Superstrings, lattice and AdS/CFT

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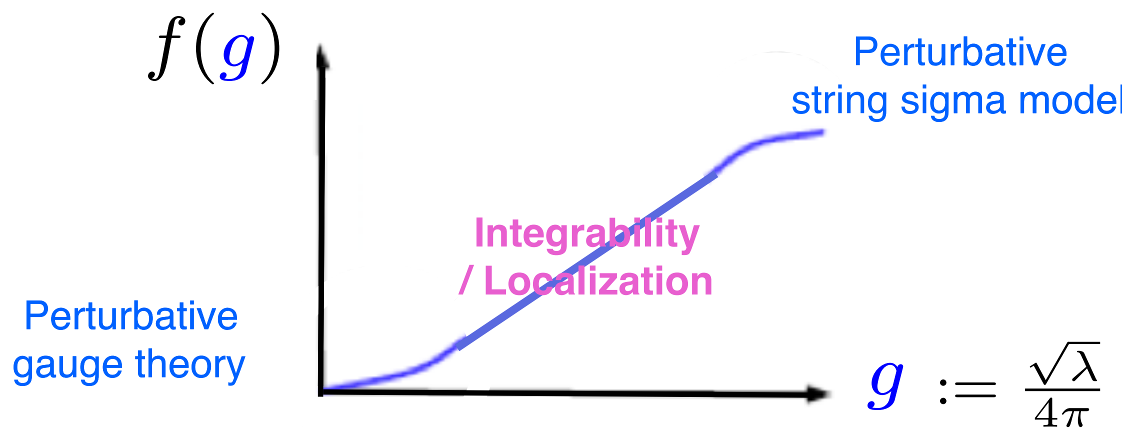
CONVEGNO NAZIONALE DI FISICA TEORICA

AdS/CFT and exact results

Impressive progress in obtaining results exact in the coupling (here, planar $\mathrm{AdS}_5/\mathrm{CFT}_4$)

$$f(g) = c\,g + d + \frac{e}{g} + \ldots$$



$$f(g) = a\,g^2 + b\,g^4 + \ldots$$

$$g := \frac{\sqrt{\lambda}}{4\pi} = \frac{\sqrt{g_{YM}^2 N}}{4\pi} \equiv \frac{R^2}{4\pi\alpha'}$$

- from integrability
- from supersymmetric localization

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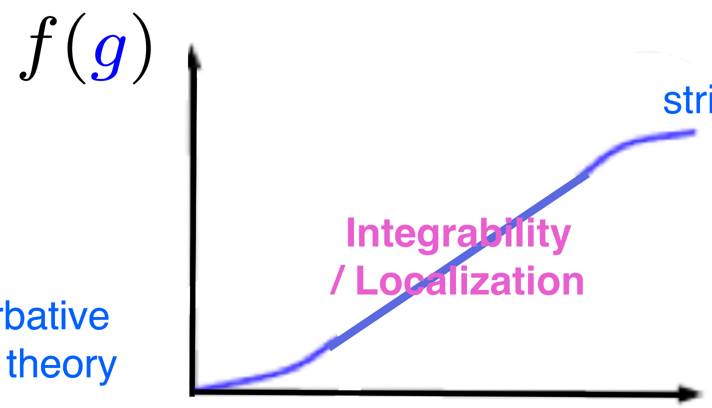
Perturbative
string sigma model



$$f(g) = a\, g^2 + b\, g^4 + \dots$$

Perturbative
gauge theory

$$g := \frac{\sqrt{\lambda}}{4\pi} = \frac{\sqrt{g^2_{YM}\, N}}{4\pi} \equiv \frac{R^2}{4\pi\alpha'}$$

- from integrability
- from supersymmetric localization

**See talks by: Alfredo Bonini, Enrico Olivucci, Michelangelo Preti,
and Lorenzo Bianchi, Sara Bonansea, Francesco Galvagni, Luca Griguolo
and Francisco Morales (plenary, Friday)**

Motivation

Impressive progress in obtaining results exact in the coupling (here, planar AdS_5/CFT_4)

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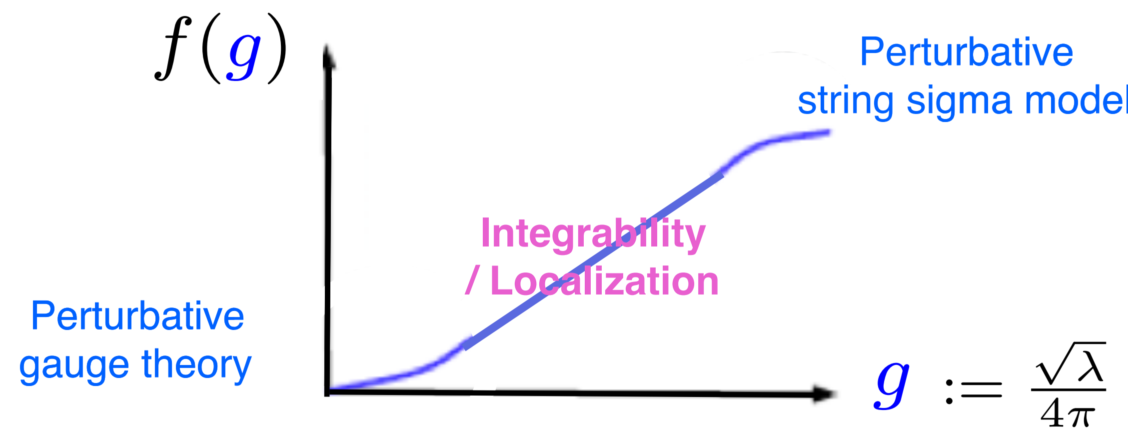

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$$g := \frac{\sqrt{\lambda}}{4\pi} = \frac{\sqrt{g_{YM}^2\,N}}{4\pi} \equiv \frac{R^2}{4\pi\alpha'}$$

- from integrability (assumed)
- from supersymmetric localization (BPS observable)

In the world-sheet string theory integrability only classically, localization not formulated.

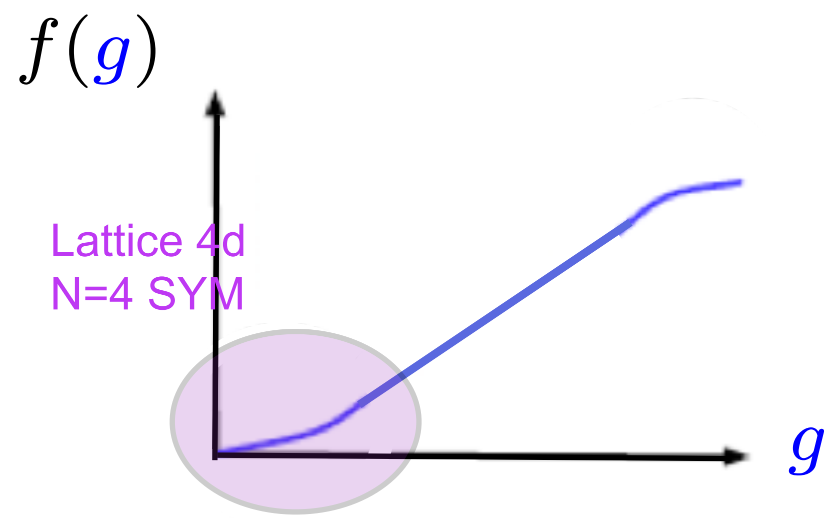
Green-Schwarz superstring in AdS backgrounds with RR fluxes: complicated interacting 2d field theory which has subtleties also perturbatively.

Call for genuine 2d QFT to cover the finite-coupling region.

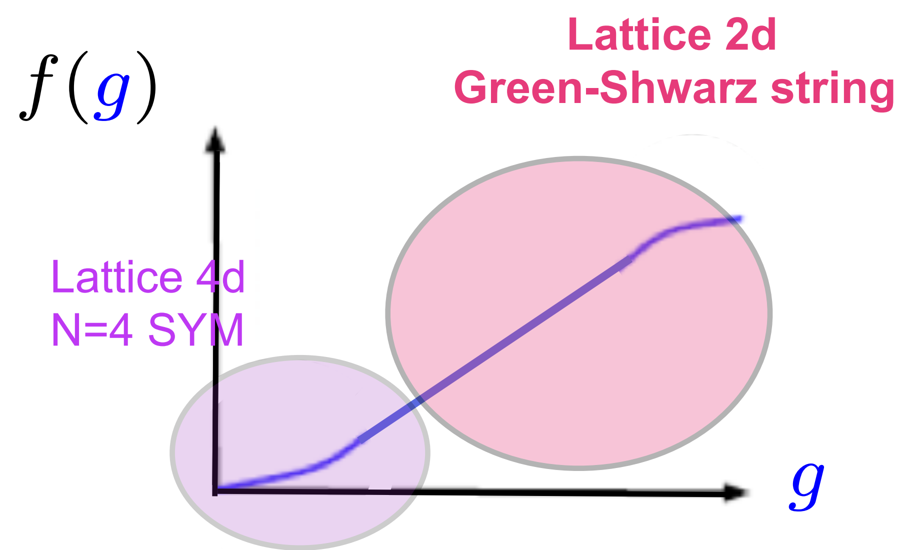
Lattice techniques in AdS/CFT

Consolidated program on 4d CFT side,
subtleties with supersymmetry,
control on the perturbative region.

[Catterall, Damgaard, DeGrand, Giedt, Schaich…]



Lattice techniques in AdS/CFT



[previous study: Roiban McKeown 2013]

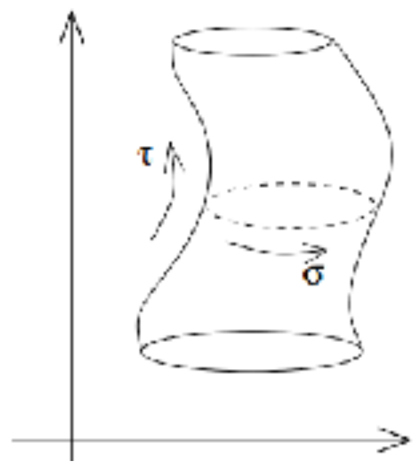
Features:

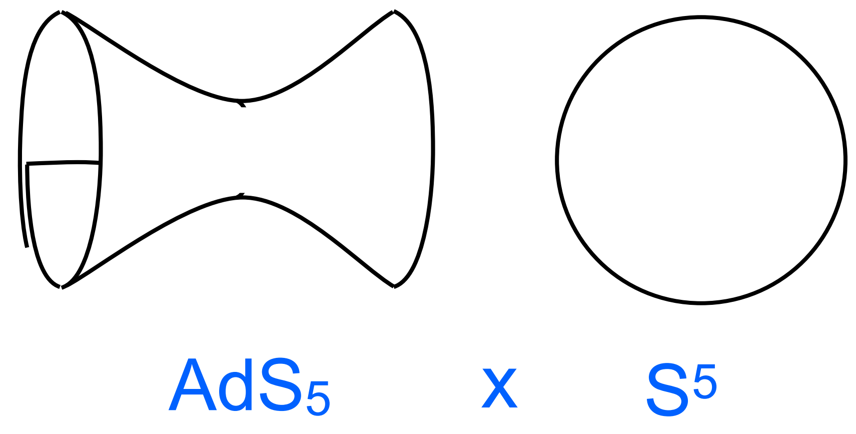
- 2d: computationally cheap
- no supersymmetry (only as flavour symmetry, Green-Schwarz)
- all gauge symmetries are fixed (no formulation à la Wilson), only scalar fields (some of which anti-commuting)

Non-trivial 2d qft with strong coupling analytically known, finite-coupling (numerical) prediction.

The model in perturbation theory

Green-Schwarz string in $AdS_5 \times S^5$





[Metsaev Tseytlin 1998]

$$S = g \int d\tau d\sigma \left[\partial_a X^\mu \partial^a X^\nu G_{\mu\nu} + \bar\theta \Gamma (D + F_5) \theta \partial X + \bar\theta \partial\theta \bar\theta \partial\theta + \dots \right]$$

Symmetries:

- global $PSU(2,2|4)$, local bosonic (diffeomorphism) and fermionic (κ-symmetry)
- classical integrability

manifest when written as sigma-model action on $G/H = \frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$.

Green-Schwarz string in $AdS_5 \times S^5$ + RR flux perturbatively

Highly non-linear, to quantize it use semiclassical methods

$$X = X_{\rm cl} + \tilde{X} \quad \longrightarrow \quad \Gamma = g\left[\Gamma_0 + \frac{\Gamma_1}{g} + \frac{\Gamma_2}{g^2} + \dots\right]$$

- General analysis of fluctuations in terms of background geometry,

  [Drukker Gross Tseytlin 00] [Buchbinder Tseytlin 14] [**VF** Giangreco Griguolo Seminara Vescovi 15]

- Explicit analytic form of one-loop partition function $Z = \det O_F / \sqrt{\det O_B}$ for a class of effectively one-dimensional problems.

  Several "vacua" (GKP string, quark-antiquark potential, generalized cusp) have been "solved" this way at one loop, and agree with predictions.

  [Drukker Gross Tseytlin Frolov **VF** Beccaria Dunne Giangreco, Ohlson Sax, Griguolo Seminara Vescovi ....]

