Topological properties of CP^{N-1} models in the large-N limit

Claudio Bonanno† (claudio.bonanno@pi.infn.it)
In collaboration with Claudio Bonati† and Massimo D'Elia†
† University of Pisa and INFN Pisa

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Role of $\mathbb{C}P^{N-1}$ in non-perturbative QCD

The $2d$ CP^{N-1} models share many fundamental properties with QCD: confinement, asymptotic freedom, topologically-stable instantons, θ-vacua...

These theories admit an analytic solution in the large-N limit. They have been employed as a theoretical laboratory for the study of non-perturbative features of QCD (e. g. Witten, 1979).

The CP^{N-1} have been also extensively studied numerically through Monte Carlo simulations:

- lattice CP^{N-1} simulations need low numerical effort,
- CP^{N-1} models are ideal test-bed for new algorithms to solve LQCD non-trivial computational problems,
- possibility of a comparison between numerical and analytic large-N results.

Topology and θ-dependence

In the CP^{N-1} models one can introduce a topological charge Q and a corresponding θ-term in the action.

This work focuses on the study of the θ-dependence of the vacuum energy (density):

$$f(\theta) \equiv -\frac{1}{V}\log Z(\theta) = \frac{1}{2}\chi\theta^2\left(1 + \sum_{n=1}^{\infty} b_{2n}\theta^{2n}\right).$$

The coefficients of the expansion are related to the cumulants k_m of the probability distribution of Q:

$$\left.\frac{d^m f}{d\theta^m}\right|_{\theta=0} = -\frac{i^m}{V}k_m \implies \begin{cases} \chi = \dfrac{\langle Q^2\rangle\,|_{\theta=0}}{V}, \\ b_2 = \left.\dfrac{-\langle Q^4\rangle + 3\langle Q^2\rangle^2}{12\langle Q^2\rangle}\right|_{\theta=0} \ldots \end{cases}$$

The study of $f(\theta)$ is of particular relevance in QCD and in $SU(N)$ gauge theories:

- θ-dependence of pure Yang-Mills enters η' physics,
- $f_{QCD}(\theta)$ enters axion phenomenology and, thus, the resolution of the strong-CP problem.

In QCD and Yang-Mills, χ and b_{2n} cannot be computed analytically from first principles. Besides, θ-dependence is a non-perturbative feature.
$\implies$ Numerical MC simulations on the lattice have become one of the most reliable tools to study this issue.

This constitutes a strong motivation to perform a similar numerical study for the lattice CP^{N-1} models.

Goals

Currently, the topological susceptibility of the CP^{N-1} models and its large-N asymptotic behaviour have been checked numerically quite well.

The goals of this work are:

- extension of the lattice measure of the vacuum energy $f(\theta)$ to higher orders in θ;
- extension of the study of the large-N limit of $f(\theta)$ and comparison with analytic predictions.

The total action is:

$$S = S_0 + S_{topo} = \beta E - i\theta Q \qquad (\beta \equiv 1/g).$$

$S \in \mathbb{C} \implies P \propto \exp\{-S\}$ is not a proper probability distribution when $\theta \neq 0$.

To measure χ and b_{2n}, related to the derivatives of f in $\theta = 0$, one can limit to make simulations at $\theta = 0$.

The non-topological action S_0 is linear in the fields, therefore, it is easy to implement a local algorithm to sample P. We adopted the standard over-heat-bath update algorithm.

Computational problems

This set-up suffers from two computational problems:

- Critical Slowing Down (CSD) of topological modes,
- difficulties in measuring high-order cumulants of Q.

1) When approaching the continuum limit ($\xi_L \to \infty$), the machine time needed to change the topological charge of a field configuration exponentially grows with ξ_L and with N.

This is due to the impossibility of changing the winding number of a configuration with a continuum deformation.

2) The measure of high-order cumulants of Q becomes very noisy for large lattice sizes.

This happens because the Gaussian behaviour is dominant in the thermodynamic limit for the central limit theorem.

