Twining Genera of $K3$

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Outlines of the Talk

- Introduction to NLSMs on K3
- Definition of the Twining Genera of $K3$.
- Brief discussion on their physical and mathematical implications.

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Superconformal Field Theories

Quick Introduction

Superconformal Field Theories are supersymmetric extensions of Conformal Field Theories.
We will focus in particular on $\mathcal{N} = 4$ SCFTs in $D = 2$, in which the Virasoro algebra is extended by the presence of four 3/2-spin supercurrents $G(z)$ and three 1-spin currents $J(z)$.

String Theory point of view

The most common examples of $2D$-SCFTs relevant in the String Theory context are *Non-Linear σ-Models*. Explicitly we have:

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \left[\left(\delta^{ab} G_{\mu\nu}(X) + i\epsilon^{ab} B_{\mu\nu}(X) \right) \partial_a X^\mu \partial_b X^\nu \right]$$

We can clearly notice that the model is completely determined by fixing the couple $(G_{\mu\nu},\, B_{\mu\nu})$.

Target-Space

We want now to consider a $\mathcal{N} = (4,4)$ $2D$-SCFTs whose target space is a K3 surface.
Such theories arise when we compactify Type IIA and Type IIB Theories on the 10D space-time:

$$M_{10} = \mathcal{M}_{1,3} \times K3 \times \mathbb{T}^2$$

NLSMs on K3

K3 Surfaces

K3 surfaces are interesting because their (restricted) Holonomy group is $SU(2)$ and thus only half of the number of starting supercharges is preserved.
This means that we can have:

$$\underbrace{\text{Type II A-B}}_{D=10} \xrightarrow{K3\times\mathbb{T}^2} \underbrace{\mathcal{N}=4\text{ SUGRA}}_{D=4}$$

What can we compute?

In this context, the relevant physical quantities that can be potentially computed are:

- Partition Function
- Witten Index
- Elliptic Genus
- Twining Genus

In Formulae (1)

- The Partition Function is defined, as usual, as:

$$Z(q,\bar{q}) = \mathrm{Tr}_{\mathcal{H}}[e^{-2\pi(\tau_2 H - i\tau_1 P)}] = \ldots = \mathrm{Tr}_{\mathcal{H}}[q^{L_0}\ \bar{q}^{\bar{L}_0}]$$

- The Witten Index:

$$Z = \mathrm{Tr}_{RR}[(-1)^{F+\bar{F}} q^{L_0}\ \bar{q}^{\bar{L}_0}] = \ldots = \chi_{ts}$$

with $\tau = \tau_1 + i\tau_2$ the modular parameter of the torus, $q = e^{2\pi i\tau}$, $\bar{q} = e^{2\pi i\bar{\tau}}$ and $F, \bar{F}$ the left- and right-moving fermion numbers.

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In Formulae (2)

- The Elliptic Genus is defined as:

$$Z(q,y) = \mathrm{Tr}_{RR}[(-1)^{F+\bar{F}} q^{L_0} \; \bar{q}^{\bar{L}_0} y^Q]$$

with $y = e^{2\pi i z}$, $z \in \mathbb{R}$ and Q the generator of a $u(1)$ algebra contained in the $su(2)$ R-symmetry algebra belonging to the $\mathcal{N} = (4,4)$ superconformal algebra.

- Finally, supposing that a given NLSM on K3 has a discrete symmetry g commuting with the $\mathcal{N} = (4,4)$ algebra, the corresponding Twining Genus can be defined as:

$$Z_g(q,y) = \mathrm{Tr}_{RR}[(-1)^{F+\bar{F}} q^{L_0} \; \bar{q}^{\bar{L}_0} y^Q g]$$

Here come the issues...

Everything we introduced above is not explicitly computable for a NLSM whose target-space is a GENERIC K3 surface.
We need therefore to simplify the problem by choosing a NLSM defined on a "treatable" target-space.

One possibility is thus to choose a special point of the moduli space of K3 for which $K3 \simeq \mathbb{T}^4/\mathbb{Z}_2$.

$K3$ as Torus Orbifold

Symmetries Classification

All such (discrete) symmetries have been classified, for generic NLSMs on K3, in [Gaberdiel, Hohenegger, Volpato, 1106.4315]. Thanks to the proposed Classification Theorem, in [Cheng, Harrison, Volpato, Zimet, 1612.04404], it is shown that there are exactly 81 different Twining Genera for K3.