  In BPS cases (e.g. dual to circular Wilson loop) more care needed:

  - avoid measure ambiguities, considering ratio of partition functions
  - choose suitable regularization scheme

  [Kruczenski Tirziu 08] [Kristjansen Makeenko 12] [Buchbinder Tseytlin 14]
  [**VF,** Giangreco, Griguolo, Seminara, Vescovi 15] [Pando-Zayas Trancanelli et al.16]
  [**VF,** Vescovi, Tseytlin 17] [Cagnazzo, Medina-Rincon, Zarembo 17] [Medina-Rincon, Tseytlin, Zarembo 18]

Green-Schwarz string in $AdS_5 \times S^5$ + RR flux perturbatively

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2 loops is current limit: "homogenous" configs, "AdS light-cone" gauge-fixing

[Uvarov, 09,10][Uvarov, 09,10]

[Giombi Ricci Roiban Tseytlin 09] [Bianchi² Bres **VF** Vescovi 14]

Check of exact predictions based on integrability and localization [Gromov, Syzov 14]

and check of quantum consistency (UV finiteness) of certain string actions. [Uvarov 09,10]

Green-Schwarz string in $AdS_5 \times S^5$ + RR flux perturbatively

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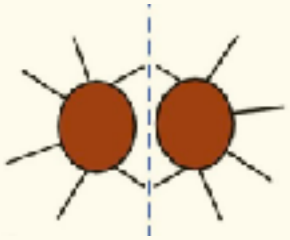
$$X = X_{\rm cl} + \tilde{X} \quad \longrightarrow \quad \Gamma = g\left[\Gamma_0 + \frac{\Gamma_1}{g} + \frac{\Gamma_2}{g^2} + \ldots\right]$$

Efficient alternative to Feynman diagrams for on-shell objects (worldsheet S-matrix)



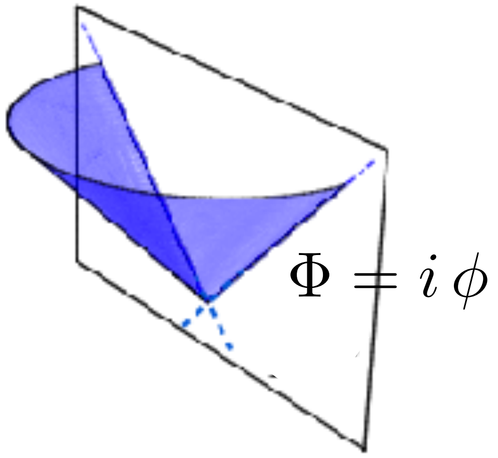
unitarity cuts (on-shell methods) in d=2

[Bianchi **VF** Hoare 2013][Engelund Roiban 2013] [Bianchi Hoare 14]

Beyond perturbation theory

with L. Bianchi, M. S. Bianchi, B. Leder, P. Töpfer, E. Vescovi

The cusp anomaly of $\mathcal{N}=4$ SYM from string theory



Completely solved via integrability. [Beisert Eden Staudacher 2006]

Expectation value of a light-like cusped Wilson loop

$$\langle W[C_{\rm cusp}]\rangle \sim e^{-f(g)\,\phi\,\ln\frac{L_{\rm IR}}{\epsilon_{\rm UV}}}$$

AdS/CFT

$$Z_{\rm cusp} = \int [D\delta X][D\delta\theta]\, e^{-S_{\rm IIB}(X_{\rm cusp}+\delta X,\delta\theta)}$$

String partition function with “cusp” boundary conditions.

[Giombi Ricci Roiban Tseytlin 2009]

$X_{\rm cusp}$ is the minimal surface

$$ds^2_{AdS_5} = \frac{dz^2 + dx^+dx^- + dx^*dx}{z^2} \qquad x^\pm = x^3 \pm x^0 \qquad x = x^1 + i\,x^2$$

$$z = \sqrt{\frac{\tau}{\sigma}} \qquad x^+ = \tau \qquad x^- = -\frac{1}{2\sigma} \qquad x^+x^- = -\frac{1}{2}z^2$$

ending on a null cusp, since $x^+x^- = 0$ at the boundary $z = 0$.

Remark

Completely solved via integrability.

- In general, no quest here for integrability-preserving discretization.
  We use an integrable model for establishing a benchmark of the method,
  (we'll actually break manifest symmetries, let alone hidden ones!)
  the integrability prediction as final check for standard lattice field theory methods.

- That the dispersion relation for string world-sheet excitations on a BMN vacuum

$$\epsilon^2 = 1 + 16\, g^2 \sin^2\left(\frac{p}{4g}\right)$$

is lattice-like plays no role here.

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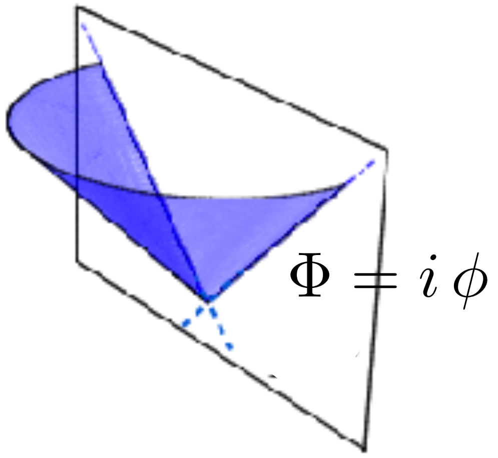

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$$Z_{\rm cusp} = \int [D\delta X][D\delta\theta]\, e^{-S_{\rm IIB}(X_{\rm cusp}+\delta X,\delta\theta)} = e^{-\Gamma_{\rm eff}} \equiv e^{-f(g)\,V_2}$$

String partition function with “cusp” boundary conditions.

Perturbatively

$$f(g)|_{g\to 0} = 8g^2\left[1 - \frac{\pi^2}{3}g^2 + \frac{11\,\pi^4}{45}g^4 - \left(\frac{73}{315} + 8\,\zeta_3\right)g^6 + \ldots\right]$$ [Bern et al. 2006]

$$f(g)|_{g\to\infty} = 4g\left[1 - \frac{3\ln 2}{4\pi}\frac{1}{g} - \frac{K}{16\pi^2}\frac{1}{g^2} + \ldots\right]$$ [Gubser Klebanov Polyakov 02] [Frolov Tseytlin 02][Giombi et al. 2009]

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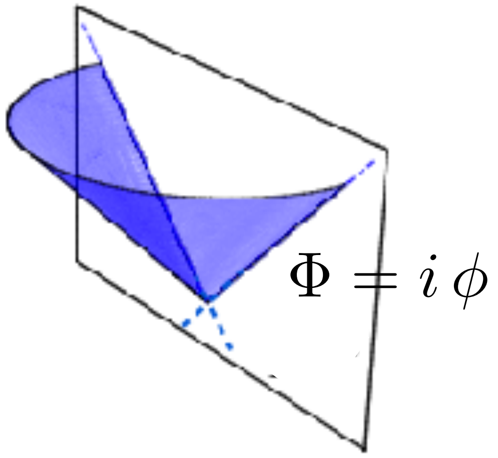

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String partition function with “cusp” boundary conditions.

A lattice approach prefers expectation values

$$\langle S_{\rm cusp}\rangle = \frac{\int [D\delta X][D\delta\Psi]\, S_{\rm cusp}\, e^{-S_{\rm cusp}}}{\int [D\delta X][D\delta\Psi]\, e^{-S_{\rm cusp}}} = -g\,\frac{d\ln Z_{\rm cusp}}{dg} \equiv g\,\frac{V_2}{8}\,f'(g)$$

$$S_{\rm cusp} = g\int \mathcal{L}_{\rm cusp}$$

Green-Schwarz string in the null cusp background

The (AdS lightcone) gauge-fixed action for fluctuations above the null cusp is

$$S_{\text{cusp}} = g \int dt ds \mathcal{L}_{\text{cusp}}$$

[Giombi Ricci Roiban Tseytlin 2009]

$$\mathcal{L}_{\text{cusp}} = |\partial_t x + \tfrac{1}{2}x|^2 + \tfrac{1}{z^4}|\partial_s x - \tfrac{1}{2}x|^2 + \left(\partial_t z^M + \tfrac{1}{2}z^M + \tfrac{i}{z^2} z_N \eta_i \left(\rho^{MN}\right)^i{}_j \eta^j\right)^2 + \tfrac{1}{z^4}\left(\partial_s z^M - \tfrac{1}{2}z^M\right)^2$$
$$+ i\left(\theta^i \partial_t \theta_i + \eta^i \partial_t \eta_i + \theta_i \partial_t \theta^i + \eta_i \partial_t \eta^i\right) - \tfrac{1}{z^2}\left(\eta^i \eta_i\right)^2$$
$$+ 2i\Big[\tfrac{1}{z^3} z^M \eta^i \left(\rho^M\right)_{ij}\left(\partial_s \theta^j - \tfrac{1}{2}\theta^j - \tfrac{i}{z}\eta^j\left(\partial_s x - \tfrac{1}{2}x\right)\right) + \tfrac{1}{z^3} z^M \eta_i (\rho^\dagger_M)^{ij}\left(\partial_s \theta_j - \tfrac{1}{2}\theta_j + \tfrac{i}{z}\eta_j\left(\partial_s x - \tfrac{1}{2}x\right)^*\right)\Big]$$

- 8 bosons: x, x^*, z^M $(M = 1, \cdots, 6)$, $z = \sqrt{z_M z^M}$;
- 8 fermions: $\theta^i = (\theta_i)^\dagger$, $\eta^i = (\eta_i)^\dagger$, $i = 1, 2, 3, 4$, complex Graßmann;
- ρ^M are off-diagonal blocks of $SO(6)$ Dirac matrices
- $(\rho^{MN})^i_j$ are the $SO(6)$ generators

Remnant global symmetry is $SO(6) \times SO(2)$.

Fermionic interactions at most quartic.

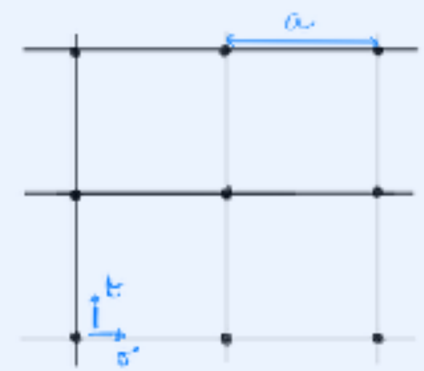
Lattice QFT basics

Discretize Euclidean worldsheet in a grid of lattice spacing a, size $L = N\,a$.

Fields $\phi \equiv \phi_n$ defined at $\xi = (an_1, an_2) \equiv a\,n$.

a) natural cutoff $-\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}$

b) path integral measure $[D\phi] = \prod_n d\phi_n$.



Then $\int \prod_n d\phi_n\, e^{-S_{\rm discr}}$ via Monte Carlo: generate an ensamble $\{\Phi_1, \ldots, \Phi_K\}$

of field configurations, each weighted by $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]}}{Z}$.

Ensemble average $\quad \langle A \rangle = \int [D\Phi]\, P[\Phi]\, A[\Phi] = \frac{1}{K} \sum_{i=1}^{K} \, A[\Phi_i] + \mathcal{O}\big(\frac{1}{\sqrt{K}}\big)$

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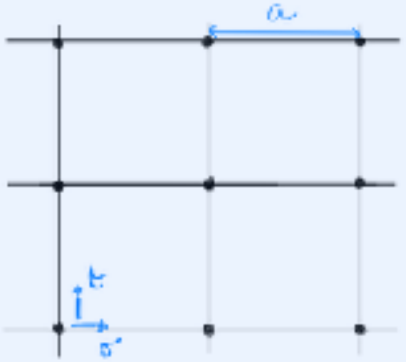

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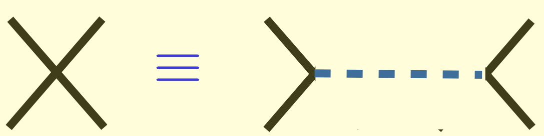
Graßmann-odd fields are formally integrated out: $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]}\det \mathcal{O}_F}{Z}$

- action must be quadratic in fermions



Introduce auxiliary fields (complex bosons)

- determinant must be positive definite

$$\det O_F \quad \longrightarrow \quad \sqrt{\det(O_F^\dagger\, O_F)} \equiv \int D\zeta\, D\bar{\zeta}\;\, e^{-\int d^2\xi\, \bar{\zeta}(O_F^\dagger O_F)^{-\frac{1}{2}}\zeta}$$

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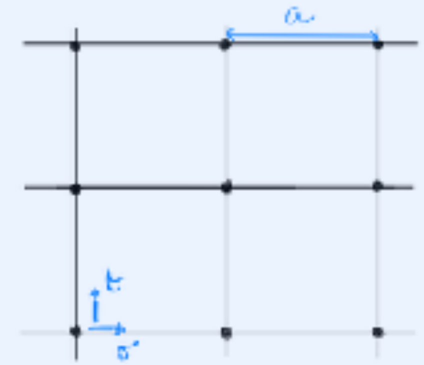

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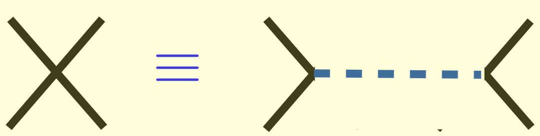

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$$\mathrm{Pf}\, O_F \;\longrightarrow\; (\det O_F^\dagger\, O_F)^{\frac{1}{4}} \equiv \int D\zeta\, D\bar\zeta\; e^{-\int d^2\xi\, \bar\zeta\, (O_F^\dagger O_F)^{-\frac{1}{4}}\, \zeta}$$

Linearization

Four-fermion interactions

Linearization via Hubbard-Stratonovich transformation

$$\exp\Big\{-\,g\int dt\,ds\,\mathcal{L}_4\Big\}\sim\int d\phi\,d\phi^M\,\exp\Big\{-\,g\int dt\,ds\,\mathcal{L}_{\rm aux}\Big\}$$

$$\exp\Big\{-\,g\int dtds\Big[-\tfrac{1}{z^2}\,(\eta^i\eta_i)^2+\Big(\tfrac{i}{z^2}z_N\eta_i\rho^{MN\,i}{}_j\eta^j\Big)^2\Big]\Big\}$$
$$\sim\int D\phi D\phi^M\,\exp\Big\{-\,g\int dtds\,[\tfrac{1}{2}\phi^2+\tfrac{\sqrt{2}}{z}\phi\,\eta^2+\tfrac{1}{2}(\phi_M)^2-i\,\tfrac{\sqrt{2}}{z^2}\phi^M\,(\tfrac{i}{z^2}z_N\eta_i\rho^{MN\,i}{}_j\eta^j)]\Big\}\,.$$

- $+7$ bosonic auxiliary fields ϕ, ϕ^M $(M=1,\cdots,6)$

Four-fermion interactions

Linearization via Hubbard-Stratonovich transformation

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$$\exp\Big\{-g\int dtds\Big[-\tfrac{1}{z^2}\,(\eta^i\eta_i)^2+\Big(\tfrac{i}{z^2}z_N\eta_i\rho^{MN\,i}{}_j\eta^j\Big)^2\Big]\Big\}$$

hermitian

$$\sim\int D\phi D\phi^M\,\exp\Big\{-g\int dtds\,[\tfrac{1}{2}\phi^2+\tfrac{\sqrt{2}}{z}\phi\,\eta^2+\tfrac{1}{2}(\phi_M)^2-i\,\tfrac{\sqrt{2}}{z^2}\phi^M\,\big(\tfrac{i}{z^2}z_N\eta_i\rho^{MN\,i}{}_j\eta^j\big)]\Big\}\,.$$

- $+7$ bosonic auxiliary fields ϕ, ϕ^M $(M=1,\cdots,6)$
- $\mathcal{L}_{\rm aux}$ is not hermitian, $e^{-\frac{b^2}{4\,a}}=\int dx\,e^{-a\,x^2+i\,b\,x}$, $b\in\mathbb{R}$.