Numerical strategies

To obtain a precise measure of $f(\theta)$ we need to adopt numerical strategies to improve the efficiency of local MC simulations.

In this work we applied:

- imaginary-θ method to avoid the sign problem and improve measure accuracy of cumulants; (Panagopoulos and Vicari, 2011)
- simulated tempering algorithm to dampen the CSD of topological modes. (Marinari and Parisi, 1992; Vicari, 1993)

Analytic continuation

Being the θ-dependence of the theory analytic around $\theta = 0$, one can continue the path integral for imaginary angles:

$$\theta \equiv -i\theta_I \implies S_{top} = -i\theta Q = -\theta_I Q \in \mathbb{R}.$$

Now $P \propto \exp\{-S\}$ is a proper probability distribution.

The vacuum energy can be continued too:

$$f(\theta_I) = f(\theta = -i\theta_I) = -\frac{1}{2}\chi\theta_I^2\left(1 + \sum_{n=1}^{\infty}(-1)^n b_{2n}\theta_I^{2n}\right).$$

$\implies$ the measure of χ and of the b_{2n} coefficients can be extracted from $f(\theta_I)$.

Imaginary-θ fit

The θ_I-dependence of the cumulants of Q is related to $f(\theta_I)$:

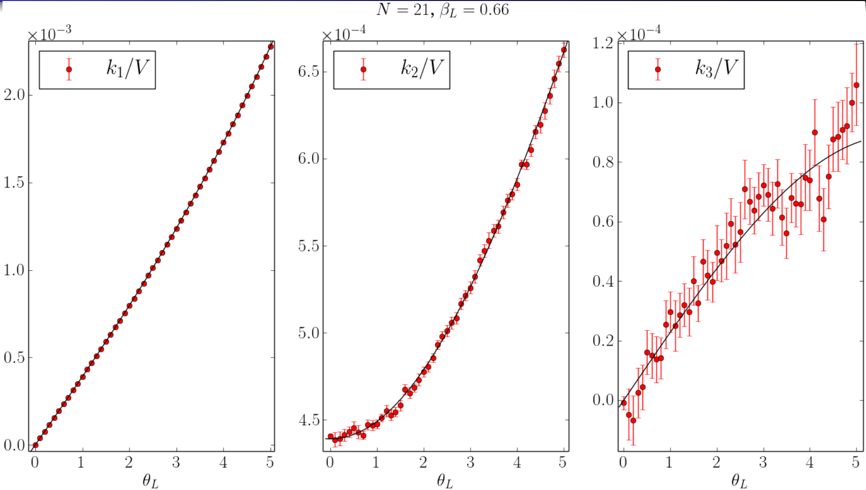
$$\frac{d^m f(\theta_I)}{d\theta_I^m} = -\frac{1}{V} k_m(\theta_I),$$

A global fit of the θ_I-dependence of the cumulants, which can be measured on the lattice, leads to an improved measure of χ and the b_{2n}:

$$\frac{k_1(\theta_I)}{V} = \chi\theta_I\left[1 - 2b_2\theta_I^2 + 3b_4\theta_I^4 + O(\theta_I^5)\right],$$
$$\frac{k_2(\theta_I)}{V} = \chi\left[1 - 6b_2\theta_I^2 + 15b_4\theta_I^4 + O(\theta_I^5)\right],$$
$$\frac{k_3(\theta_I)}{V} = \chi\left[-12b_2\theta_I + 60b_4\theta_I^3 + O(\theta_I^4)\right]...$$

On the lattice: $\theta_I = Z_\theta \theta_L$.

