Poincaré Symmetry shapes the Massive 3-point Amplitude

(and BMS?)

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Motivations

On-shell techniques

for scattering amplitudes...

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On-shell techniques

for scattering amplitudes

WHY?

Motivations

- Efficiency

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- ▷ Efficiency
- ▷ Underlying simplicity (symmetries)

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- ▷ Efficiency
- ▷ Underlying simplicity (symmetries)
- ▷ Beyond the perturbative regime

e.g.: Parke-Taylor formula for MHV gluon tree-level amplitudes

$$M_n(\ldots, i^-, \ldots, j^-, \ldots) = \frac{\langle i, j\rangle^4}{\langle 1,2\rangle\langle 2,3\rangle \cdots \langle n-1, n\rangle\langle n,1\rangle}$$

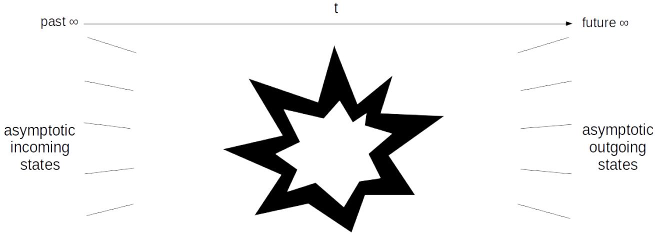
proved by induction with BCFW recursion relations for any n, while increasingly painful for increasing n with Feynman graphs...

n	4	5	6	$\cdots$	8
#	4	25	220	$\cdots$	10525900

Philosophy

Reconstructing the amplitude with the minimal amount of information.

Amplitude is an asymptotic object



The states of the Hilbert space (particles) are identified by the symmetry of space-time (Wigner classification)

Uniqueness of the 3-p Amplitude

Lorentz structure of three-point amplitudes is fixed by symmetry (at any order in perturbation theory!). Theory dependent information is all in the coupling constants.

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Uniqueness of Poincaré invariant 3-point amplitude in *complex spinor formalism* for **massless** legs, up to *one* constant (coupling).

[Benincasa-Cachazo '07]

Uniqueness of Poincaré invariant 3-point amplitude in *complex spinor formalism* for **any masses**, up to *some* constants.

[E. Conde, AM arxiv/1601.08113]

Outline

Poincaré in 4 dim

Casimir operators

P^2	square of translation generator	$\longrightarrow$ *mass*
W^2	square of Pauli-Lubanski operator	$\longrightarrow$ *spin*

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$W_\lambda = \epsilon_{\lambda\mu\nu\rho} M^{\mu\nu} P^\rho$ generator of the Little Group

$$\mathrm{LG}_p = \left\{ \Lambda_p \in L_+^\uparrow \middle/ \Lambda_p p = p \right\}$$

LG for massless representations

$$p \xrightarrow{L_p} k = (E, 0, 0, E)$$

$\Longrightarrow \ \mathrm{LG}_k \equiv \mathrm{ISO}_2:$ Isometries in 2 dim. euclidean space

$$\Lambda_k |k; a\rangle = e^{i\alpha A} e^{i\beta B} e^{i\theta J_3} |k; a\rangle$$

If $\alpha, \beta \neq 0 \Rightarrow$ continuous spin

J_3 admits for discrete eigenvalues: $\pm h \ \longrightarrow$ *helicity*

$$J_3 \, |p; h\rangle = h \, |p; h\rangle$$

LG for massive representations

$$P \xrightarrow{L_P} K = (m, 0, 0, 0)$$

$\Longrightarrow$ $\mathrm{LG}_K \equiv \mathrm{SO}(3)$: 3-dim. spatial rotations

$$J_0 \, |P; s, \sigma\rangle = \sigma \, |P; s, \sigma\rangle$$

$$J_\pm |P; s, \sigma\rangle = \sigma_\pm |P; s, \sigma \pm 1\rangle$$

$$\sigma \in \{-s, \ldots, +s\}$$

$$\sigma_\pm = \sqrt{(s \mp \sigma)(s \pm \sigma + 1)}$$

How is this story helpful to constrain the amplitude?

$$|p;a\rangle \quad \longrightarrow \quad M_n \sim \bigotimes_{i=1}^{n} |p_i;a_i\rangle$$