Green-Schwarz string in the null cusp background

After linearization the Lagrangian reads ($m \sim P_+$)

$$\mathcal{L}_{\rm cusp} = |\partial_t x + \frac{m}{2} x|^2 + \frac{1}{z^4}|\partial_s x - \frac{m}{2} x|^2 + (\partial_t z^M + \frac{m}{2} z^M)^2 + \frac{1}{z^4}(\partial_s z^M - \frac{m}{2} z^M)^2$$
$$+ \frac{1}{2}\phi^2 + \frac{1}{2}(\phi_M)^2 + \psi^T O_F \psi \,,$$

where $\psi \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$ and

$$O_F = \begin{pmatrix} 0 & i\partial_t & -\mathrm{i}\rho^M \left(\partial_s + \frac{m}{2}\right) \frac{z^M}{z^3} & 0 \\ \mathrm{i}\partial_t & 0 & 0 & -\mathrm{i}\rho^\dagger_M \left(\partial_s + \frac{m}{2}\right) \frac{z^M}{z^3} \\ \mathrm{i}\frac{z^M}{z^3}\rho^M \left(\partial_s - \frac{m}{2}\right) & 0 & 2\frac{z^M}{z^4}\rho^M \left(\partial_s x - m\frac{x}{2}\right) & i\partial_t - A^T \\ 0 & \mathrm{i}\frac{z^M}{z^3}\rho^\dagger_M \left(\partial_s - \frac{m}{2}\right) & \mathrm{i}\partial_t + A & -2\frac{z^M}{z^4}\rho^\dagger_M \left(\partial_s x^* - m\frac{x}{2}^*\right) \end{pmatrix}$$

$$A = \frac{1}{\sqrt{2}z^2}\phi_M \rho^{MN} z_N - \frac{1}{\sqrt{2}z}\phi + \mathrm{i}\,\frac{z_N}{z^2}\rho^{MN}\,\partial_t z^M$$

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where $\psi \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$ and

$$O_F = \begin{pmatrix} 0 & i\partial_t & -\mathrm{i}\rho^M\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} & 0 \\ \mathrm{i}\partial_t & 0 & 0 & -\mathrm{i}\rho^\dagger_M\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} \\ \mathrm{i}\frac{z^M}{z^3}\rho^M\left(\partial_s - \frac{m}{2}\right) & 0 & 2\frac{z^M}{z^4}\rho^M\left(\partial_s x - m\frac{x}{2}\right) & i\partial_t - A^T \\ 0 & \mathrm{i}\frac{z^M}{z^3}\rho^\dagger_M\left(\partial_s - \frac{m}{2}\right) & \mathrm{i}\partial_t + A & -2\frac{z^M}{z^4}\rho^\dagger_M\left(\partial_s x^* - m\frac{x}{2}^*\right) \end{pmatrix}$$

$$A = \frac{1}{\sqrt{2}z^2}\phi_M \rho^{MN} z_N - \frac{1}{\sqrt{2}z}\phi + \mathrm{i}\,\frac{z_N}{z^2}\rho^{MN}\,\partial_t z^M$$

As $A^\dagger \neq A$, Pfaffian is complex: $\mathrm{Pf}(\mathcal{O}_F) = e^{i\theta}\,(O_F O_F{}^\dagger)^{\frac{1}{4}}$.

Phase problem

Even with $\mathrm{Pf}(\mathcal{O}_F) = e^{i\theta}\,(O_F {O_F}^\dagger)^{\frac{1}{4}}$, vev's can be still obtained via reweighting:

$$\langle \mathcal{A} \rangle = \frac{\int D\Phi\, \mathcal{A}\, \mathrm{Pf}(O_F)\, e^{-S[\Phi]}}{\int D\Phi\ \mathrm{Pf}(O_F)\, e^{-S[\Phi]}}$$

$$= \frac{\int D\Phi\, D\zeta\, D\bar\zeta\, \mathcal{A}\, e^{i\theta}\, e^{-S[\Phi]-\int d^2\xi\, \bar\zeta (\mathcal{O}_F \mathcal{O}_F^\dagger)^{-\frac{1}{4}} \zeta}}{\int D\Phi\, D\zeta\, D\bar\zeta\, e^{i\theta}\, e^{-S[\Phi]-\int d^2\xi\, \bar\zeta (\mathcal{O}_F \mathcal{O}_F^\dagger)^{-\frac{1}{4}} \zeta}} = \frac{\langle \mathcal{A}\ e^{i\theta} \rangle_{\theta=0}}{\langle e^{i\theta} \rangle_{\theta=0}}$$

It gives meaningful results as long as the phase does not averages to zero.

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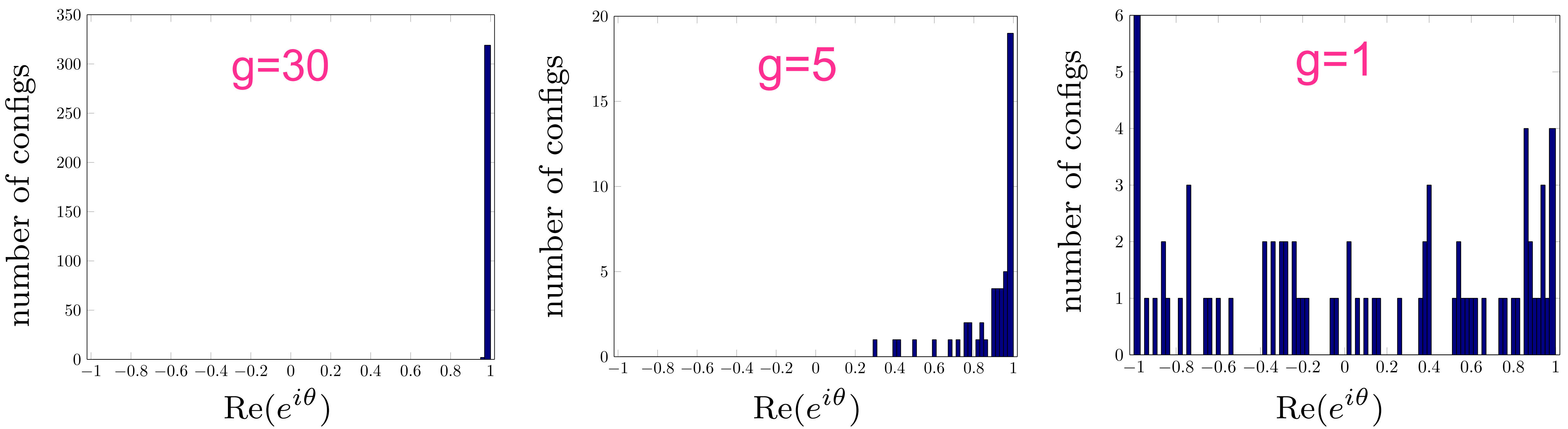

Dedicated algorithms: active field of study, no general proof of convergence.

Alternative linearization

The phase is implicit in the linearization, like $e^{-\frac{b^2}{4\,a}} = \int dx\, e^{-a\,x^2 + i\,b\,x}$

Consider a simple SO(4) invariant four-fermion interaction **[Catterall 2015]**

$$\mathcal{L}_{4F} = \frac{1}{2}\epsilon_{abcd}\,\psi^a(x)\,\psi^b(x)\,\psi^c(x)\,\psi^d(x) \equiv \Sigma^{ab}\,\widetilde{\Sigma}^{ab}$$

where $\Sigma^{ab} = \psi^a\psi^b$, $\widetilde{\Sigma}^{ab} = \frac{1}{2}\epsilon_{abcd}\,\psi^c\,\psi^d$. Introducing $\Sigma^{ab}_{\pm} = \frac{1}{2}\left(\Sigma^{ab} \pm \widetilde{\Sigma}^{cd}\right)$, rewrite

$$\mathcal{L}_{4F} = \pm\, 2\left(\Sigma^{ab}_{\pm}\right)^2$$

just exploiting the Graßmann character of the underlying fermions.

$$\pm\,\Sigma^{ab}_{\pm}\,\Sigma^{ab}_{\pm} = \pm\frac{1}{4}\left[\Sigma^{ab} \pm \frac{1}{2}\epsilon_{abcd}\,\Sigma^{cd}\right]\left[\Sigma^{ab} \pm \frac{1}{2}\epsilon_{abef}\,\Sigma^{ef}\right]$$

$$= \pm\frac{1}{4}\left[\Sigma^{ab}\Sigma^{ab} \pm \frac{1}{2}\epsilon_{abcd}\left(\Sigma^{ab}\Sigma^{cd} + \Sigma^{cd}\Sigma^{ab}\right) + \frac{1}{4}\epsilon_{abcd}\,\epsilon_{abef}\,\Sigma^{cd}\Sigma^{ef}\right]$$

since $\psi^a\psi^b\psi^a\psi^b = 0$

$$= \frac{1}{4}\left(\Sigma^{ab}\widetilde{\Sigma}^{ab} + \widetilde{\Sigma}^{ab}\Sigma^{ab}\right) = \frac{1}{2}\Sigma^{ab}\widetilde{\Sigma}^{ab}$$

Alternative linearization

In our case, $(\rho^M)^{im}(\rho^M)^{kn} = 2\epsilon^{imkn}$,

$$\mathcal{L}_{F4} = -\frac{1}{z^2}(\eta^2)^2 + \frac{1}{z^2}\big(i\,\eta_i(\rho^{MN})^i{}_j n^N \eta^j\big)^2$$

Alternative linearization

In our case, $(\rho^M)^{im}(\rho^M)^{kn} = 2\epsilon^{imkn}$, we analogously rewrite

$$\mathcal{L}_{F4} = -\frac{1}{z^2}(\eta^2)^2 \mp \frac{2}{z^2}(\eta^2)^2 \mp \frac{1}{z^2}\Sigma_{\pm}{}_i^j \Sigma_{\pm}{}_j^i$$

$$\Sigma_i{}^j = \eta_i \eta^j \,, \qquad \widetilde{\Sigma}_j{}^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l \,, \qquad \Sigma_{\pm}{}_i^j = \Sigma_i^j \pm \widetilde{\Sigma}_i^j$$

Choosing the good sign ($-$), new set of $1 + 16$ real auxiliary fields

$$\mathcal{L}_{\rm aux} = \frac{12}{z}\eta^2 \phi + 6\phi^2 + \frac{2}{z}\Sigma_{\pm}{}_j^i \phi_i^j + \phi_j^i \phi_i^j \qquad \mathcal{L}^\dagger_{\rm aux} = \mathcal{L}_{\rm aux}$$

Antisymmetry and Γ_5-hermiticity ($\Gamma_5^\dagger \Gamma_5 = \mathbb{1}$, $\Gamma_5^\dagger = -\Gamma_5$)

$$O_F^\dagger = \Gamma_5 \, O_F \, \Gamma_5 \,, \qquad O_F^T = -O_F$$

ensure positive-definite determinant $({\rm Pf} O_F)^2 = \det O_F \geq 0$, and a real Pfaffian.

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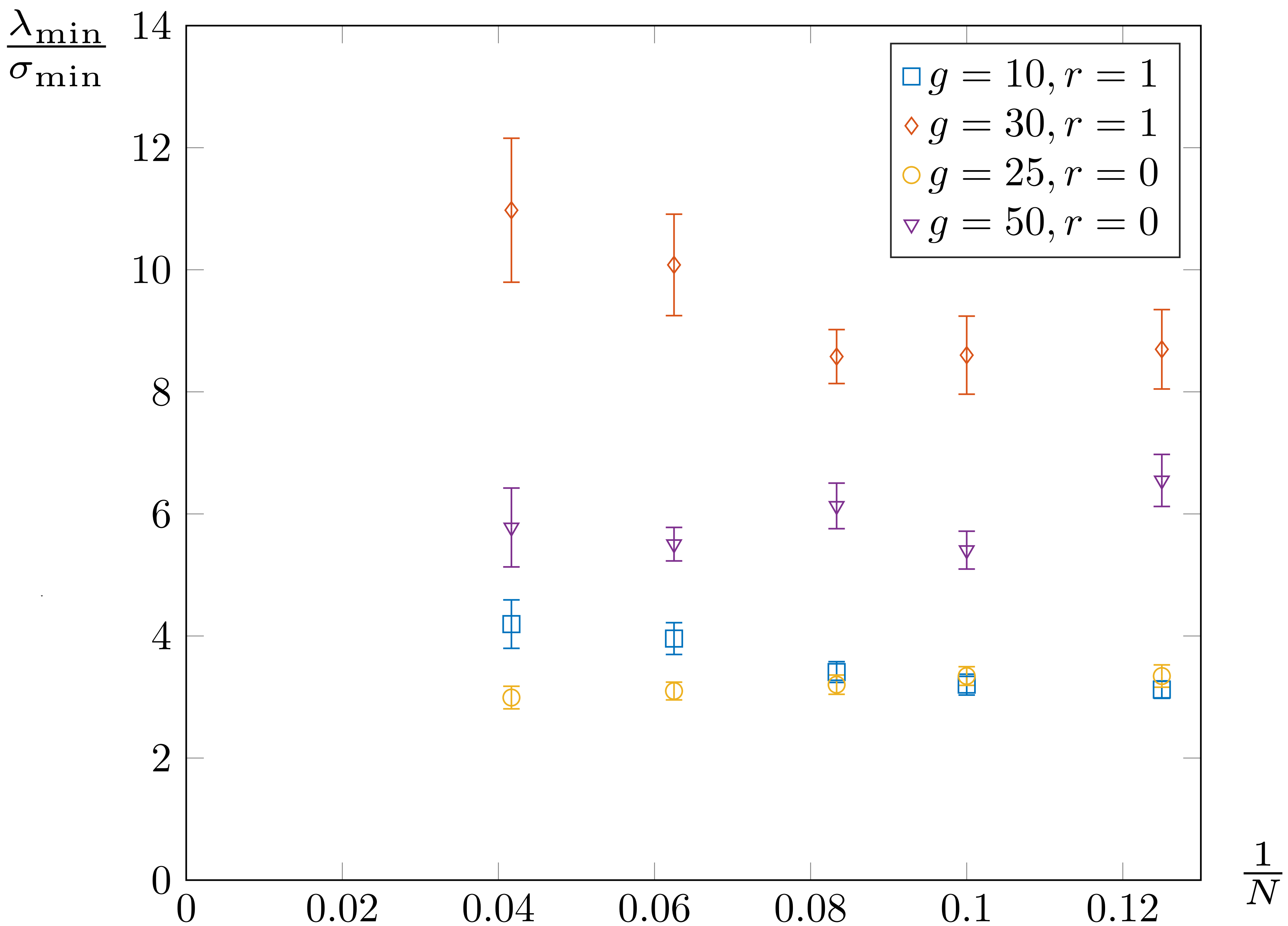
ensure positive-definite determinant $(\text{Pf} O_F)^2 = \det O_F \geq 0$, and a real Pfaffian.

In simpler models with four-fermion interactions, similar manipulations ensure a positive definite Pfaffian. [Catterall 2016, Catterall and Schaich 2016]
Here, gain in computational costs but Pf$O_F = \pm\sqrt{\det O_F}$.

Where are we sign-problem free?



Eigenvalue distribution of fermionic operators well separated from zero, no sign problem for $g \geq 10$, where nonperturbative physics is captured.