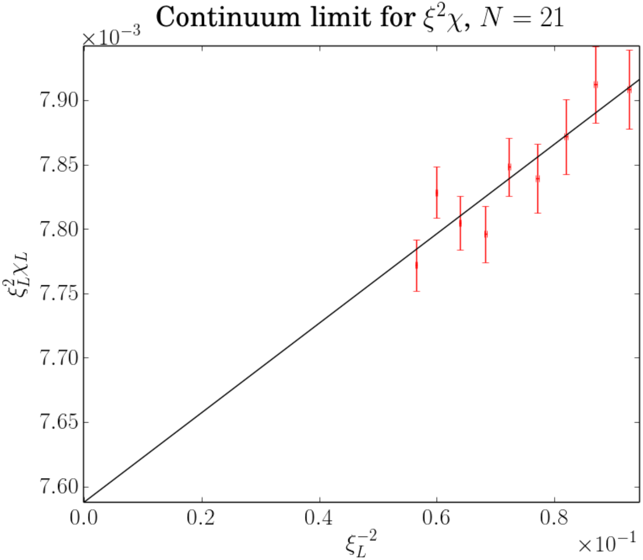
Imaginary-θ fit results

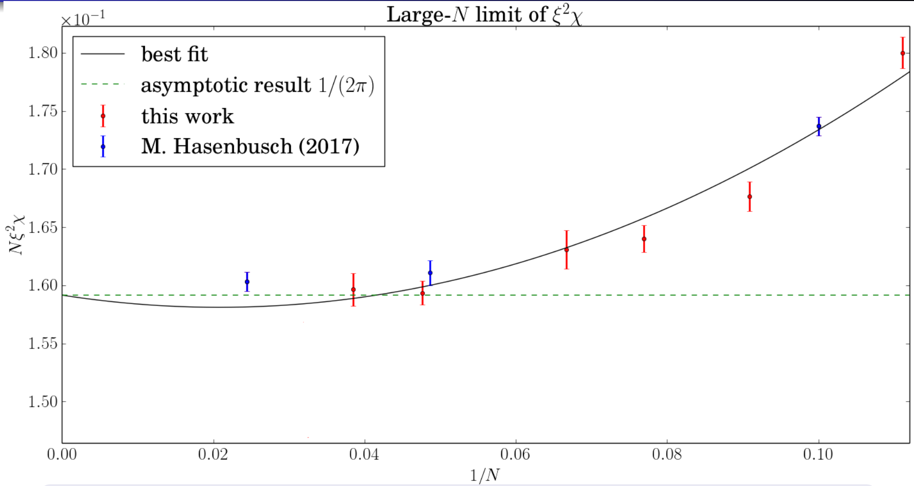


$\mathcal{O}(\beta = 0.66, N = 21)$	Standard method	Imaginary-θ	Gain
$\chi \cdot 10^4$	$4.401(11)$	$4.3908(58)$	~ 2
$b_2 \cdot 10^3$	$-5.36(40)$	$-4.958(76)$	~ 5
$b_4 \cdot 10^5$	-11 ± 21	$-1.27(20)$	~ 100

Continuum limit

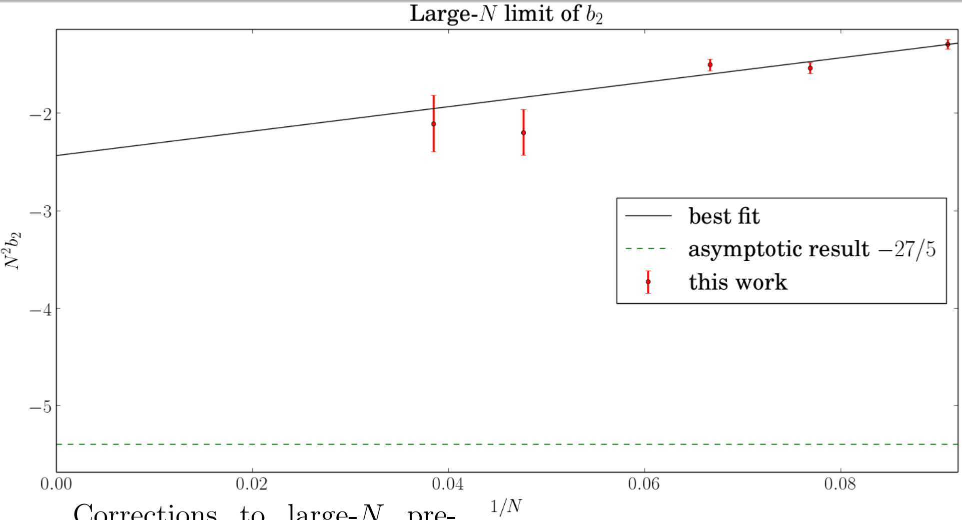
Linear corrections in the lattice spacing ($\sim \xi_L^{-1}$) are killed by the adoption of the $O(a)$ improved lattice action.