Symmetries of the Orbifold case

Let us now focus only on symmetries which are realized on NLSMs on $\mathbb{T}^4/\mathbb{Z}_2$.
Such symmetries can be explicitly obtained by starting from discrete symmetries g of generic NLSMs on $\mathbb{T}^4$ (classified in [Volpato, 1403.2410]), commuting with the non trivial element $h \in \mathbb{Z}_2$.

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Twining Genus for $\mathbb{T}^4/\mathbb{Z}_2$

Knowing that fields defined on $\mathbb{T}^4/\mathbb{Z}_2$ must satisfy the conditions

$$X_i(\sigma + 2\pi R, \tau) = \begin{cases} +X_i(\sigma,\tau) & \text{(Untw-Sector)} \\ -X_i(\sigma,\tau) & \text{(Tw-Sector)}, \end{cases}$$

the Twining Genus on $\mathbb{T}^4/\mathbb{Z}_2$ becomes:

$$Z_g(q,y) = \mathrm{Tr}_{untw}\left[\frac{(\mathbb{1}+h)}{2}\; q^{L_0}\bar{q}^{\bar{L}_0}(-1)^{F+\bar{F}}y^Q g\right] + \mathrm{Tr}_{tw}\left[\frac{(\mathbb{1}+h)}{2}\; q^{L_0}\bar{q}^{\bar{L}_0}(-1)^{F+\bar{F}}y^Q g\right]$$

An Explicit Formula

The final formula for a generic symmetry g, realized on a Torus Orbifold, whose eigenvalues are $\zeta_L, \zeta_L^{-1}, \zeta_R, \zeta_R^{-1}$, with $\zeta_{L,R} = e^{2\pi i r_{L,R}}$ and both $(\neq 1)$, is:

$$
\begin{aligned}
Z_g^{orb}(\tau, z) = \frac{1}{2}\Bigg[& \frac{\theta_1(z + r_L|\tau)\ \theta_1(z - r_L|\tau)}{\theta_1(r_L|\tau)\ \theta_1(-r_L|\tau)} \cdot (2 - \zeta_L - \zeta_L^{-1})(2 - \zeta_R - \zeta_R^{-1}) \\
& + \frac{\theta_2(z + r_L|\tau)\ \theta_2(z - r_L|\tau)}{\theta_2(r_L|\tau)\ \theta_2(-r_L|\tau)} \cdot (2 + \zeta_L + \zeta_L^{-1})(2 + \zeta_R + \zeta_R^{-1}) \\
& + \left(\frac{\theta_4(z + r_L|\tau)\ \theta_4(z - r_L|\tau)}{\theta_4(r_L|\tau)\ \theta_4(-r_L|\tau)} + \frac{\theta_3(z + r_L|\tau)\ \theta_3(z - r_L|\tau)}{\theta_3(r_L|\tau)\ \theta_3(-r_L|\tau)} \right) \cdot \mathrm{Tr}_{\mathcal{H}_{gs}^{tw}}[\rho(g)] \Bigg]
\end{aligned}
$$

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Examples

- *Elliptic Genus*: $g = \mathbb{1}$ $(r_{L,R} = 0)$

$$Z^{orb}(\tau, z) = 8 \cdot \left[\frac{\theta_2(z|\tau)^2}{\theta_2(0|\tau)^2} + \frac{\theta_4(z|\tau)^2}{\theta_4(0|\tau)^2} + \frac{\theta_3(z|\tau)^2}{\theta_3(0|\tau)^2} \right]$$

- (New) $\pi_{g_1} = 1^8\ 2^{-8}\ 4^8$ of Table 3, p.46, in [Paquette, Volpato, Zimet, 1702.05095], with the eigenvalues $\zeta_{L,R} = -1$, chosen in Table 2, p.20, from [Volpato, 1403.2410]:

$$Z^{orb}_{g_1}(\tau, z) = 8 \cdot \frac{\theta_1(z + \frac{1}{2}|\tau)\ \theta_1(z - \frac{1}{2}|\tau)}{\theta_1(\frac{1}{2}|\tau)\ \theta_1(-\frac{1}{2}|\tau)},$$

Possible Applications

The computation of the Twining Genera of K3 may be interesting:

- To study the hidden symmetries of $\mathcal{N} = 4$ SUGRA in $D = 4$.
- To exstimate the entropy of some kinds of Supersymmetric Black Holes.
- To verify the conjectured values of the 81 Twining Genera listed in [Paquette, Volpato, Zimet, 1702.05095].
- To have a better comprehension of the *Mathieu Moonshine Phenomenon*.