Discretization

Guiding lines for discretization

- Lattice perturbation theory $\xrightarrow{a\to 0}$ continuum perturbation theory
- Preserve the symmetries of the model
- No complex phases

Guiding lines for discretization

- Lattice perturbation theory $\xrightarrow{a\to 0}$ continuum perturbation theory

In the continuum, the free kinetic part of the fermionic operator

$$K_F = \begin{pmatrix} 0 & -p_0 \mathbb{1} & (p_1 - i\frac{m}{2})\rho^M u_M & 0 \\ -p_0 \mathbb{1} & 0 & 0 & (p_1 - i\frac{m}{2})\rho^\dagger_M u^M \\ -(p_1 + i\,\frac{m}{2})\rho^M u_M & 0 & 0 & -p_0 \mathbb{1} \\ 0 & -(p_1 + i\,\frac{m}{2})\rho^\dagger_M u^M & -p_0 \mathbb{1} & 0 \end{pmatrix}$$

gives the contribution $\det K_F = \left(p_0^2 + p_1^2 + \frac{m^2}{4}\right)^8$ to the one-loop partition function

$$\Gamma^{(1)} = -\ln Z^{(1)} = \frac{V_2}{a^2}\frac{1}{2}\int_{-\pi}^{\pi}\frac{dp_0 dp_1}{(2\pi)^2}\ln\Big[\frac{(p_0^2+p_1^2+m^2)(p_0^2+p_1^2+\frac{m^2}{2})^2(p_0^2+p_1^2)^5}{(p_0^2+p_1^2+\frac{m^2}{4})^8}\Big]$$

$$= -\frac{3\ln 2}{8\pi} m^2\, V_2$$

Guiding lines for discretization

- Lattice perturbation theory $\xrightarrow{a \to 0}$ continuum perturbation theory

A naive discretization $p_\mu \to \mathring{p}_\mu \equiv \frac{1}{a}\sin(a\, p_\mu)$ leads to fermion doublers,

$$K_F = \begin{pmatrix} 0 & -\mathring{p}_0 \mathbb{1} & (\mathring{p}_1 - i\frac{m}{2})\rho^M u_M & 0 \\ -\mathring{p}_0 \mathbb{1} & 0 & 0 & (\mathring{p}_1 - i\frac{m}{2})\rho_M^\dagger u^M \\ -(\mathring{p}_1 + i\frac{m}{2})\rho^M u_M & 0 & 0 & -\mathring{p}_0 \mathbb{1} \\ 0 & -(\mathring{p}_1 + i\,\frac{m}{2})\rho_M^\dagger u^M & -\mathring{p}_0 \mathbb{1} & 0 \end{pmatrix}$$

spoiling UV finiteness (effective 2d supersymmetry).

A Wilson-like fermion discretization

- Lattice perturbation theory $\stackrel{a\to 0}{\longrightarrow}$ continuum perturbation theory
- Preserve $SO(6)$, breaks $U(1) \sim SO(2)$
- No complex phases: $(O_F^W)^\dagger = \Gamma_5\, O_F^W\, \Gamma_5\,,\ (O_F^W)^T = -O_F^W$

Add to the action a "Wilson term", $K_F + W \equiv K_F^W$

$$K_F{}^W = \begin{pmatrix} W_+ & -\mathring{p}_0 \mathbb{1} & (\mathring{p}_1 - i\frac{m}{2})\rho^M u_M & 0 \\ -\mathring{p}_0 \mathbb{1} & -W_+^\dagger & 0 & (\mathring{p}_1 - i\frac{m}{2})\rho_M^\dagger u^M \\ -(\mathring{p}_1 + i\frac{m}{2})\rho^M u_M & 0 & W_- & -\mathring{p}_0 \mathbb{1} \\ 0 & -(\mathring{p}_1 + i\,\frac{m}{2})\rho_M^\dagger u^M & -\mathring{p}_0 \mathbb{1} & -W_-^\dagger \end{pmatrix}$$

where $W_\pm = \frac{r}{2}\left(\hat{p}_0^2 \pm i\,\hat{p}_1^2\right)\rho^M u_M$, $|r| = 1$, and $\hat{p}_\mu \equiv \frac{2}{a}\sin\frac{p_\mu a}{2}$, leads to

$$\Gamma^{(1)}_{\text{LAT}} = \frac{V_2}{2\,a^2}\int_{-\pi}^{+\pi}\frac{d^2p}{(2\pi)^2}\ln\Big[\frac{4^8(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2})^5(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2}+\frac{M^2}{8})^2(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2}+\frac{M^2}{4})}{\left(\sin^2 p_0+\sin^2 p_1+\frac{M^2}{4}+4\sin^4\frac{p_0}{2}+4\sin^4\frac{p_1}{2}\right)^8}\Big]$$

$$\stackrel{a\to 0}{\longrightarrow}\ -\frac{3\ln 2}{8\pi}\,V_2\, m^2\,,$$

cusp anomaly at strong coupling $(\,|r| = 1, M = m\,a.)$

Simulations, continuum limit: measurements

Parameter space, continuum limit ($a \to 0$)

- Two bare parameters, $g = \frac{\sqrt{\lambda}}{4\pi}$ and $P^+ \sim m$, assume the only additional scale is a

$$F_{\rm LAT} = F_{\rm LAT}\left(g, M, N\right) \qquad M = m\,a\,, \qquad N = \frac{L}{a}$$

- The continuum limit must be taken along a line of constant physics: curve in $\{g, M, N\}$ where physical quantities are kept fixed as $a \to 0$.

E.g.
$$\begin{aligned} m_x^2 &= \frac{m^2}{2}\left(1 - \frac{1}{8\,g} + \mathcal{O}(g^{-2})\right) \\ L^2\,m_x^2 &= \text{const} \qquad \longrightarrow \qquad (L\,m)^2 \equiv (N M)^2 = \text{const}\,. \end{aligned}$$

- For a generic observable

finite lattice spacing (~a) effects

finite volume (~ m L) effects

$$F_{\rm LAT} = F_{\rm LAT}(g, M, N) = F(g) + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(e^{-MN}\right)$$

Recipe: fix g, fix MN large enough, evaluate $F_{\rm LAT}$ for $N = 6, 8, 10, 12, 16, \ldots$;
Obtain $F(g)$ extrapolating to $N \to \infty$.

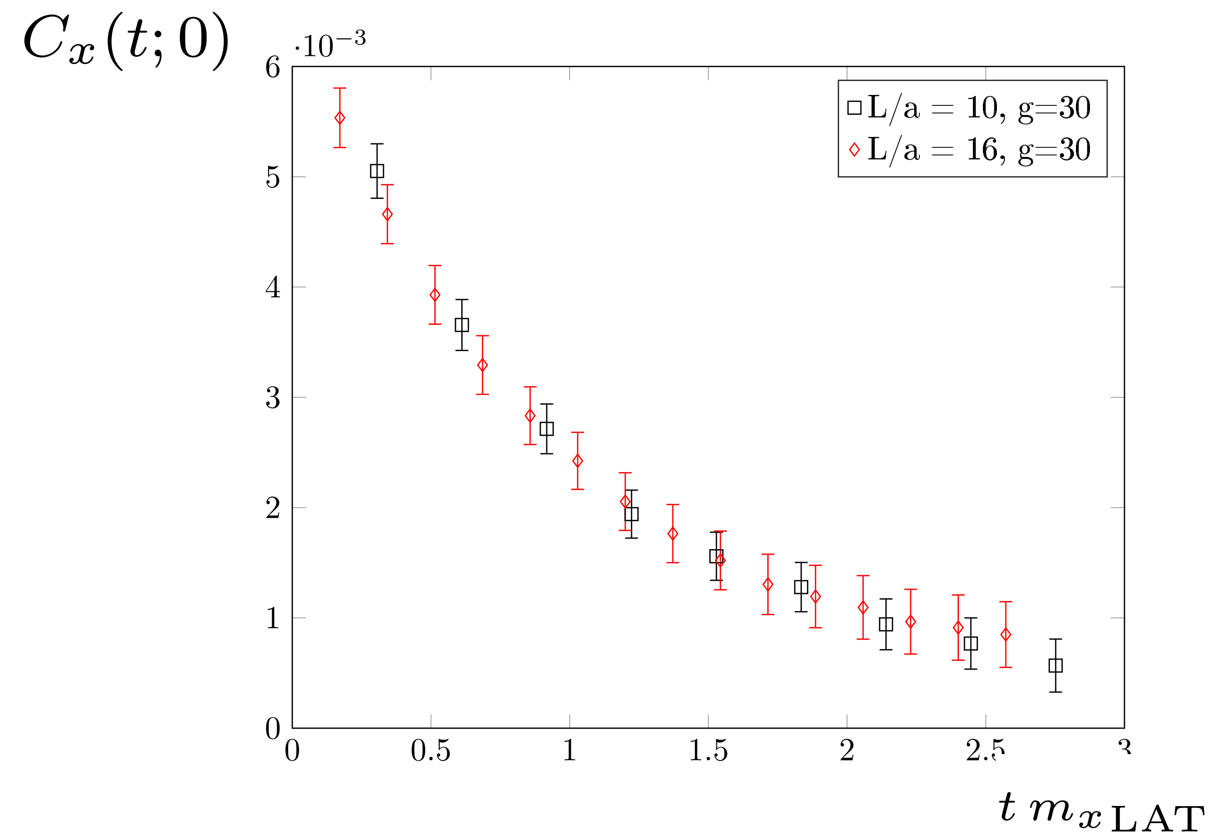
Measurement I: $\langle x, x^* \rangle$ correlator

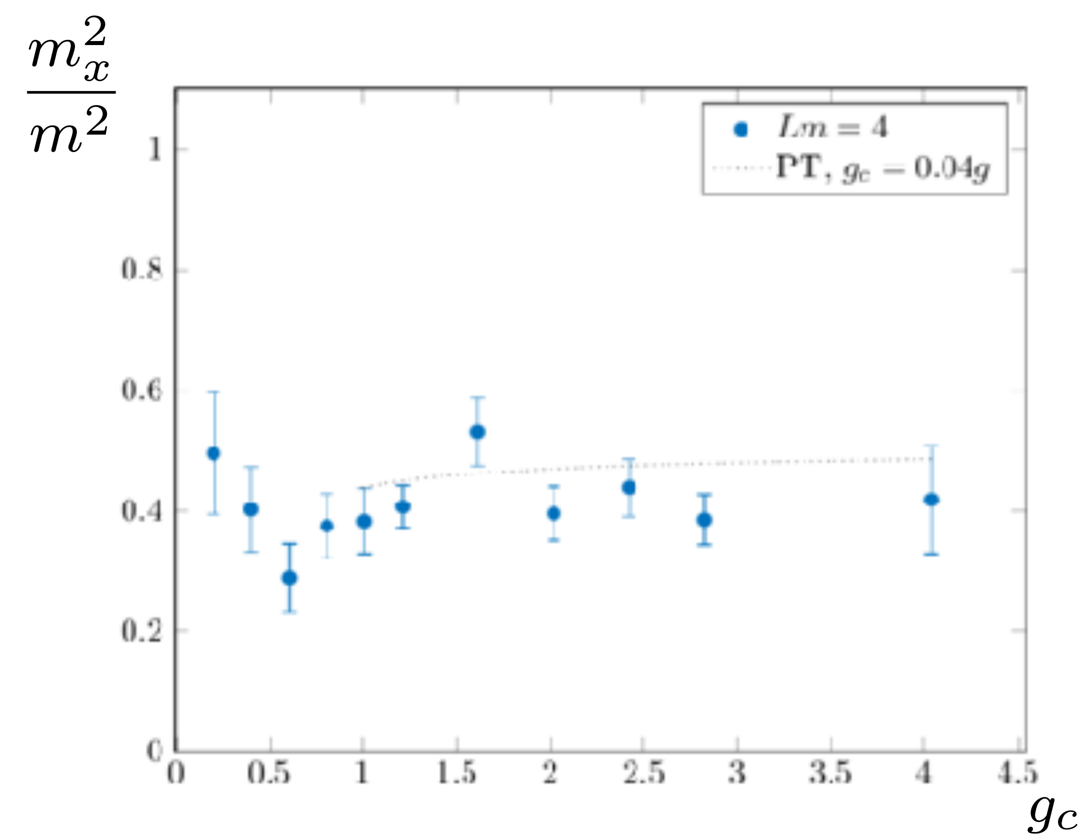
From the correlator of the x fields

$$C_x(t;0) = \sum_{s_1,\, s_2} \langle x(t,\, s_1) x^*(0,\, s_2) \rangle \underset{t \gg 1}{\sim} e^{-t\, m_{x\,\mathrm{LAT}}}$$

extract the x-mass

$$m_{x\,\mathrm{LAT}} = \lim_{t\to\infty} m_x^{\mathrm{eff}} \equiv \lim_{t\to\infty} \frac{1}{a} \log \frac{C_x(t;0)}{C_x(t+a;\, 0)}$$



