New Frontiers in Theoretical Physics
XXXVI Convegno Nazionale di Fisica Teorica 25 May 2018 - Cortona

Entropy for a Rotating Black Hole

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Introduction

[J. D. Bekenstein, Phys. Rev. D 7, 949 (1973). S. W. Hawking, Comm. Math. Phys. 43, 199 (1975).]

The prescription for assigning a Bekenstein-Hawking entropy

$$S_{BH} = \frac{A}{4l_P^2}$$

to a black hole of surface area A was first inferred in the mid '70s from the formal similarities between black hole dynamics and thermodynamics, combined with Hawking's discovery that black hole radiates thermally with a characteristic (Hawking) temperature

$$T_H = \frac{\hbar\kappa}{2\pi} \qquad\qquad \kappa_0 \quad \Leftrightarrow \quad surface\ gravity$$

Gerard 't Hooft, in 1985 considered the statistical thermodynamics of quantum fields in the Hartle-Hawking state (i.e. having the Hawking temperature T_H at large radii) propagating on a fixed Schwarzschild background of mass M. To control divergences, 't Hooft introduced a brick wall" with radius slightly larger than the gravitational radius 2MG. He found, in addition to the expected volume-dependent thermodynamical quantities describing hot fields in a nearly at space, additional contributions proportional to the area. These contributions are, however, also proportional to α^{-2}, where α is the proper distance from the horizon, and thus diverge in the limit $\alpha \longrightarrow 0$. For a specic choice of α, he recovered the Bekenstein-Hawking formula [G. 't Hooft, Nucl. Phys. B256, 727 (1985).]

How it works?? Consider a massless scalar field in the following background

$$ds^2 = -\exp\left(-2\phi\left(r\right)\right)dt^2 + \frac{dr^2}{1-\frac{b\left(r\right)}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

- $b(r)$ is the shape function
- $\phi(r)$ is the redshift function

$$b\left(r_0\right) = r_0$$
$$r \in \left[r_0, +\infty\right)$$

From the equation of motion, we can define an r-dependent radial wave number

$$k^2\left(r,l,\omega_{nl}\right)=\frac{1}{1-\frac{b(r)}{r}}\left[\exp\left(2\Lambda(r)\right)\frac{\omega_{nl}^2}{1-\frac{b(r)}{r}}-\frac{l(l+1)}{r^2}\right]$$

$$\left\{\begin{array}{l}\nu\left(l,\omega\right)=\frac{1}{\pi}\int_{r_0+h}^{R}\sqrt{k^2\left(r,l,\omega\right)}dr\\ \\ \tilde{g}\left(\omega\right)=\int_0^{l_{\max}}dl\left(2l+1\right)\nu\left(l,\omega\right)\end{array}\right.\Rightarrow\quad\text{Free energy}\qquad \mathrm{F}=\frac{1}{\beta}\int_0^{\infty}\ln\left(1-\exp\left(-\beta\omega\right)\right)\frac{d\tilde{g}\left(\omega\right)}{d\omega}d\omega$$

$$\text{The entropy is simply S}=\beta^2\frac{\partial F}{\partial\beta}\Rightarrow \mathrm{S}=\frac{16\pi^3}{90\beta^3}\int_{r_0+h}^{R}\frac{\exp\left(2\Lambda(r)\right)}{\left(1-\frac{b(r)}{r}\right)^2}r^2dr$$

In proximity of r_0

$$S=\frac{16\pi^3}{90\beta^3}\frac{r_0^2}{4\kappa_0^3}\frac{\exp\left(-\Lambda\left(r_0\right)\right)}{\alpha^2}=\frac{A}{90}\left(\frac{T}{\kappa_0/2\pi}\right)^3\frac{\exp\left(-\Lambda\left(r_0\right)\right)}{4\pi\alpha^2}$$

$$Identification\quad \frac{\exp\left(-\Lambda\left(r_0\right)\right)}{90\pi\alpha^2}=\frac{1}{l_P^2}\Rightarrow S=\frac{A}{4l_P^2}$$

What happens with Rotations???

E. Young, D. Lee and M. Yoon, Class.Quant.Grav. 26, 155011 (2009); arXiv:0811.3294 [hep-th].
M. H. Lee and J. K. Kim, Phys. Rev. D 54, 3904 (1996); arXiv:9603055 [hep-th].
M. Kenmoku, K. Ishimoto, K. Nandi and K. Shigemoto, Phys.Rev. D73 (2006) 064004; arXiv:0510012 [gr-qc].
J. Jing and M. Yan, Phys.Rev. D61 (2000) 044016; arXiv:9907011 [gr-qc].