$$(N\xi^2\chi)_{theo} \simeq 0.1592 - 0.0606/N + O(1/N^2),$$
$$(N\xi^2\chi)_{MC} = 0.1600(8) + e_2/N + e_3/N^2,$$
$$\text{Best fit: } |e_2| \lesssim 10^{-1}, \quad e_3 = O(1).$$

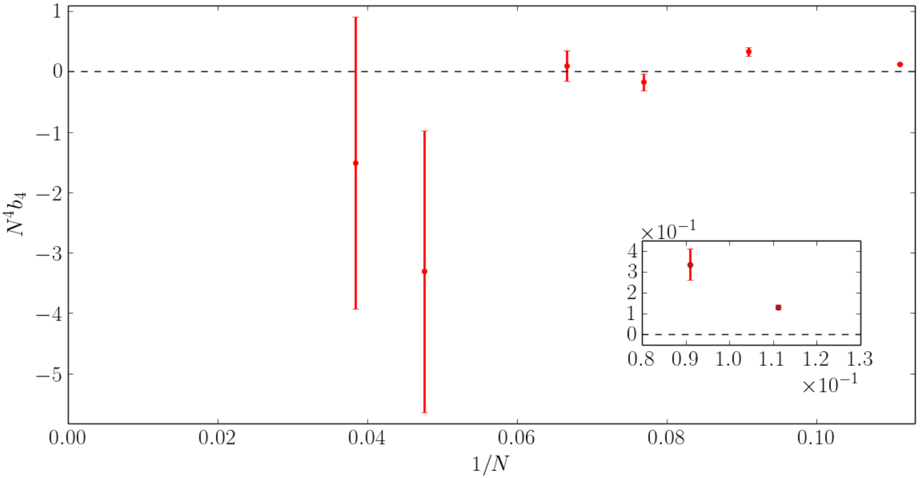
Large-N limit of b_2



Corrections to large-N predicted behaviour (cf. Del Debbio et al., 2006; Bonati et al., 2016) are stille large in this range of N.

$$(N^2 b_2)_{theo} = -5.4 + O(1/N)$$
$$(N^2 b_2)_{MC} = -2.4(5) + O(1/N)$$

Results for b_4



Summary and future perspective

Summarizing, this work consists in:

- application of imaginary-θ method and of simulated tempering algorithm to lattice CP^{N-1} models to improve measure accuracy of topological observables,
- lattice determination of χ, b_2 and b_4 for $N \in [9, 31]$,
- numerical study of the large-N limit of χ and b_2 and comparison with analytic predictions.

In the next future we plan to:

- improve the study of the large-N limit of χ and b_2 including larger Ns and improving measure accuracy,
- try other proposed algorithm to improve this analysis.

Thank you for your attention!

Used definitions

Continuum Euclidean action:

$$S_0 = N\beta \int d^2x \, \bar{D}_\mu \bar{z} D_\mu z,$$

where $D_\mu = \partial_\mu + iA_\mu$.