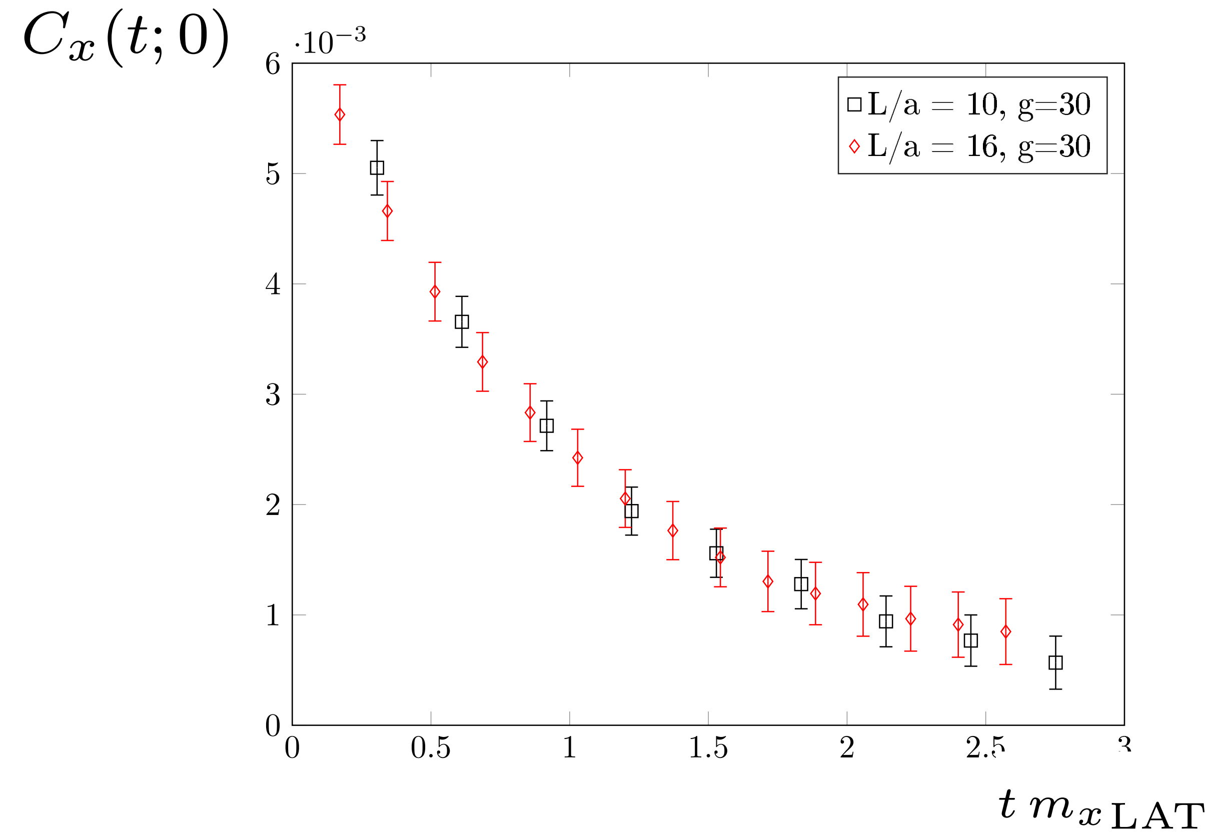


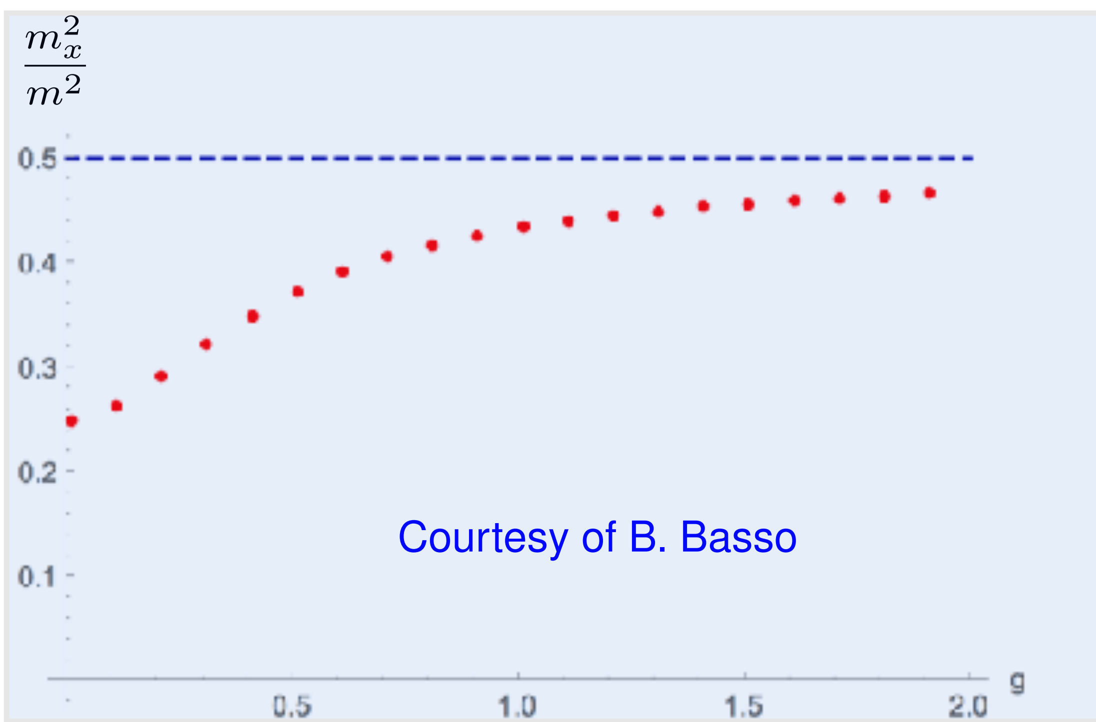
Measurement I: $\langle x, x^* \rangle$ correlator

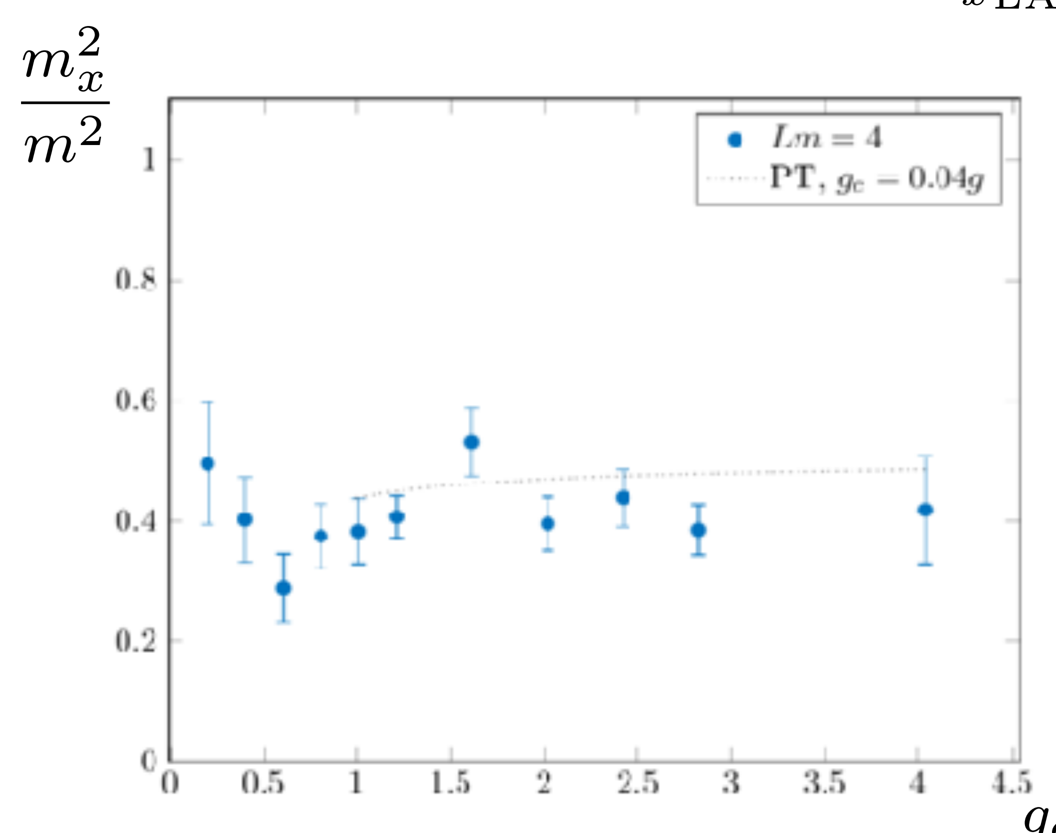
From the correlator of the x fields

$$C_x(t;0) = \sum_{s_1, s_2} \langle x(t,\, s_1) x^*(0,\, s_2) \rangle$$

$$\overset{t \gtrsim 1}{} \quad e^{-t\, m_{x\,\mathrm{LAT}}}$$







Consistent with large g prediction, no clear signal of bending down.

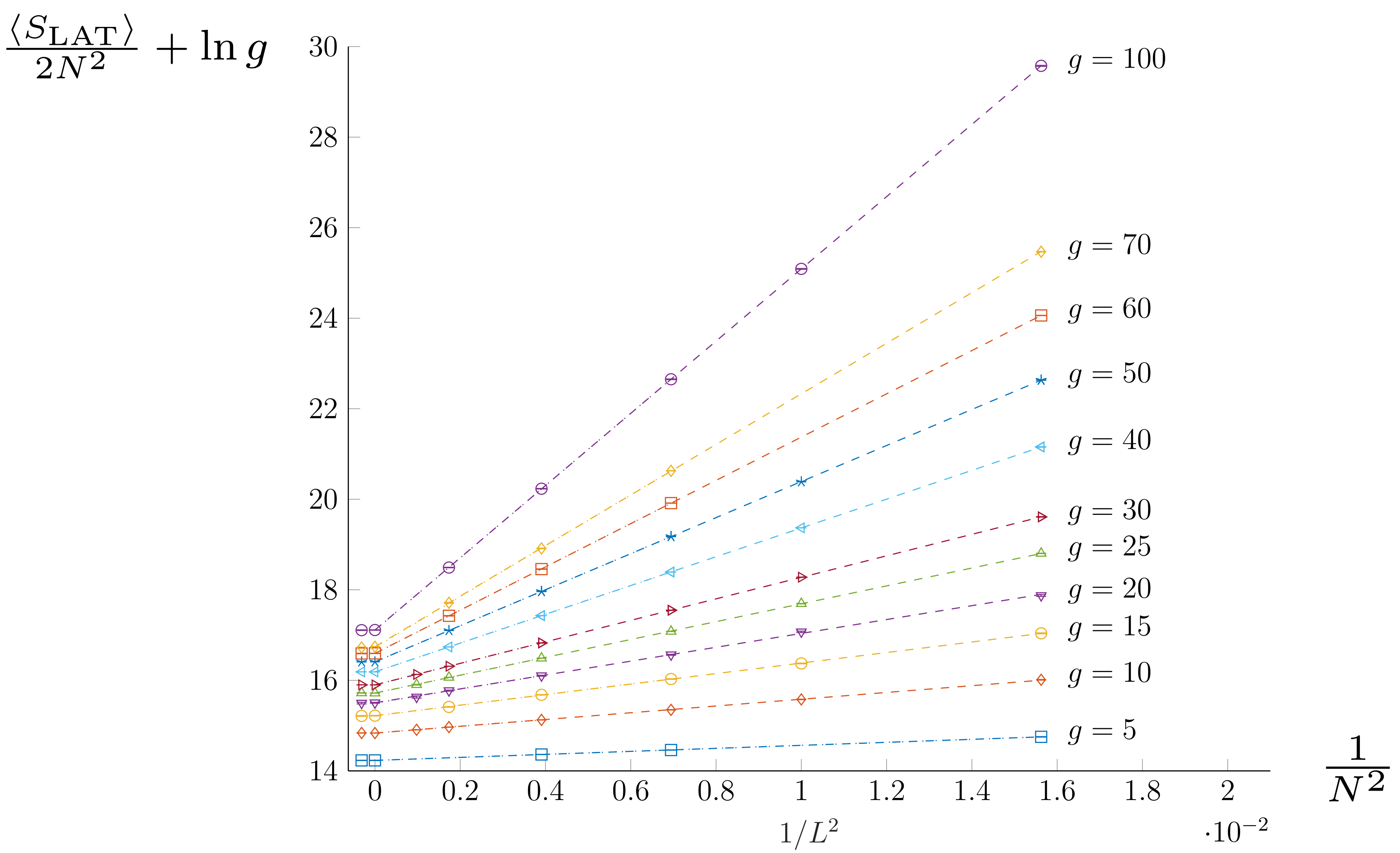
No infinite renormalization occurring.

Measurement II: (derivative of the) cusp anomaly

We measure $\langle S_{\rm cusp}\rangle \equiv g\,\frac{V_2\,m^2}{8}\,f'(g)$. At large g,

$$\langle S_{\rm LAT}\rangle \equiv g\,\frac{N^2\,M^2}{4}\,4 + \frac{c}{2}(2N^2)$$

quadratic divergences appear, with $c = \mathrm{n}_{bos} = 8 + 17 = 25$.