Massless Scalar Field in cylindrical Minkowski Space

$$F = \frac{1}{\beta}\int_0^\infty dn(E,m)\ln\left(1-e^{-\beta(E-m\Omega_0)}\right) \Rightarrow \mathrm{F} \approx -\frac{A\pi^4}{90\beta^4\Omega_0^2\varepsilon} \qquad \mathrm{A}=\int dz d\phi \qquad \Omega_0 > 0$$

Leading

Massless Scalar Field in a Kerr Background

Order

$$F = \frac{1}{\beta}\int_0^\infty dn(E,m)\ln\left(1-e^{-\beta(E-m\Omega_0)}\right) \Rightarrow \mathrm{F} \approx -\frac{\zeta(4)}{\pi\beta^4\varepsilon}\int d\theta \frac{\left(\mathrm{r}_+^2+a^2\right)^4\sin\theta}{\left(\mathrm{r}_+-r_-\right)^2\Sigma_+}$$

Eliminating the brick wall from QFT Procedures

•Renormalization of Newton's constant [L. Susskind and J. Uglum, Phys. Rev. D50, 2700 (1994). J. L. F. Barbon and R. Emparan, Phys. Rev. D52, 4527 (1995) 1995), hep-th/9502155. E. Winstanley, Phys. Rev. D63, 084013 (2001) 2001), hep-th/0011176.]
•Pauli-Villars regularization [J.-G. Demers, R. Lafrance and R. C. Myers, Phys. Rev. D52, 2245 (1995), gr-qc/9503003. D. V. Fursaev and S. N. Solodukhin, Phys. Lett. B365, 51 (1996), hep-th/9412020. S. P. Kim, S. K. Kim, K.-S. Soh and J. H. Yee, Int. J. Mod. Phys. A12, 5223 (1997) gr-qc/9607019.]

Eliminating the brick wall using Generalized Uncertainty Principle

Counting of Quantum States Modified by GUP

$$\Delta x \Delta p \geq \hbar \frac{\lambda_P^2}{\hbar} \left(\Delta p \right)^2$$

$\lambda_P \Leftrightarrow$ Planck Length

$\hbar \Leftrightarrow$ Planck Constant

$$\frac{d^3 x d^3 p}{\left(2\pi\hbar \right)^3 \left(1 + \lambda_P^2 p^2 \right)^3}$$

- X. Li, Phys. Lett. B 540, 9 (2002), gr-qc/0204029.
- Z. Ren, W. Yue-Qin and Z. Li-Chun, Class. Quant. Grav. 20 (2003), 4885.
- G. Amelino-Camelia, Class.Quant.Grav. 23, 2585 (2006), gr-qc/0506110.
- G. Amelino-Camelia, Gen.Rel.Grav. 33, 2101 (2001), gr-qc/0106080.

Eliminating the brick wall using Gravity's Rainbow

Gravity's Rainbow

Doubly Special Relativity

G. Amelino-Camelia, Int.J.Mod.Phys. D 11, 35 (2002); gr-qc/001205.
G. Amelino-Camelia, Phys.Lett. B 510, 255 (2001); hep-th/0012238.

$$E^2 g_1^2\left(E/E_P\right) - p^2 g_2^2\left(E/E_P\right) = m^2$$

$$\lim_{E/E_P\to 0} g_1\left(E/E_P\right) = \lim_{E/E_P\to 0} g_2\left(E/E_P\right) = 1$$

Curved Space Proposal → *Gravity's Rainbow*
[J. Magueijo and L. Smolin, Class. Quant. Grav. 21, 1725 (2004) arXiv:gr-qc/0305055].