Continuum Euclidean charge:

$$Q = \frac{1}{4\pi} \int d^2x \, \epsilon_{\mu\nu} F_{\mu\nu}$$

Lattice action

$O(a)$ Symanzik-improved action:

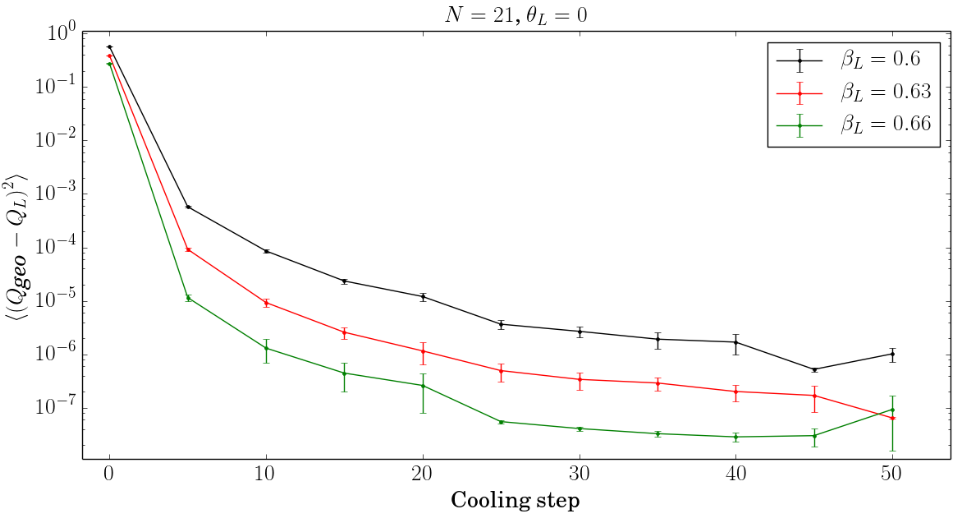
$$S_0^{(L)} = -\frac{8}{3} N\beta_L \sum_{x,\mu} \Re[\bar{U}_\mu(x)\bar{z}(x+\hat{\mu})z(x)]$$
$$+\frac{1}{6} N\beta_L \sum_{x,\mu} \Re[\bar{U}_\mu(x+\hat{\mu})\bar{U}_\mu(x)\bar{z}(x+2\hat{\mu})z(x)].$$

Possible discretizations of Q:

- $Q_L = \frac{1}{2\pi} \sum_x \Im\{\Pi_{12}(x)\}$, (Non-geometric)
- $Q_{geo} = \frac{1}{2\pi} \sum_x \Im\left[\log\left(\Pi_{12}(x)\right)\right]$, (Geometric)

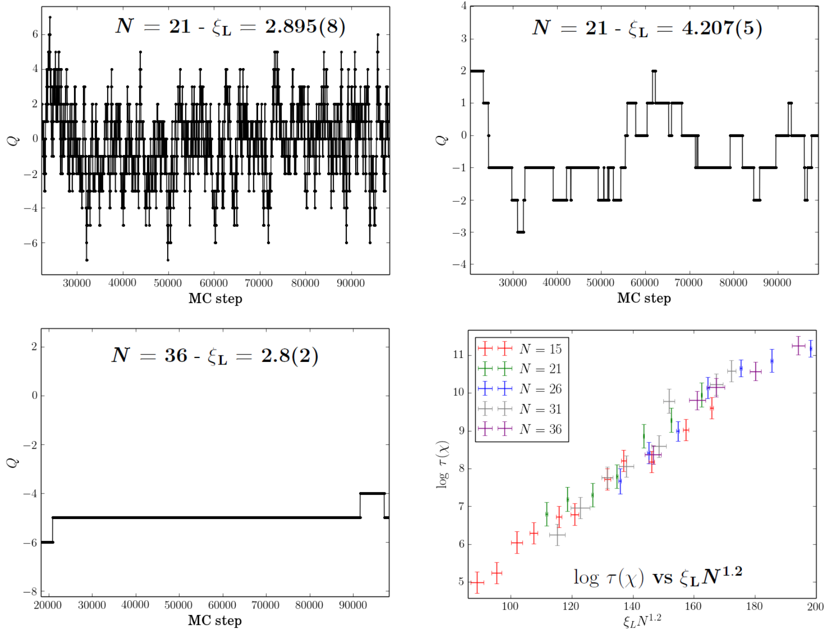
$$\left(\Pi_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x+\hat{\mu})\bar{U}_\mu(x+\hat{\nu})\bar{U}_\nu(x)\right)$$