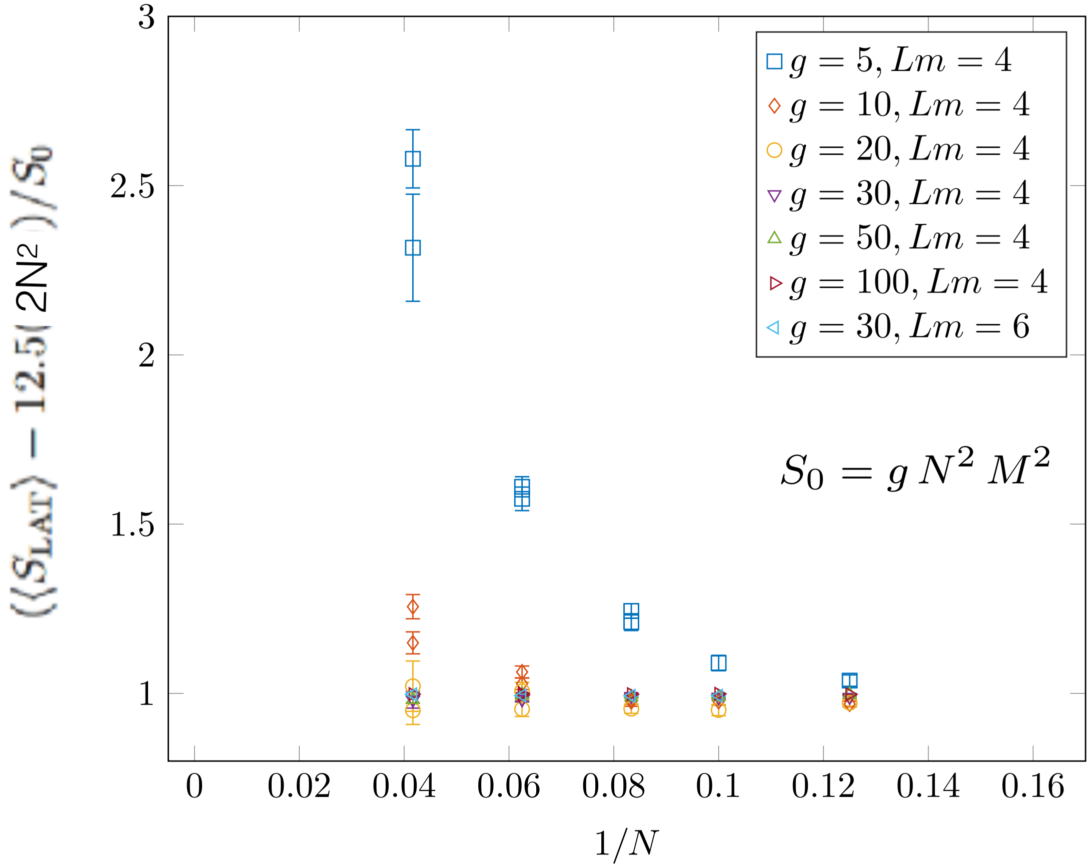


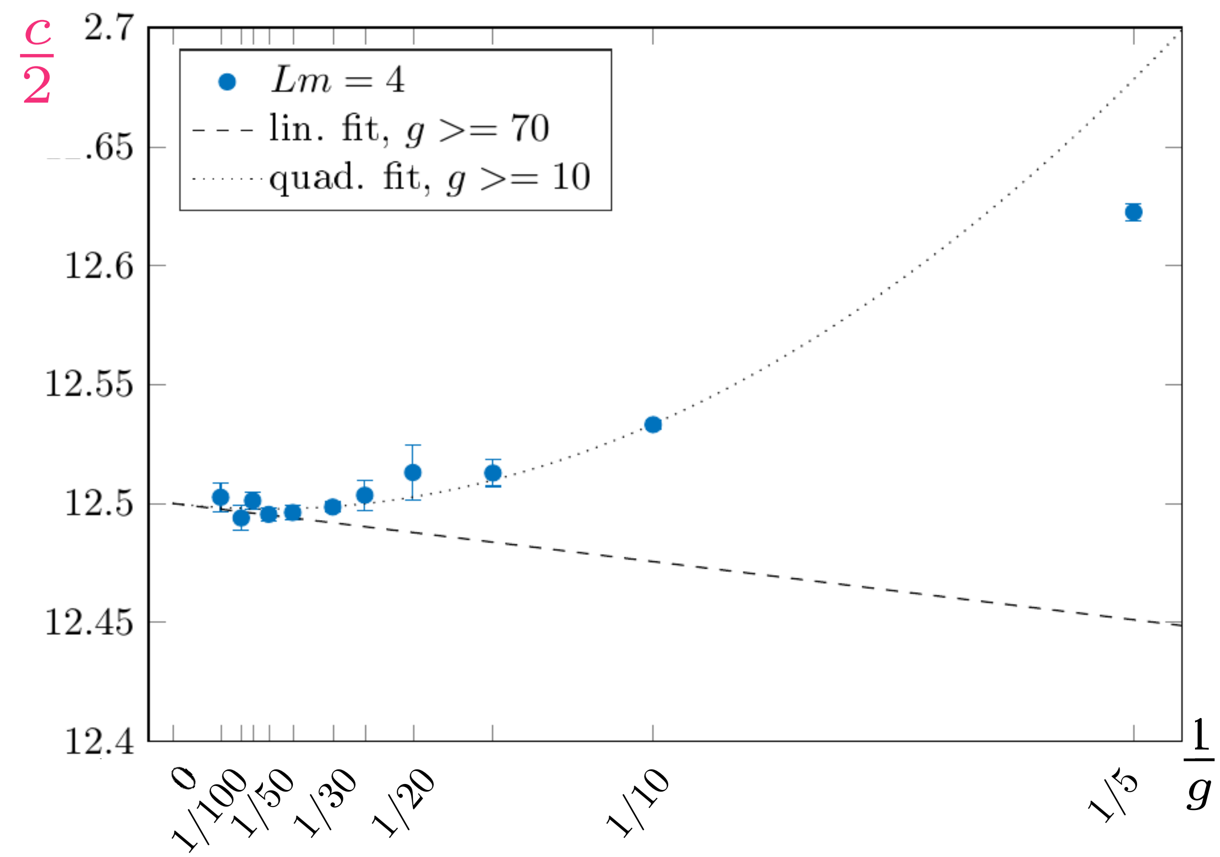
Indeed, $\langle S\rangle = -\frac{\partial \ln Z}{\partial \ln g}$ and $Z \sim \Pi_{\mathrm{n_{bos}}}(\det g\,\mathcal{O})^{-\frac{1}{2}}$, so for each bosonic species there is a factor $\sim g^{-\frac{(2N^2)}{2}}$. In lattice codes, coupling omitted from fermionic part.

Measurement II: (derivative of the) cusp anomaly

We measure $\langle S_{\rm cusp}\rangle \equiv g\, \frac{V_2\, m^2}{8}\, f'(g)$. At finite g,

$$\langle S_{\rm LAT}\rangle \equiv g\, \frac{N^2\, M^2}{4}\, f'_{\rm LAT}(g) + \frac{c(g)}{2}(2N^2)$$





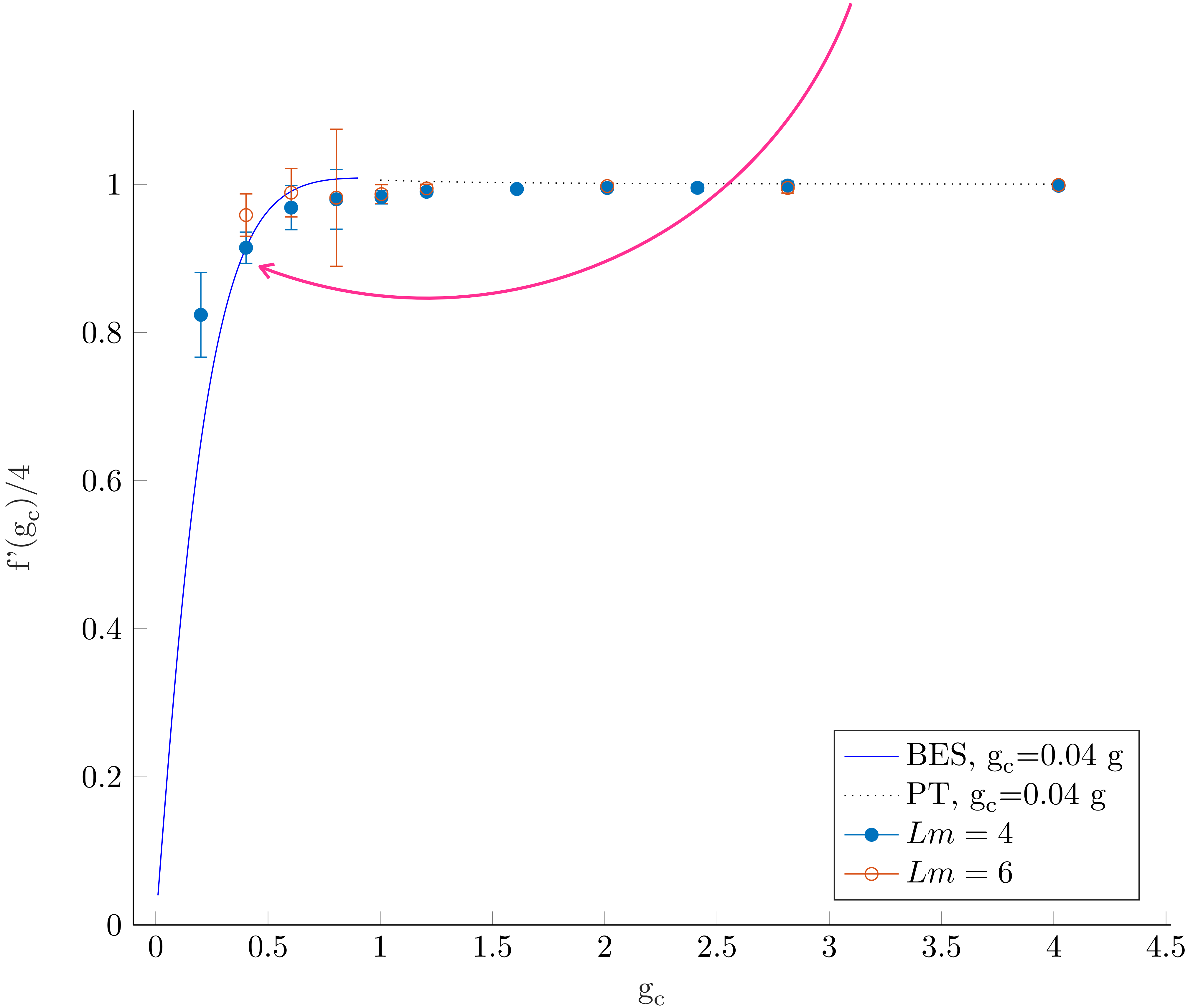
In continuum, existing power divergences are set to zero (dim. reg.)

Here, expected mixing of the Lagrangian with lower dimension operator

$$\mathcal{O}(\phi(s))_r = \sum_{\alpha:[O_\alpha]\leq D} Z_\alpha\, \mathcal{O}_\alpha(\phi(x))\,, \qquad Z_\alpha \sim \Lambda^{(D-[\mathcal{O}_\alpha])} \sim a^{-(D-[\mathcal{O}_\alpha])}$$

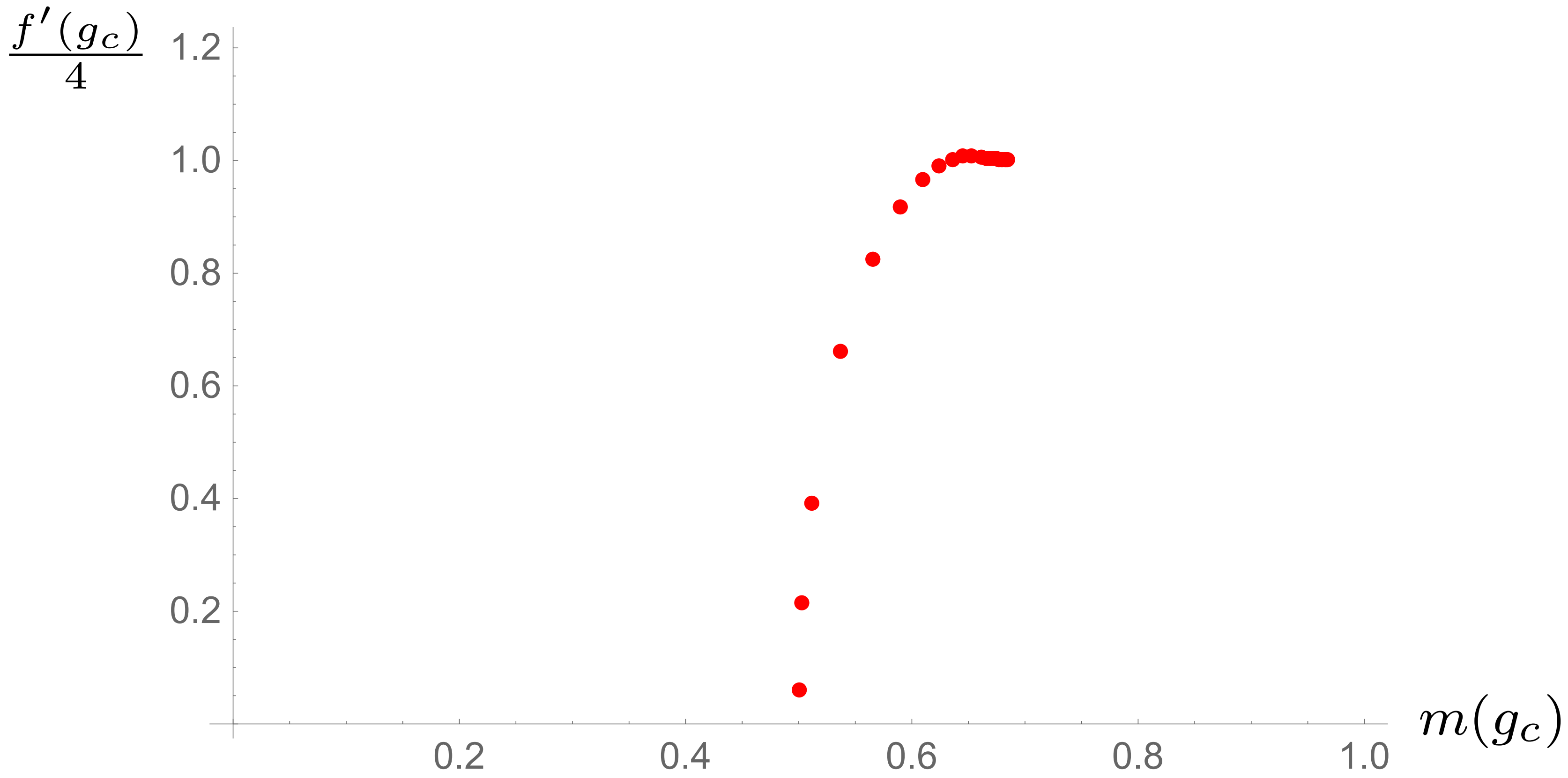
Measurement II: (derivative of the) cusp anomaly

To compare, assume $g = \alpha\, g_c$: then from $f'(g) = f'(g_c)_c$ is $g_c = 0.04g$.



In progress

- The relation among g_c and g may be non-trivial. Then the cusp may be “declared” as the coupling, and e.g. mass measurements plotted against it.



- We are observing an unexpected splitting in the fermionic masses ($m_F^2 = \frac{1}{2}$) related to the $U(1)$-breaking of the discretization.
  The corresponding Ward identity may be used as renormalization condition, a single tuning is expected.

- We are extending our simulations to $g \leq 5$.

On the CFT side

Strong sign problem at strong coupling ($\lambda \gg 1$), one tuning.

The control is in the perturbative region (matching with NNLO).

Courtesy of David Schaich

Coupling dependence of Coulomb coefficient

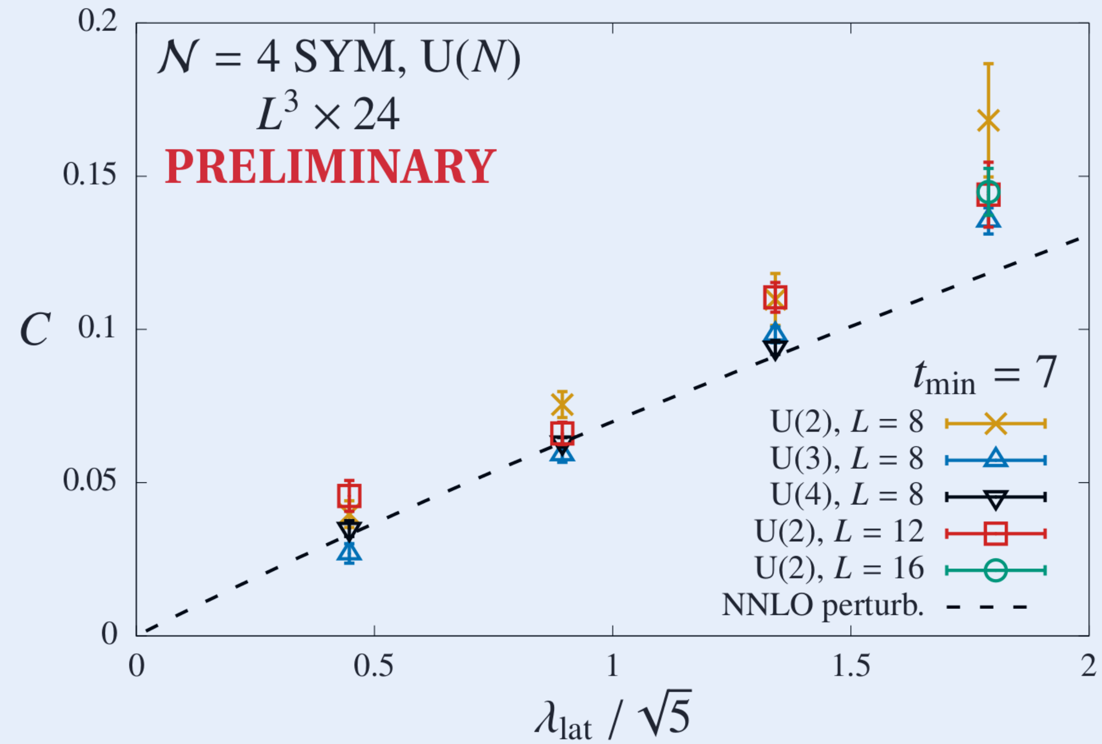
Fit $V(r)$ to Coulombic
or confining form

$$V(r) = A - C/r$$

$$V(r) = A - C/r + \sigma r$$

C is Coulomb coefficient

σ is string tension



$V(r)$ is Coulombic at all λ:
fits to confining form produce vanishing string tension

C for U(4) in good agreement with perturbation theory for $\lambda \lesssim 3/\sqrt{5}$

U(2) and U(3) results less stable — working on further improvements

Conclusions

Solving a non-trivial 4d QFT is **hard** $\longrightarrow$ reduce the problem via AdS/CFT:

solve (finding a good regulator for) a non-trivial 2d QFT.