$P \quad \longrightarrow \quad$ momentum conservation $\quad \longrightarrow \quad M_n \propto \delta\big(\sum_i p_i\big)$

$W \quad \longrightarrow \quad$ little group scaling "spin conservation" $\quad \longrightarrow \quad$ LG equations

LG equations in the massless case

From the LG action on the states descends the LG action on the amplitude

$$\begin{aligned} e^{i\theta J_3}|p;h\rangle &= e^{i\theta h}|p;h\rangle \\ &\Downarrow \\ e^{i\theta J_3^j}M_n(\{p_i,h_i\}) &= e^{i\theta h_j}M_n(\{p_i,h_i\}) \end{aligned}$$

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\begin{aligned}
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&\Downarrow \\
e^{i\theta J_3^j} M_n(\{p_i,h_i\}) &= e^{i\theta h_j} M_n(\{p_i,h_i\})
\end{aligned}
$$

The infinitesimal version of this equation,

$$J_3^j M_n(\{p_i,h_i\}) = h_j M_n(\{p_i,h_i\})$$

yields strong constraints on the amplitude, and it is actually enough to fully fix the 3-point one.

LG equations for massive particles

$$J_0^I \, M_n(\lambda_j\tilde{\lambda}_j; a_k) = \sigma_I \, M_n(\lambda_j\tilde{\lambda}_j; a_k)$$ equivalent of helicity eq.

$$J_\pm^I \, M_n(\lambda_j\tilde{\lambda}_j; ..., \sigma_I, ...) = \sigma_I^\pm \, M_n(\lambda_j\tilde{\lambda}_j; ..., \sigma_I \pm 1, ...)$$

$j = 1, \ldots, n + \#$ of massive particles

The latter equations relate different amplitudes!

Equations for the "lowest-spin" amplitude

Solution: let's take $\sigma_I = -s_I$ for every massive particle. (helicities of possible massless legs are still free to vary)
Then

$$J_-^I \, M_n = 0$$

$$J_0^I \, M_n = -s_I \, M_n$$

$$(J_+^I)^{2s_I+1} M_n = 0$$

The third equation is not as simple as the others, let's keep it for the end. So

2 eq.s for every massive leg + 1 eq. for every massless leg

Spinor-Helicity formalism...

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$$\mathrm{L}_+^\uparrow(\mathbb{R}) \xrightarrow[\text{1 to 2}]{\text{homomorph.}} \mathrm{SL}(2,\mathbb{C})$$

$$\mathrm{L}_+(\mathbb{C}) \xrightarrow[\text{1 to 2}]{\text{homomorph.}} \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$$

$$p_\mu \longrightarrow p_{a\dot{a}} = \sigma^\mu_{a\dot{a}} p_\mu \qquad \sigma^\mu = (\mathbb{I}, \vec{\sigma})$$

$$\Lambda_\mu{}^\nu p_\nu \longrightarrow \zeta_a{}^b\, p_{b\dot{b}}\, \eta^{\dot{b}}{}_{\dot{a}}$$

reality condition: $\eta \equiv \zeta^\dagger$

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$$\boxed{p_\mu p^\mu = \det |p_{a\dot{a}}|}$$

… FOR MASSLESS MOMENTA

$$\det |p_{a\dot{a}}| = p_\mu p^\mu = 0$$

$$\Downarrow$$

$$\boxed{p_{a\dot{a}} = \lambda_a \otimes \tilde{\lambda}_{\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}}$$

reality condition: $\tilde{\lambda}_{\dot{a}} \equiv (\lambda_a)^*$

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We define:

$$\langle \lambda, \mu \rangle = \epsilon^{ab} \lambda_b \mu_a$$

$$[\tilde{\lambda}, \tilde{\mu}] = \tilde{\lambda}_{\dot{a}} \epsilon^{\dot{a}\dot{b}} \tilde{\mu}_{\dot{b}}$$

$$\langle i, j \rangle [i, j] \equiv \langle \lambda_i, \lambda_j \rangle [\tilde{\lambda}_i, \tilde{\lambda}_j] = 2\, p_i \cdot p_j$$

… for massive momenta

A time-like momentum can be always decomposed into two light-like ones

$$P = \lambda\tilde{\lambda} + \mu\tilde{\mu}$$

$$\text{with} \quad P^2 = -m^2 = \langle\lambda, \mu\rangle[\tilde{\lambda}, \tilde{\mu}]$$

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- The on-shell condition was built-in in the spinor formalism for massless particles, here we have to impose it

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But we can still keep the advantage of having LG differential equations in a simple and effective form!