Cooling



$N = 21, \theta_L = 0$
$\beta_L = 0.6$
$\beta_L = 0.63$
$\beta_L = 0.66$
$\langle (Qgeo - Q_L)^2 \rangle$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
0
10
20
30
40
50
Cooling step

Topological Charge Freezing



Determination of the free energy

An estimation of the free energy is needed to avoid non-ergodicity:

$$P \to P' \propto e^{-\beta E + \theta_I Q_L + F(\beta, \theta_I)}$$

To estimate $F(\beta, \theta_I)$ one can use these two relations:

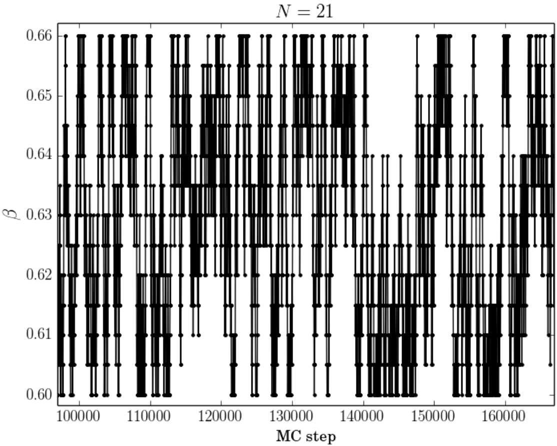
- $\dfrac{\partial F}{\partial \beta} = \langle E \rangle$
- $\dfrac{\partial F}{\partial \theta_I} = -\langle Q_L \rangle$

Both $\langle E \rangle$ and $\langle Q_L \rangle$ can be easily measured in a MC simulation. Then, with a numerical integration, one can obtain F.

Simulated tempering set-up

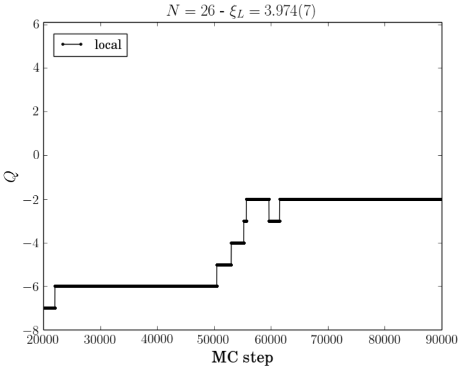
To get an efficient set-up, the β interval has to be chosen accurately.

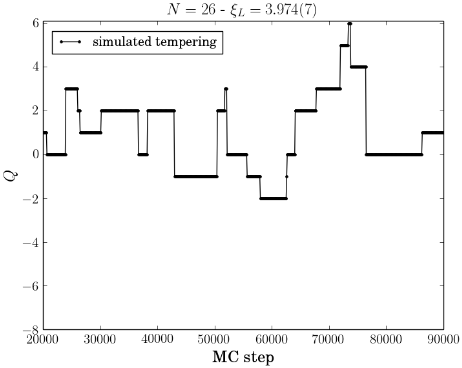
- $\beta_{min} \to$ local algorithm decorrelates fast,
- $\beta_{max} \to$ how close one wants to get to the continuum limit,
- $\delta\beta \to$ reasonable acceptance of change of β.



The correct choice of $\delta\beta$ is obtained when there is a reasonable overlap between the probability distributions of the energy at different temperatures.

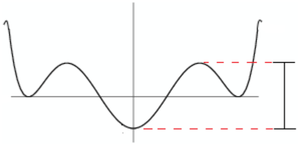
Evolution of Q: local vs simulated tempering





The simulated tempering algorithm

The simulated tempering consists in promoting the temperature T as a dynamical variable.



The system heats up during its evolution and can escape from the local minima in which it is trapped.

In the case of the CP^{N-1} models, one can promote both β and θ_I to dynamical variables:

$$P \propto \exp\{-S\} = \exp\{-\beta E + \theta_I Q\}.$$

- When β decreases, the algorithm changes Q more easily.
- When θ_I increases, higher-charge configurations are more probable to realize.