$$G_{\mu\nu}(E) = 8\pi G(E) T_{\mu\nu}(E) + g_{\mu\nu}\Lambda(E)$$
$G(E) \to G(0)$ when $E \to E_P$
$\Lambda(E) \to \Lambda(0)$ when $E \to E_P$

$$ds^2 = -\left(1-\frac{2MG(0)}{r}\right)\frac{d\tilde{t}^2}{g_1^2\left(E/E_P\right)} + \frac{d\tilde{r}^2}{\left(1-\frac{2MG(0)}{r}\right)g_2^2\left(E/E_P\right)} + \frac{\tilde{r}^2}{g_2^2\left(E/E_P\right)}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Gravity's Rainbow

$$ds^2 = -\frac{N^2(t)}{g_1^2(E/Ep)}dt^2 + \frac{a^2(t)}{g_2^2(E/Ep)}d\Omega_3^2 \quad \Leftrightarrow \text{Distorted } \textit{FLRW metric}$$

$$-\left(1 - \frac{2M(v)G}{r}\right)\frac{dv^2}{g_1^2\left({}^{E}/_{E_P}\right)} + 2\frac{dvdr}{g_1\left({}^{E}/_{E_P}\right)g_2\left({}^{E}/_{E_P}\right)} + \frac{r^2}{g_2^2\left({}^{E}/_{E_P}\right)}d\Omega_2^2 \quad \Leftrightarrow Vaidya$$

Rotating
Heat bath

$$ds^2 = -\frac{dt^2}{g_1^2\left({}^{E}/_{E_P}\right)} + \frac{dr^2}{g_2^2\left({}^{E}/_{E_P}\right)} + \frac{r^2 d\phi^2}{g_2^2\left({}^{E}/_{E_P}\right)} + \frac{dz^2}{g_2^2\left({}^{E}/_{E_P}\right)}$$

Gravity's Rainbow

Comoving Frame

$$ds^2(E) = -\left(1-\Omega_0^2 r^2\right)\frac{dt^2}{g_1^2(E)} - \frac{2\Omega_0 r^2 d\phi' dt}{g_1(E)\,g_2(E)} + \frac{dr^2}{g_2^2(E)} + \frac{r^2 d\phi'^2}{g_2^2(E)} + \frac{dz^2}{g_2^2(E)}.$$

Kerr Frame

$$ds^2 = g_{tt}\frac{dt^2}{g_1^2(E/E_P)} + 2g_{t\phi}\frac{dt\,d\phi}{g_1(E/E_P)\,g_2(E/E_P)} + g_{\phi\phi}\frac{d\phi^2}{g_2^2(E/E_P)} + g_{rr}\frac{dr^2}{g_2^2(E/E_P)} + g_{\theta\theta}\frac{d\theta^2}{g_2^2(E/E_P)}$$

$$g_{tt} = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}, \qquad g_{t\phi} = -\frac{a\sin^2\theta\left(r^2+a^2-\Delta\right)}{\Sigma},$$

$$g_{\phi\phi} = \frac{\left(r^2+a^2\right)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta, \qquad g_{rr} = \frac{\Sigma}{\Delta}, \qquad g_{\theta\theta} = \Sigma,$$

$$\Delta = r^2 - 2MGr + a^2, \qquad \Sigma = r^2 + a^2\cos^2\theta$$

Eliminating the brick wall using Gravity's Rainbow

Generalization of the Curved Space Proposal for a S.S.Metric

[R.Garattini P.L.B. B685 (2010) 329 e-Print: arXiv:0902.3927 [gr-qc]]

$$ds^2 = -\exp\left(-2\Lambda\left(r\right)\right)\frac{dt^2}{g_1^2\left(E/E_P\right)} + \frac{dr^2}{\left(1-\frac{b\left(r\right)}{r}\right)g_2^2\left(E/E_P\right)} + \frac{r^2}{g_2^2\left(E/E_P\right)}d\theta^2 + \frac{r^2}{g_2^2\left(E/E_P\right)}\sin^2\theta d\varphi^2$$

- $b(r)$ is the shape function
- $\Lambda\ (r)$ is the redshift function

$$b\left(r_0\right) = r_0 \quad r \in \left[r_0, +\infty\right)$$

Eliminating the brick wall using Gravity's Rainbow

From the equation of motion, we can define an r-dependent radial wave number

$$k^2\left(r,l,E\right)=\frac{1}{1-\frac{b(r)}{r}}\left[\exp\left(2\Lambda\left(r\right)\right)\frac{E^2\tilde{h}^2\left(E/E_P\right)}{1-\frac{b(r)}{r}}-\frac{l\left(l+1\right)}{r^2}\right]$$

$$\tilde{h}\left(E/E_P\right)=\frac{g_1\left(E/E_P\right)}{g_2\left(E/E_P\right)}$$

