pi}\times\frac{\prod_{i<j}^{n} g(a_i-a_j)g(c_i-c_j)\prod_{i<j}^{2n} g(b_i-b_j)}{\prod_{j=1}^{2n}\left(\prod_{i=1}^{n} f(a_i-b_j)f(c_i-b_j)\prod_{l=1}^{2n} f\left(u_l-b_j\right)\right)}}$$

Residues evaluation and Young diagrams

- Integrals over $a, c \Rightarrow \int \prod_{i=1}^{2n} \frac{db_i}{2\pi} [\delta_{2n}(b_1, \ldots, b_{2n})]^2 \prod_{i,j}^{2n} \frac{1}{f(u_i - b_j)} \prod_{i<j} \frac{b_{ij}^2}{(b_{ij}^2 + 1)}$
- Zeroes and poles structure, analogy with Nekrasov function $\mathcal{Z}$:
  - Double zeros $b_i = b_j \Rightarrow$ No residues with the same rapidity
  - Poles $b_i = u_j + i/2$ and $b_{ij} = +i \Rightarrow$ Strings in the complex plane with real part u_i and displaced by $+i$
  - $\delta_{2n}(b_1, b_1 + i, b_1 + 2i, b_4, \cdots, b_{2n}) = 0 \Rightarrow l_i \leq 2$ residues with u_i
- Symmetrization $u_i \leftrightarrow u_j \Rightarrow$ Young diagrams: for $2n$ scalars

$$\boxed{\Pi_{mat}^{(2n)} = \sum_{l_1+\ldots+l_{2n}=2n, l_i<3, l_{i+1}\leq l_i} (l_1, \ldots, l_{2n})_s = \sum_{|Y|=2n, l_i<3} (Y)_s}$$, $(Y)_s$ rational function

$\Longrightarrow$ *Example*: $\Pi_{mat}^{(2)}(u_1, u_2) = (2,0) + (1,1) + (0,2) = (2,0)_s + (1,1)_s$

- $n\ \psi\bar{\psi} \Rightarrow$ $$\boxed{\Pi_{mat}^{(n)}(u_1, \ldots, u_n, v_1, \ldots, v_n) = \sum_{l_1+\ldots+l_{2n}=n, l_i=0,1} (l_1, \ldots, l_{2n})}$$

Poles structure of Π_{mat}

▷ Asymptotic factorisation of $\Pi_{mat}^{(2n)}$ ⇒ polar structure in the complex plane:

$$\boxed{\Pi_{mat}^{(2n)}(u_1, \cdots, u_{2n}) = \frac{P^{(2n)}(u_1, \cdots, u_{2n})}{\prod_{i<j}(u_{ij}^2 + 1)(u_{ij}^2 + 4)}}$$

▷ Some properties of the polynomials $P^{(2n)}$ are known, for instance:

$$\boxed{P_{2n}(u_1 - i, u_1 + i, u_3, \ldots, u_{2n}) = 6P_{2n-2}(u_3, \ldots, u_{2n}) \prod_{j=3}^{2n}(u_{1j}^2 + 4)(u_{1j}^2 + 9)}$$

$\Longrightarrow$ Kinematic poles of the form factors

▷ Similar considerations for n couples $\psi\bar{\psi}$ ⇒ Poles and polynomials P_n

$\Longrightarrow$ Formation of the mesons and bound states, important check of the integral formula!

Summary and perspectives

Integrability + OPE series ⇒ Non-perturbative approach to Wl/amplitudes

▷ Strong coupling regime: exponential contributions

- AdS_5 minimal area: OPE resummation (gluons, fermions) ⇒ TBA-like equations for any polygon
- OPE prediction: non-pertubative correction from scalars, string on S^5 ⇒ Series for the leading coefficients J_N, any N

⟹ Extension to NMHV, subleading corrections?

▷ Form factors of the twist operator $\hat{P}$:

- Split dynamical + matrix ⇒ Simplification
- $SU(4)$ matrix part ⇒ Young diagrams representation (rational functions)

⟹ Split, properties of $\hat{P}$, integral formula of Π_{mat}

▷ Analogies with $\mathcal{N} = 2$

⟹ Relation between $\mathcal{Z}$ and $W^{(M)}$, role of integrability?

Thank you for your attention!