LG scaling for massless particles

$$e^{i\theta h_i}\, M_n\big(\lambda_j\tilde{\lambda}_j; h_j\big) \quad \longrightarrow \quad t^{-2h_i}\, M_n\big(\lambda_j\tilde{\lambda}_j; h_j\big) \qquad t \in \mathbb{C}$$

$$\left.\begin{array}{l} \lambda \longrightarrow t\,\lambda \\ \tilde{\lambda} \longrightarrow t^{-1}\tilde{\lambda} \end{array}\right\} \Rightarrow \lambda\tilde{\lambda} \longrightarrow \lambda\tilde{\lambda}$$

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LG differential equation:

$$J_3^j M_n(\{p_i, h_i\}) = h_j M_n(\{p_i, h_i\})$$

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LG differential equation:

$$\left(\lambda_i \frac{\partial}{\partial\lambda_i} - \tilde{\lambda}_i \frac{\partial}{\partial\tilde{\lambda}_i}\right) M_n\big(\lambda_j\tilde{\lambda}_j; h_j\big) = -2h_i\, M_n\big(\lambda_j\tilde{\lambda}_j; h_j\big)$$

3-point massless amplitude

[Benincasa-Cachazo '07]

$$\left(\lambda_i \frac{\partial}{\partial \lambda_i} - \tilde{\lambda}_i \frac{\partial}{\partial \tilde{\lambda}_i}\right) M_3\left(\lambda_j \tilde{\lambda}_j; h_j\right) = -2h_i\, M_3\left(\lambda_j \tilde{\lambda}_j; h_j\right)$$

3 equations for 6 variables

$$\begin{gathered} x_1 = \langle 2,3\rangle \;,\; x_2 = \langle 3,1\rangle \;,\; x_3 = \langle 1,2\rangle \\ y_1 = [2,3] \;,\; y_2 = [3,1] \;,\; y_3 = [1,2] \end{gathered}$$

$$\begin{aligned} M_3^{h_1,h_2,h_3} &= x_1^{h_1-h_2-h_3} x_2^{h_2-h_3-h_1} x_3^{h_3-h_1-h_2} f\left(x_1 y_1, x_2 y_2, x_3 y_3\right) \\ &= y_1^{h_2+h_3-h_1} y_2^{h_3+h_1-h_2} y_3^{h_1+h_2-h_3} \tilde{f}\left(x_1 y_1, x_2 y_2, x_3 y_3\right) \end{aligned}$$

3-point massless amplitude

... but then we have to impose momentum conservation:

$$0 = p_1^2 = (-p_2 - p_3)^2 = 2p_2 \cdot p_3 = \langle 2,3 \rangle [2,3] \;\Rightarrow x_1 = 0\,, \text{ or } \; y_1 = 0$$

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This implies:

or all x_i are zero, or all y_i are zero.

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For real kinematics ($x_i = 0 = y_i$) a 3p amplitude for massless particles is zero, so the complex 3p amplitude had better go to zero in this limit, rather than exploding.

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For real kinematics ($x_i = 0 = y_i$) a 3p amplitude for massless particles is zero, so the complex 3p amplitude had better go to zero in this limit, rather than exploding. This selects

$$M^{\{h_j\}} = g_{\mathrm{H}}\, x_1^{h_1-h_2-h_3} x_2^{h_2-h_3-h_1} x_3^{h_3-h_1-h_2} \quad \text{for} \quad h_1+h_2+h_3 < 0$$

$$M^{\{h_j\}} = g_{\mathrm{A}}\, y_1^{h_2+h_3-h_1} y_2^{h_3+h_1-h_2} y_3^{h_1+h_2-h_3} \quad \text{for} \quad h_1+h_2+h_3 > 0$$

For $h_1+h_2+h_3 = 0$ the answer is left undetermined (there are claims that such interactions cannot exist).