Free energy $$F=\frac{2}{\pi\beta}\int_0^\infty \ln\left(1-\exp\left(-\beta E\right)\right)\frac{d}{dE}\left(\frac{1}{3}E^3\tilde{h}^3\left(E/E_P\right)\right)dE\int_{r_0+h}^{R}\frac{\exp\left(3\Lambda\left(r\right)\right)}{\left(1-\frac{b(r)}{r}\right)^2}r^2dr$$

Assumption $\Rightarrow r_0+h=r_0+h\left(E/E_P\right)=r_0\left(1+\sigma\left(E/E_P\right)\right)$

Modification of the brick wall due to the rainbow's functions

In proximity of the throat, we consider the approximate free energy

Free energy $$F_{r_0}=\frac{2r_0^4}{\pi\beta}\frac{\exp\left(3\Lambda\left(r_0\right)\right)}{\left(1-b'\left(r_0\right)\right)^2}\int_0^\infty\frac{\ln\left(1-\exp\left(-\beta E\right)\right)}{r_0\sigma\left(E/E_P\right)}\frac{d}{dE}\left(\frac{1}{3}E^3\tilde{h}^3\left(E/E_P\right)\right)dE$$

Int. by parts $$F_{r_0}=-\frac{2r_0^4}{\pi\beta}\frac{\exp\left(3\Lambda\left(r_0\right)\right)}{\left(1-b'\left(r_0\right)\right)^2}\int_0^\infty\frac{d}{dE}\left[\frac{\ln\left(1-\exp\left(-\beta E\right)\right)}{r_0\sigma\left(E/E_P\right)}\right]\left(\frac{1}{3}E^3\tilde{h}^3\left(E/E_P\right)\right)dE$$

This is possible if $\tilde{h}\left(E/E_P\right)$ is rapidly convergent $\rightarrow$ Good choice $\tilde{h}\left(E/E_P\right)=\exp\left(-E/E_P\right)$

Property $\Rightarrow\sigma\left(E/E_P\right)\rightarrow 0$ when $E/E_P\rightarrow 0$

$$\sigma\left(E/E_P\right)=\tilde{h}^\delta\left(E/E_P\right)\left(\frac{E}{E_P}\right)^\alpha$$

Assumption on σ(E/E$_P$)

Two interesting cases

Case a) $\delta=0$ $\alpha>0$

Case b) $\delta>0$ $\alpha>0$

Case a)

In the limit $\beta E_P >> 1$

$$S = \frac{A_{r_0} E_P^2}{4} \frac{\exp\left(2\Lambda\left(r_0\right)\right)}{1 - b'\left(r_0\right)} \frac{2}{9\pi}$$

To recover the area law, we have to set

$$\frac{\exp\left(2\Lambda\left(r_0\right)\right)}{1 - b'\left(r_0\right)} = \frac{9\pi}{2}$$

This corresponds to a redefinition of the time variable with respect to the Schwarzschild time.

The internal energy $U = \frac{M}{4}$ where we have used

$r_0 = 2MG$ and $\beta = 8\pi MG$

Discrepancy of a factor of 3/2 with the ’t Hooft result

Rotations and Gravity's Rainbow

For a S.S.M

Free energy $$F=\frac{2}{\pi\beta}\int_0^\infty \ln\left(1-\exp\left(-\beta E\right)\right)\frac{d}{dE}\left(\frac{1}{3}E^3h^3\left(E/E_P\right)\right)dE\int_{r_+ + r_0\sigma\left(E/E_P\right)}^{R}\frac{\exp\left(3\Lambda\left(r\right)\right)}{\left(1-\frac{b\left(r\right)}{r}\right)^2}r^2dr$$

Property $\Rightarrow \sigma\left(E/E_P\right)\rightarrow 0$ when $E/E_P\rightarrow 0$

For a Rotating Heat Bath

$$F=\frac{1}{6\beta}\int_0^R rdr\int dzd\phi\int_{-1}^{1}du\int_0^\infty \frac{d}{dE}\left(h^3\left(E/E_P\right)E^3\right)\ln\left(1-e^{-\beta E\left(1-h\left(E/E_P\right)ru\Omega_0\right)}\right)dE$$

For a Comoving Frame

$$F=\frac{A}{3\beta}\int_0^{\Omega_0^{-1}}\frac{rdr}{\left(1-\Omega_0^2r^2\right)^2}\int_0^\infty \frac{d}{dE}\left(h^3\left(E/E_P\right)E^3\right)\ln\left(1-e^{-\beta E}\right)dE$$