- I presented a study of lattice field theory methods for *gauge-fixed* string σ-models relevant in AdS/CFT: address ab initio, non-perturbative calculations within them.
  - The model – GS string on GKP vacuum – is amenable to study using standard techniques (Wilson-like fermion discretizations, RHMC algorithm).
  - We observe good agreement with expectation at large g, and indications of non-perturbative physics;

  Ongoing work on several open questions, which include the proper continuum limit.

- Future: different backgrounds/gauge-fixing/observables . . .

- Non-perturbative definition of string theory?
  For sure, suitable framework for first principle statements (proofs of AdS/CFT) and (potentially) very efficient tool in numerical holography.

Thanks for your attention.

Extra-slides

Boundary conditions

Fluctuations must vanish at the AdS boundary (two sides of the grid)

$$\tilde{X}(t=-\infty, s) = 0 = \tilde{X}(t, s=+\infty)$$

and be free to fluctuate elsewhere. Field redefinitions adopted in the continuum lead to exotic (unstable) boundary conditions.

So far we used periodic BC for all the fields (antiperiodic temporal BC for fermions).

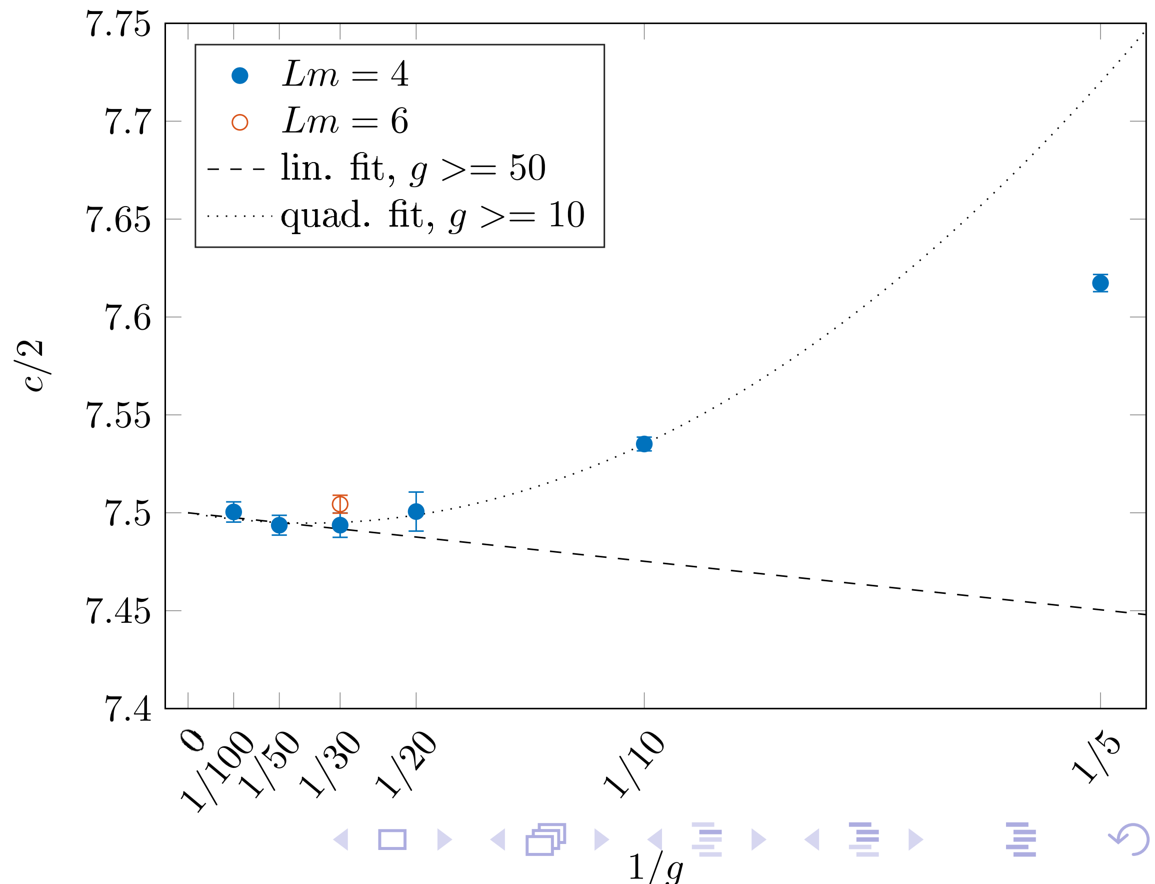
and evaluated finite volume effects $\sim e^{-m\,L} \equiv e^{-M\,N}$.

Most run are done at $M\,N = 4$ ($e^{-4} \simeq 0.02$),

some at $M\,N = 6$ ($e^{-6} \simeq 0.002$).

Appear to play a role only in evaluating

the coefficient of divergences.



A remark on numerics

The most difficult part of the algorithm is the inversion of the fermionic matrix

$$|\mathrm{Pf}\, O_F| \equiv (\det O_F^\dagger O_F)^{\frac{1}{4}} \equiv \int d\zeta d\bar\zeta\, e^{-\int d^2\xi\, \bar\zeta\, (O_F^\dagger O_F)^{-\frac{1}{4}}\, \zeta} \,.$$

The RHMC (Rational Hybrid Montecarlo) uses a rational approximation

$$\bar\zeta\, (O_F^\dagger O_F)^{-\frac{1}{4}}\, \zeta = \alpha_0\, \bar\zeta\, \zeta + \sum_{i=1}^{P} \bar\zeta\, \frac{\alpha_i}{O_F^\dagger O_F + \beta_i}\, \zeta$$

with α_i and β_i tuned by the range of eigenvalues of O_F.

Defining $s_i \equiv \frac{1}{O_F^\dagger O_F + \beta_i}\, \zeta$, one solves

$$(O_F^\dagger O_F + \beta_i)\, s_i = \zeta\,, \qquad\qquad i = 1, \ldots, P.$$

with a (multi-shift conjugate) solver for which

$$\text{number of iterations} \sim \lambda_{\min}^{-1}$$

In our case the spectrum of O_F has very small eigenvalues.

And: $O_F = \left| \begin{array}{cc} & i\partial_t \\ i\frac{z^M}{z^3}\rho^M & (\partial_s - \frac{m}{2}) \end{array} \right.$

Alternative linearization

Γ_5-hermiticity and antisymmetry hold now for the full operator (including aux. fields)

$$O_F^\dagger = \Gamma_5 \, O_F \, \Gamma_5 \, , \qquad\qquad O_F^T = -O_F$$

Pfaffian is real, $(\mathrm{Pf} O_F)^2 = \det O_F \geq 0$, but not positive definite, $\mathrm{Pf} O_F = \pm \det O_F$.

Gain in computational costs: for large values of N (finer lattices) the algorithm for evaluating complex determinants is very inefficient. Now just a sign flip.

$$\langle \mathcal{O} \rangle_{\text{reweight}} = \frac{\langle \mathcal{O} \, e^{i\theta} \rangle_{\theta=0}}{\langle e^{i\theta} \rangle_{\theta=0}} \qquad \longrightarrow \qquad \langle \mathcal{O} \rangle_{\text{reweight}} = \frac{\langle \mathcal{O} \, w \rangle_{\sqrt{\det O_F}}}{\langle w \rangle_{\sqrt{\det O_F}}}$$

where $w = \pm 1$, and $\sqrt{\det O_F} = (\det O_F^\dagger \, O_F)^{\frac{1}{4}}$.

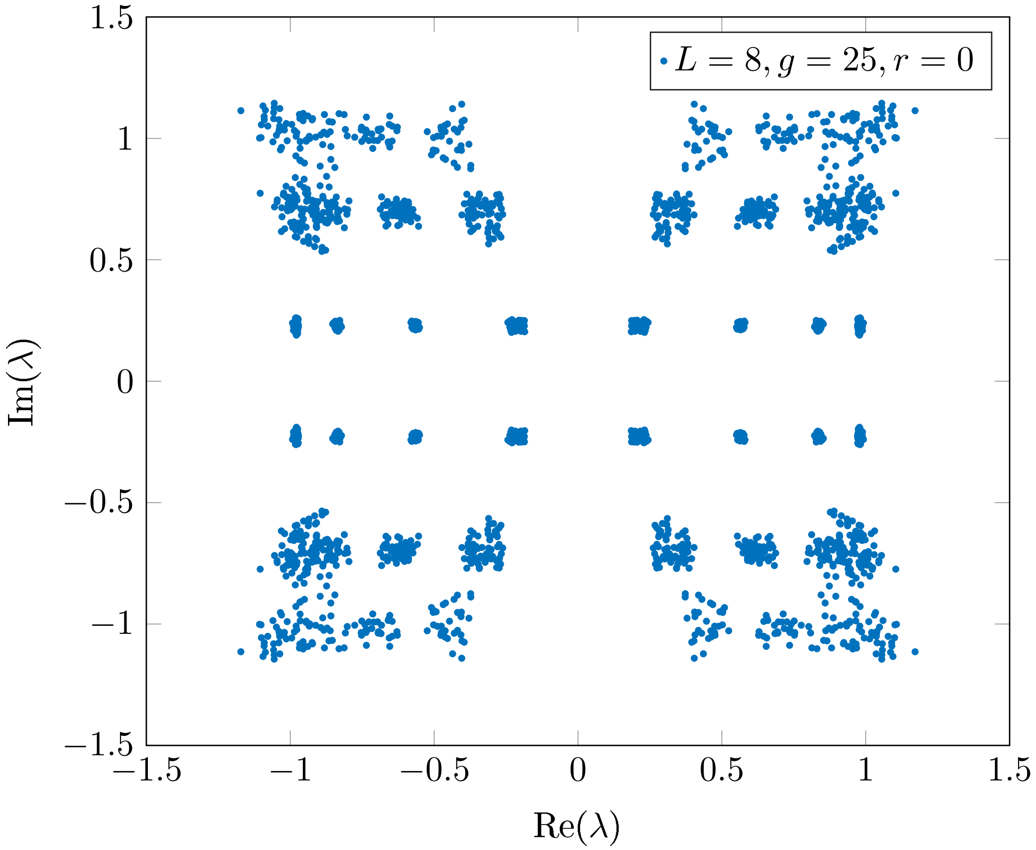
In simpler models with four-fermion interactions, similar manipulations ensure a definite positive Pfaffian. There real, antisymmetric operator with doubly degenerate eigenvalues: quartets $(ia, ia, -ia, -ia) \, , a \in \mathbb{R}$.

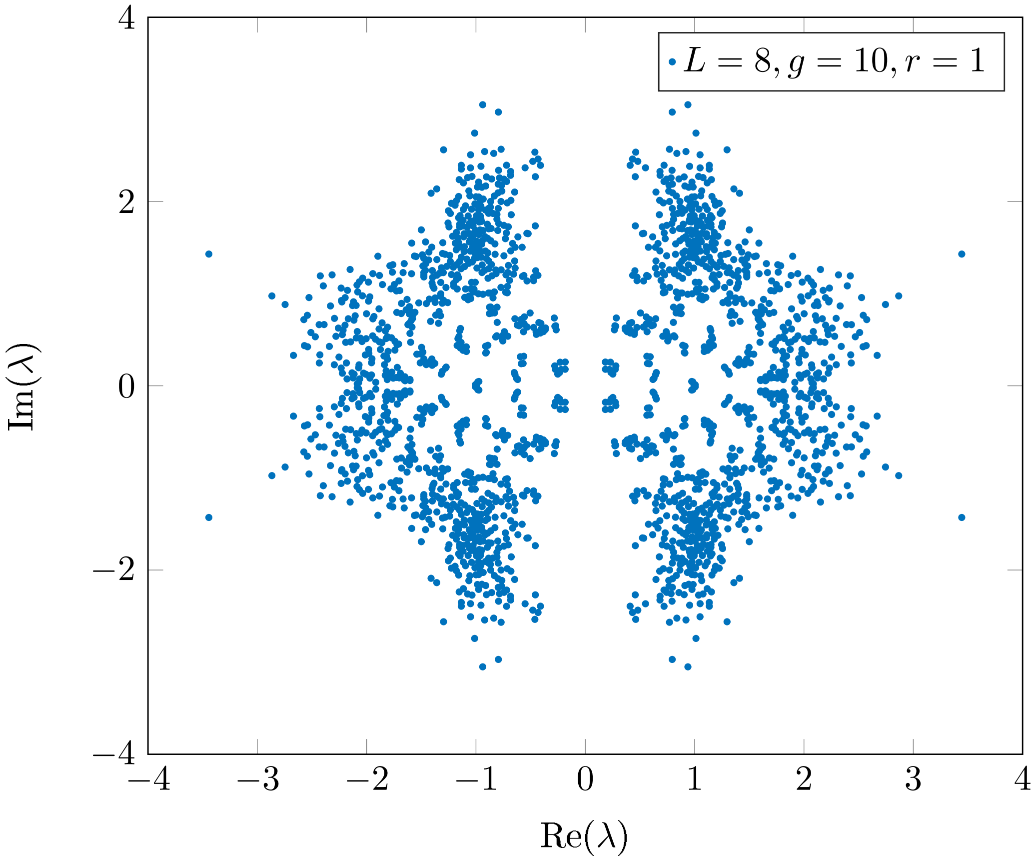
[Catterall 2016, Catterall and Schaich 2016]

Spectrum of O_F

From Γ_5-hermiticity and antisymmetry,

$$\mathcal{P}(\lambda) = \det(O_F - \lambda \mathbb{1}) = \det(\Gamma_5 (O_F - \lambda \mathbb{1}) \Gamma_5)$$
$$= \det(O_F^\dagger + \lambda \mathbb{1}) = \det(O_F + \lambda^* \mathbb{1})^* = \mathcal{P}(-\lambda^*)^*$$





Spectrum characterized by quartets $\{\lambda, -\lambda^*, -\lambda, \lambda^*\}$.