Massive LG equations in spinor formalism

Take the transformation

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \to U \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \qquad \begin{pmatrix} \tilde{\lambda} & \tilde{\mu} \end{pmatrix} \to \begin{pmatrix} \tilde{\lambda} & \tilde{\mu} \end{pmatrix} U^\dagger \qquad U \in \mathrm{U}(2)$$

under which the massive momentum is invariant (LG)

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under which the massive momentum is invariant (LG), then

$$\begin{aligned} J_+ &= -\mu \frac{\partial}{\partial \lambda} + \tilde{\lambda} \frac{\partial}{\partial \tilde{\mu}} \\ J_0 &= -\frac{1}{2}\left(\lambda \frac{\partial}{\partial \lambda} - \tilde{\lambda} \frac{\partial}{\partial \tilde{\lambda}} - \mu \frac{\partial}{\partial \mu} + \tilde{\mu} \frac{\partial}{\partial \tilde{\mu}} \right) \\ J_- &= -\lambda \frac{\partial}{\partial \mu} + \tilde{\mu} \frac{\partial}{\partial \tilde{\lambda}} \end{aligned}$$

Massive LG equations in spinor formalism

and so

$$J_0^I \, M_n^{lw} = -s_I \, M_n^{lw}$$

$$J_-^I \, M_n^{lw} = 0$$

Massive LG equations in spinor formalism

and so

$$\left(\lambda_I \frac{\partial}{\partial \lambda_I} - \tilde{\lambda}_I \frac{\partial}{\partial \tilde{\lambda}_I} - \mu_I \frac{\partial}{\partial \mu_I} + \tilde{\mu}_I \frac{\partial}{\partial \tilde{\mu}_I}\right) M_n^{lw} = 2s_I \, M_n^{lw}$$

$$\left(\lambda_I \frac{\partial}{\partial \mu_I} - \tilde{\mu}_I \frac{\partial}{\partial \tilde{\lambda}_I}\right) M_n^{lw} = 0$$

1-massive 2-massless 3-point amplitude

$$p_1 = \lambda_1 \tilde{\lambda}_1 \quad p_2 = \lambda_2 \tilde{\lambda}_2 \quad P_3 = \lambda_3 \tilde{\lambda}_3 + \lambda_4 \tilde{\lambda}_4$$

$$\langle 3, 4 \rangle [3, 4] = -{m_3}^2$$

$$1 + 1 + 2 \ = \ 4 \text{ eq.s}$$

$$\frac{4 \cdot 3}{2} \text{ angle prod.s} + \frac{4 \cdot 3}{2} \text{ square prod.s} = 12 \text{ spinor prod.s}$$

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momentum conservation: $12 \longrightarrow 6$

mass on-shell condition: $6 \longrightarrow 5$

1-massive 2-massless 3-point amplitude

$$M^{h_1,h_2,-s_3} = \langle 1,2\rangle^{-s_3-h_1-h_2}\langle 2,3\rangle^{h_1-h_2+s_3}\langle 3,1\rangle^{h_2-h_1+s_3}\ f_1\big(\langle 3,4\rangle\big)$$

all angle products!

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$\langle 3, 4\rangle$ in the real-momenta limit is essentially the mass, so f_1 can be reduced to dimensionless constant

$$f_1(\langle 3, 4\rangle) = g\, m_3^{1+h_1+h_2-s_3-[g]}\, \tilde{f}_1\left(\frac{\langle 3, 4\rangle}{m_3}\right)$$

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This amplitude is physically allowed for real momenta! (contrarily to the massless one)

$\Rightarrow$ full non-perturbative result

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If we apply the third LG equation

$$(J_+^3)^{2s_3+1} M^{h_1,h_2,-s_3} = 0$$

we get the following condition on the allowed helicities

$$|h_1 - h_2| \le s_3$$

1-massive 2-massless 3-point amplitude

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"conservation of the spin"!

2-massive 1-massless 3-point amplitude

$$P_1 = \lambda_1\tilde{\lambda}_1 + \lambda_5\tilde{\lambda}_5 \quad P_2 = \lambda_2\tilde{\lambda}_2 + \lambda_4\tilde{\lambda}_4 \quad p_3 = \lambda_3\tilde{\lambda}_3$$

$$\langle 1,5\rangle[1,5] = -{m_1}^2 \quad \langle 2,4\rangle[2,4] = -{m_2}^2$$

$$2+2+1 \;=\; 5 \text{ eq.s}$$

$$5\cdot 4 \;=\; 20 \text{ spinor prod.s}$$

momentum conservation: $20 \longrightarrow 10$

mass on-shell conditions: $10 \longrightarrow 8$

2-massive 1-massless 3-point amplitude

$$M^{-s_1,-s_2,h_3} =$$

$$\langle 1,2\rangle^{s_1+s_2+h_3}\,\langle 3,1\rangle^{s_1-s_2-h_3}\,\langle 2,3\rangle^{s_2-s_1-h_3}\;f_2\Big(\langle 1,5\rangle,\langle 2,4\rangle,\frac{[4,5]}{\langle 1,2\rangle}\Big)$$