Kerr Black Hole Entropy and Gravity's Rainbow

Kerr metric

$$C\left(r_{+},\theta\right)=\frac{\pi\left(r_{+}^{2}+a^{2}\right)^{2}}{ar_{+}r_{0}\kappa^{2}}\arctan\left(\frac{a}{r_{+}}\right)$$

$\kappa \longrightarrow$ surface gravity

Non-Superradiant modes

$$F_{r_{+}}^{NSR}=-\frac{C\left(r_{+},\theta\right)}{16\pi^{2}\beta}\int_{0}^{1/h}du\left(1-u\right)^{2}\int_{0}^{\infty}\frac{d}{dE}\left[\frac{\left(Eh\left(E/E_{P}\right)\right)^{3}}{\sigma\left(E/E_{P}\right)}\right]\ln\left(1-e^{-\beta E\left(1-h\left(E/E_{P}\right)u\right)}\right)dE$$

Superradiant modes

$$F_{r_{+}}^{SR}=\frac{C\left(r_{+},\theta\right)}{16\pi^{2}\beta}\int_{1/h}^{\infty}du\left(1-u\right)^{2}\int_{0}^{\infty}\frac{d}{dE}\left[\frac{\left(Eh\left(E/E_{P}\right)\right)^{3}}{\sigma\left(E/E_{P}\right)}\right]\ln\left(1-e^{-\beta E\left(h\left(E/E_{P}\right)u-1\right)}\right)dE$$

Kerr metric $\longrightarrow$ ZAMO

$$F=\frac{1}{8\pi^{2}\beta}\int d\theta d\tilde{\phi}\int_{0}^{\infty}\ln\left(1-\exp\left(-\beta E\right)\right)\frac{d}{dE}\left(\frac{1}{3}E^{3}h^{3}\left(E/E_{P}\right)\right)dE\int_{r_{+}+r_{0}\sigma\left(E/E_{P}\right)}^{R}\left(-g^{tt}\right)^{\frac{3}{2}}\sqrt{g_{rr}g_{\theta\theta}g_{\tilde{\phi}\tilde{\phi}}}\,dr$$

Kerr Black Hole Entropy and Gravity's Rainbow

Kerr metric $\longrightarrow$ ZAMO

$$ds^2 = -N^2 \frac{dt^2}{g_1^2\left(E/E_P\right)} + g_{\phi\phi}\frac{d\tilde{\phi}^2}{g_2^2\left(E/E_P\right)} + g_{rr}\frac{dr^2}{g_2^2\left(E/E_P\right)} + g_{\theta\theta}\frac{d\theta^2}{g_2^2\left(E/E_P\right)}$$

$$N^2 = -\frac{\Delta \sin^2\theta}{g_{\tilde{\phi}\tilde{\phi}}} \to 0 \text{ when } r \to r_+$$

Constraint Equation in phase space

$$g^{\mu\nu}(E)k_\mu k_\nu = 0 \quad h\left(E/E_P\right) = \frac{g_1\left(E/E_P\right)}{g_2\left(E/E_P\right)}$$

Kerr Black Hole Entropy and Gravity's Rainbow

Kerr metric

$$F_{r_+} = -\frac{C\left(r_+,\theta\right)}{12\pi^2\beta^3}\zeta\left(3\right)E_P^{\alpha}\int_0^{\infty} dE\frac{d\left(E^{3-\alpha}h^{3-\delta}\left(E/E_P\right)\right)}{dE}\frac{\left(h\left(E/E_P\right)-1\right)}{E^2h^3\left(E/E_P\right)}$$

Kerr metric $\longrightarrow$ ZAMO

$$F_{r_H} = -\frac{C\left(r_H,\theta\right)}{24\pi^2\beta}\left[\zeta\left(2,1+\frac{3}{\beta E_P}\right)+\frac{\beta E_P}{3}\left(\gamma+\Psi\left(1+\frac{3}{\beta E_P}\right)\right)\right]$$

$$S \propto \frac{A}{4G} \quad \text{where } A = 4\pi\left(r_H^2+a^2\right)$$

Conclusions

- Application of Gravity's Rainbow can be considered to compute divergent quantum observables.
- Neither Standard Regularization nor Renormalization are required. This also happens in NonCommutative geometries and G.U.P. modifications
- The thermodynamical observables can be computed in the context of spherically symmetric backgrounds.
- Rotations can be included.

Thank You
for
Your
Attention