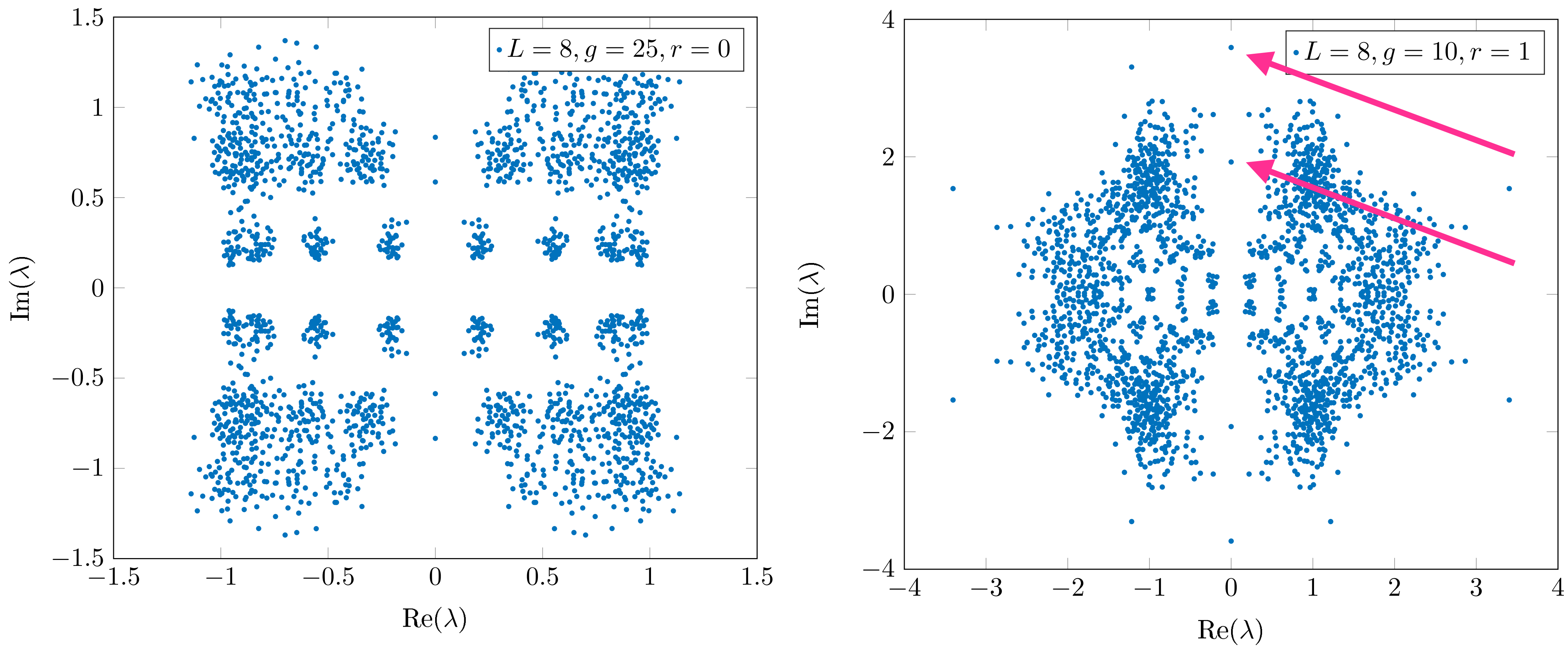
$$\det O_F = \prod_i |\lambda_i|^2 \, |\lambda_i|^2 \quad \longrightarrow \quad \mathrm{Pf}(O_F) = \pm \prod_i |\lambda_i|^2$$

Choosing a starting configuration with positive Pfaffian, no sign change possible.

Spectrum of O_F

From Γ_5-hermiticity and antisymmetry,

$$\mathcal{P}(\lambda) = \det(O_F - \lambda \mathbb{1}) = \det(\Gamma_5\,(O_F - \lambda \mathbb{1})\,\Gamma_5)$$
$$= \det(O_F^\dagger + \lambda \mathbb{1}) = \det(O_F + \lambda^* \mathbb{1})^* = \mathcal{P}(-\lambda^*)^*$$

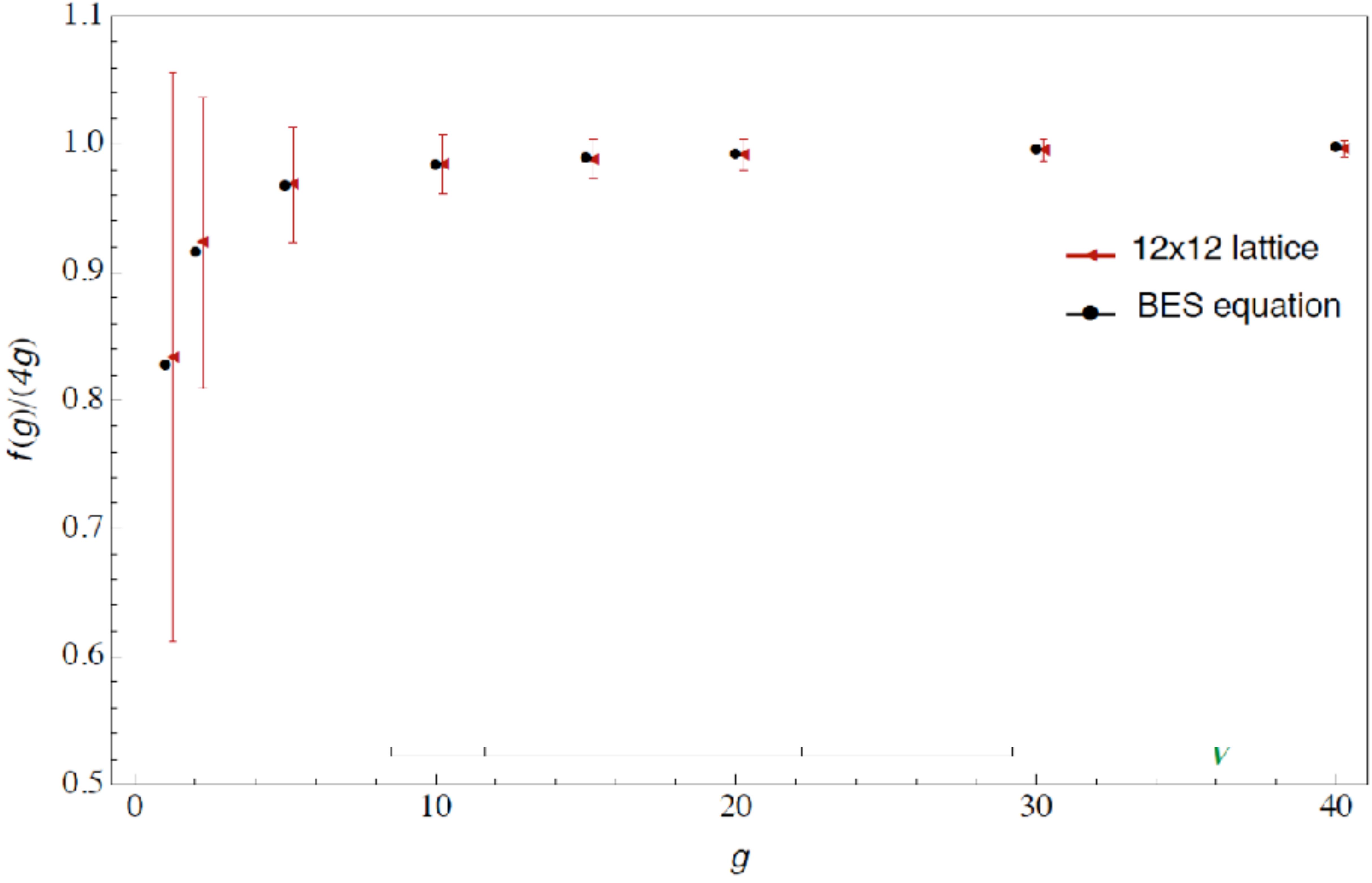


For $\lambda = \pm\lambda^*$, no four-fold property: due to zero crossings, Pfaffian may change sign.

Purely imaginary eigenvalues correspond to Yukawa-terms, even those present in the original Lagrangian: no “suitable enough” choice of auxiliary fields.

Previous study

[McKeown Roiban, arXiv: 1308.4875]



Parameters of the simulations

g	$T/a \times L/a$	Lm	am	τ_{int}^{S}	$\tau_{\text{int}}^{m_x}$	statistics [MDU]
5	16×8	4	0.50000	0.8	2.2	900
	20×10	4	0.40000	0.9	2.6	900
	24×12	4	0.33333	0.7	4.6	900,1000
	32×16	4	0.25000	0.7	4.4	850,1000
	48×24	4	0.16667	1.1	3.0	92,265
10	16×8	4	0.50000	0.9	2.1	1000
	20×10	4	0.40000	0.9	2.1	1000
	24×12	4	0.33333	1.0	2.5	1000,1000
	32×16	4	0.25000	1.0	2.7	900,1000
	48×24	4	0.16667	1.1	3.9	594,564
20	16×8	4	0.50000	5.4	1.9	1000
	20×10	4	0.40000	9.9	1.8	1000
	24×12	4	0.33333	4.4	2.0	850
	32×16	4	0.25000	7.4	2.3	850,1000
	48×24	4	0.16667	8.4	3.6	264,580
30	20×10	6	0.60000	1.3	2.9	950
	24×12	6	0.50000	1.3	2.4	950
	32×16	6	0.37500	1.7	2.3	975
	48×24	6	0.25000	1.5	2.3	533,652
	16×8	4	0.50000	1.4	1.9	1000
	20×10	4	0.40000	1.2	2.7	950
	24×12	4	0.33333	1.2	2.1	900
	32×16	4	0.25000	1.3	1.8	900,1000
	48×24	4	0.16667	1.3	4.3	150
50	16×8	4	0.50000	1.1	1.8	1000
	20×10	4	0.40000	1.2	1.8	1000
	24×12	4	0.33333	0.8	2.0	1000
	32×16	4	0.25000	1.3	2.0	900,1000
	48×24	4	0.16667	1.2	2.3	412
100	16×8	4	0.50000	1.4	2.7	1000
	20×10	4	0.40000	1.4	4.2	1000
	24×12	4	0.33333	1.3	1.8	1000
	32×16	4	0.25000	1.3	2.0	950,1000
	48×24	4	0.16667	1.4	2.4	541

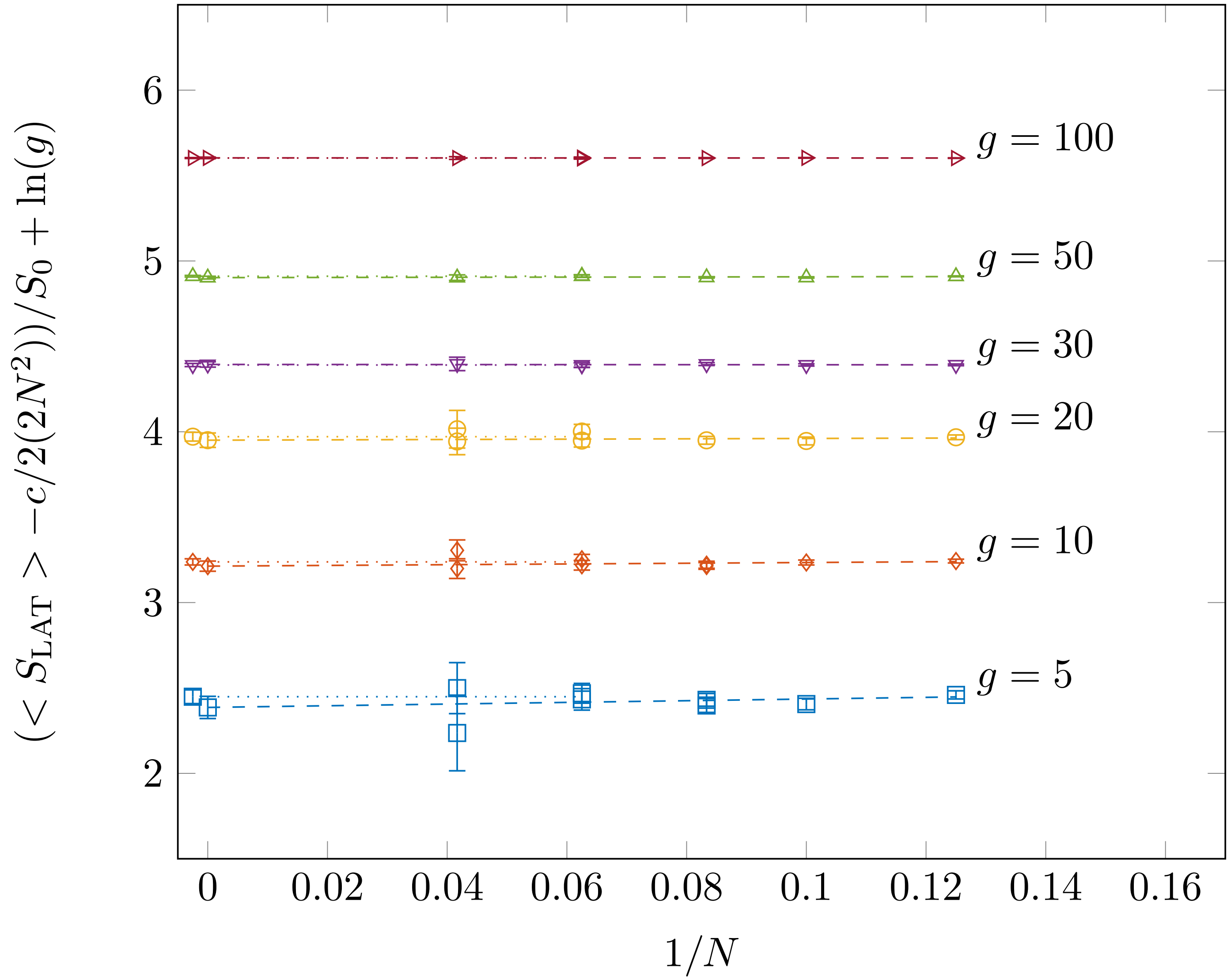
Table 1: Parameters of the simulations: the coupling g, the temporal (T) and spatial (L) extent of the lattice in units of the lattice spacing a, the line of constant physics fixed by Lm and the mass parameter $M = am$. The size of the statistics after thermalization is given in the last column in terms of Molecular Dynamic Units (MDU), which equals an HMC trajectory of length one. In the case of multiple replica the statistics for each replica is given separately. The auto-correlation times τ of our main observables m_x and S are also given in the same units.

Measurement II: (derivative of the) cusp anomaly

We proceed subtracting the continuum extrapolation of $\frac{c}{2}$ multiplied by N^2: divergences appear to be completely subtracted, confirming their quadratic nature.

Errors are small, and do not diverge for $N \to \infty$.

Flatness of data points indicates very small lattice artifacts.



We can thus extrapolate at infinite N to show the continuum limit.