Again from dimensional considerations

$$f_2 = g\,m_1^{1-s_1-s_2+h_3-[g]}\;\tilde{f}_2\Big(\frac{\langle 1,5\rangle}{m_1},\frac{\langle 2,4\rangle}{m_2},\frac{[4,5]}{\langle 1,2\rangle}\Big)$$

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Using the third LG equation: $(J_+^I)^{2s_I+1}M_n = 0$ for $I = 1,2$

$$\tilde{f}_2 = \sum_{k=0}^{2s_1} a_k\left(\frac{\langle 1,5\rangle}{m_1},\frac{\langle 2,4\rangle}{m_2}\right)\left(\frac{m_2}{m_1}+\frac{\langle 1,5\rangle}{m_1}\frac{\langle 2,4\rangle}{m_2}\frac{[4,5]}{\langle 1,2\rangle}\right)^{s_1+s_2+h_3-k}$$

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If $s_1 \neq s_2$, requiring the two different expression to be consistent, we get the following condition on the spins/helicities

$$|h_3| \leq s_1 + s_2$$

2-massive 1-massless 3-point amplitude

$$\tilde{f}_2 = \sum_{k=0}^{2s_1} a_k \left(\frac{\langle 1,5\rangle}{m_1}, \frac{\langle 2,4\rangle}{m_2}\right) \left(\frac{m_2}{m_1} + \frac{\langle 1,5\rangle}{m_1}\frac{\langle 2,4\rangle}{m_2}\frac{[4,5]}{\langle 1,2\rangle}\right)^{s_1+s_2+h_3-k}$$

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If $s_1 \neq s_2$, requiring the two different expression to be consistent, we get the following condition on the spins/helicities

$$|h_3| \leq s_1 + s_2$$

2-mass. 1-massless 3p ampl. fixed up to max. $2s_{min}+1$ const.s

Remark:

If the two massive particles are the same

$$\tilde{f}_2 = \sum_{k=0}^{2s_1} a_k \left(\frac{\langle 1,5\rangle}{m_1}, \frac{\langle 2,4\rangle}{m_2}\right) \left(\frac{m_2}{m_1} + \frac{\langle 1,5\rangle}{m_1}\frac{\langle 2,4\rangle}{m_2}\frac{[4,5]}{\langle 1,2\rangle}\right)^{s_1+s_2+h_3-k}$$

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So we cannot match, and we cannot derive any condition on spins...

But this amplitude is zero for real momenta!
So analogously to the massless case there are no constraints on spins and helicities

3-massive 3-point amplitude

$$P_1 = \lambda_1\tilde{\lambda}_1 + \lambda_3\tilde{\lambda}_3 \quad P_2 = \lambda_2\tilde{\lambda}_2 + \lambda_4\tilde{\lambda}_4 \quad P_3 = \lambda_3\tilde{\lambda}_3 + \lambda_6\tilde{\lambda}_6$$

$$\langle 1,4\rangle[1,4] = -{m_1}^2 \quad \langle 2,4\rangle[2,4] = -{m_2}^2 \quad \langle 3,6\rangle[3,6] = -{m_3}^2$$

$$2+2+2 \;=\; 6 \text{ eq.s}$$

$$6\cdot 5 \;=\; 30 \text{ spinor prod.s}$$

momentum conservation: $30 \longrightarrow 14$

mass on-shell conditions: $14 \longrightarrow 11$

3-massive 3-point amplitude

$$M^{-s_1,-s_2,-s_3} = \langle 1,2\rangle^{s_1+s_2-s_3}\,\langle 3,1\rangle^{s_3+s_1-s_2}\,\langle 2,3\rangle^{s_2+s_3-s_1}\times \times f_3\Big(\langle 1,4\rangle,\langle 2,5\rangle,\langle 3,6\rangle;\frac{[4,5]}{\langle 1,2\rangle},\frac{[6,4]}{\langle 3,1\rangle}\Big)$$

Again from dimensional considerations

$$f_3 = g\, m_1^{1-s_1-s_2-s_3-[g]}\,\tilde{f}_3\left(\frac{\langle 1,4\rangle}{m_1},\frac{\langle 2,5\rangle}{m_2},\frac{\langle 3,6\rangle}{m_3};\frac{[4,5]}{\langle 1,2\rangle},\frac{[6,4]}{\langle 3,1\rangle}\right)$$

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Here the third equations are more involved, since f_3 depends on two scaling variables. J_+^2 acts only on $\frac{[4,5]}{\langle 1,2\rangle}$, J_+^3 acts only on $\frac{[6,4]}{\langle 3,1\rangle}$, while J_+^1 acts on both.

3-massive 3-point amplitude

From $(J_+^I)^{2s_I+1} M_n = 0$ for $I = 2, 3$

$$f_3\big(\ldots;\xi_2,\xi_3\big) = \sum_{k=0}^{2s_I} c_k^{(I)}\big(\ldots;\xi_{\bar{I}}\big)\left(\langle 2,5\rangle\xi_2 + \langle 3,6\rangle\xi_3 - \frac{{m_1}^2}{\langle 1,4\rangle}\right)^{s_1-s_2-s_3+k}$$

$$\text{with}\quad \xi_2 = \frac{[4,5]}{\langle 1,2\rangle}\,,\quad \xi_3 = \frac{[6,4]}{\langle 3,1\rangle}$$

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The action of J_+^1 is more complicated...

... but can be worked out case by case

Looking backward...

We have determined all Poincaré invariant 3-point amplitudes where massive particles are involved, in spinor-helicity formalism.

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It is fixed up to some constants. *How many?*

1 massive	1 const.		
2 massive	$s_1 + s_2 - h_3 + 1$	if	$s_2 - s_1 \leq h_3 \leq s_1 + s_2$
	$2s_1 + 1$	if	$s_1 - s_2 \leq h_3 \leq s_2 - s_1$
	$s_1 + s_2 + h_3 + 1$	if	$-s_1 - s_2 \leq h_3 \leq s_1 - s_2$
3 massive	*still computing...*		

Looking backward...

We have determined all Poincaré invariant 3-point amplitudes where massive particles are involved, in spinor-helicity formalism.

We have it for the lowest value of the spin projection, but

$$J_+^I \, M_3^{\dots,-s_I,\dots} = M_3^{\dots,-s_I+1,\dots}$$

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We have determined all Poincaré invariant 3-point amplitudes where massive particles are involved, in spinor-helicity formalism.

For given known 3-point interactions these theoretical expressions reproduce existing results.

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Moreover...

- Constructive SM: all 3-p vertices of SM reproduced
[N. Christensen, B. Field 1802.00448]

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- Constructive SM: all 3-p vertices of SM reproduced
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- Massive Higher-Spins amplitudes in 4 dimensions
  [E. Conde *et al.* 1605.07402]

...and going further

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Next level

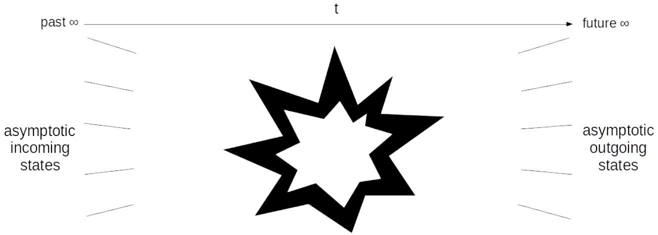
Poincaré irreps classification

$$\Big\downarrow$$

constraints on the amplitude

Next level

Amplitude is an **asymptotic** object



The states of the Hilbert space (particles) are identified by the symmetry of space-time (Wigner classification)

Next level: Asymptotic Symmetries

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Poincaré irreps classification $\quad L \ltimes T_4$

$$\downarrow$$

constraints on the amplitude

Next level: Asymptotic Symmetries

BMS irreps classification $\quad \mathrm{L} \ltimes \mathrm{ST}$

$\downarrow$

(super)constraints on the amplitude (?)

Next level: Asymptotic Symmetries

BMS irreps classification $\quad \mathrm{L} \rtimes \mathrm{ST}$

Note: one may wish to include superrotations, but Wigner classification not known for that

Next level: Asymptotic Symmetries

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$$L \ltimes ST$$

↙

same LG as Poincaré

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$$\mathrm{L} \rtimes \mathrm{ST}$$

↙ same LG as Poincaré

↘ enhancement to infinite symmetries

Next level: Asymptotic Symmetries

W.I.'s of large "gauge" transformations

$\downarrow$ [Strominger *et al.*]

soft theorems

$$\lim_{\epsilon\to 0} A_{n+1}(\epsilon q, p_1, \dots, p_n) = \left(\epsilon^{-1} S^{(0)} + \epsilon^0 S^{(1)} + \ \cdots\right) A_n(p_i)$$

$S^{(0)}$ leading soft factor, $S^{(1)}$ sub-leading soft factor, ...

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- large "gauge" W.I. $\longleftrightarrow$ soft theorems up to $(\text{sub})^s$-order ?
- higher orders in the soft expansion (ϵ^k, $k \geq s$) ?

Next level: Asymptotic Symmetries

higher orders in the soft expansion?

they do not give soft theorems (no universally determined factors), but they should give anyway constraints on the amplitude

Hopes

- *hope 1:* spinor-helicity formalism as effective in implementing **higher-order soft constraints** as for LG constraints

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- *hope 1:* spinor-helicity formalism as effective in implementing **higher-order soft constraints** as for LG constraints

- *hope 2:* it may provide a general fashion (naturally generalizable to arbitrary spin) for deriving **higher-spin sub-leading soft theorems** (*ref.* Carlo's talk of yesterday)

Work in progress...

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Grazie per l'Attenzione!

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Massive LG equations in spinor formalism

Remark!

This group of transformations has the same algebra as the LG, but it is not the largest group that leaves

$$\lambda\tilde{\lambda} + \mu\tilde{\mu}$$

invariant. Once we fix a frame by a physical LG transformation, we can still scale $\mu, \tilde{\mu}$ independently of $\lambda, \tilde{\lambda}$

$$\lambda\tilde{\lambda} + \mu\, t\, t^{-1}\tilde{\mu}$$

non-physical transformation $\Rightarrow$ amplitude should be invariant

Kinematic Constraints

$$\sum_{i=1}^{n} \lambda_i \tilde{\lambda}_i = 0 \qquad (n = 3 + \#\text{ of mass. particles})$$

Schouten identity (linear dependency in 2 dim. vector sp.):

$$\langle j,k\rangle \lambda_i + \langle k,i\rangle \lambda_j + \langle i,j\rangle \lambda_k = 0$$

Choose λ_1 and λ_2, and express all the spinor products in term of

$$\langle 1,2\rangle\,,\ \langle 1,i\rangle\,,\ \langle 2,i\rangle\,, \qquad \text{with } i = 3,\ldots,n$$

and then we use momentum conservation for the tilded spinors

$$\tilde{\lambda}_1 = -\sum_{i=3}^{n} \frac{\langle i,2\rangle}{\langle 1,2\rangle} \tilde{\lambda}_i \qquad \tilde{\lambda}_2 = -\sum_{i=3}^{n} \frac{\langle 1,i\rangle}{\langle 1,2\rangle} \tilde{\lambda}_i$$

Kinematic Constraints

With this we can eventually reduce the total number of independent variables to

$$\begin{cases} 2n-3+\frac{1}{2}(n-2)(n-3)=\frac{1}{2}n\,(n-1) & \text{if } n\leq 5 \\ 2n-3+\qquad 2n-7 \qquad = 2\,(2n-5) & \text{if } n>5 \end{cases}$$

1-massive 2-massless 3-point amplitude

Why $f_1(\langle 3,4\rangle)$ should be a constant?

Remove the redundancy in our description of time-like momentum! After we have fixed a LG transformation, we have still the freedom to rotate $\lambda_4\tilde{\lambda}_4$ independently of $\lambda_3\tilde{\lambda}_3$

$$\lambda_4 \to t\,\lambda_4 \qquad \tilde{\lambda}_4 \to t^{-1}\tilde{\lambda}_4$$

and we impose the amplitude to be invariant under it

$$\Rightarrow \quad f_1\big(\langle 3,4\rangle\big) \text{ constant}$$