Hyperbolic Geometry and Amplituhedra in 1+2 dimensions

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Outline

Positive Geometries and Canonical forms

What is a positive geometry?
(Arkani-Hamed et al, arXiv:1703.04541)

- $\mathrm{X} \supset \mathbb{R}(\mathrm{X}) \supset \mathrm{X}^+ \quad \mathrm{X}^+ := \{f(x) \geq 0\} \quad \partial\mathrm{X}^+ := \{f(x) = 0\}$
- $\omega_{\mathrm{X}^+} \in \Omega^n(\mathrm{X}) \quad \omega_{\mathrm{X}^+} = \frac{df}{f} \wedge \eta \quad \mathrm{Res}(\omega_{\mathrm{X}^+})|_{\partial\mathrm{X}^+} := \eta = \omega_{\partial\mathrm{X}^+}$

Positive Geometries and Canonical forms

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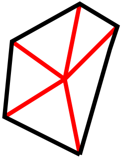
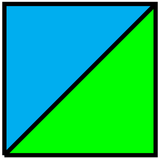
- $\phi : \mathcal{K} \to \mathrm{X}^+ \quad \partial\mathcal{K} \to \partial X^+ \quad \to \quad \omega|_\phi(K) \sim \mathcal{A}dY$

e.g. $\quad C \cdot Z = 0, \quad \sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0$

Triangulations



$$\omega_P = \sum_T \omega_T$$



$$\omega = dx \wedge dy \left(\frac{1}{x(y-1)(x-y)} - \frac{1}{y(x-1)(x-y)} \right)$$
$$= \frac{dx \wedge dy}{xy(x-1)(y-1)}$$

Feynman diagrams/ BCFW recursion relations $\Leftrightarrow$ Triangulation

ϕ^3 planar theory

Space-time scalar fields in the adjoint of $U(N) \times U(\bar{N})$.

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi_{a\bar{a}} \partial^\mu \phi^{a,\bar{a}} - \frac{\lambda}{3!} f_{abc} f_{\bar{a}\bar{b}\bar{c}} \phi^{a\bar{a}} \phi^{b\bar{b}} \phi^{c\bar{c}}$$

$$[t^a, t^b] = f_{abc} t^c \quad [\bar{t}^{\bar{a}}, \bar{t}^{\bar{b}}] = f_{\bar{a}\bar{b}\bar{c}} \bar{t}^{\bar{c}}$$

$$\begin{aligned}\mathcal{A}(a_i, \bar{a}_i, k) &= \sum \mathrm{Tr}(t_1^a t_2^a \dots) \mathrm{Tr}(\bar{t}^{\bar{a}_1} \bar{t}^{\bar{a}_2} \dots) m(k_i \cdot k_j) \\ &= \sum_{\alpha,\beta \in S_n} \mathrm{Tr}(t^{\alpha(1)} \dots t^{\alpha(n)}) \mathrm{Tr}(t^{\beta(1)} \dots t^{\beta(n)}) m_n(\alpha|\beta).\end{aligned}$$

$m_n(\alpha|\beta) \rightarrow$ sum over diagrams planar with respect to α AND β.

Cachazo-He-Yuan formulae

$$m_n(\alpha|\beta) = \oint_{\mathcal{M}_{0,n}} \frac{d^n\sigma \;\; \delta'(E_a)}{\text{Vol SL(2,}\mathbb{C})} \frac{1}{\prod \sigma_{\alpha(i)} - \sigma_{\alpha(i+1)}} \frac{1}{\prod \sigma_{\beta(i)} - \sigma_{\beta(i+1)}}$$

Scattering Equations: $E_a := \sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0$

Double Copy: $C(\alpha) := \frac{1}{\prod \sigma_{\alpha(i)} - \sigma_{\alpha(i+1)}}$ $\quad E := Pf\Psi(k_i, \epsilon_i)$

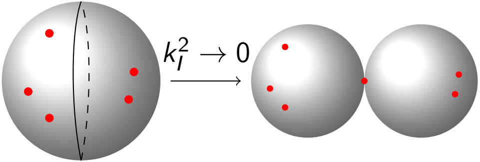
$C \times C \to \phi^3$

$C \times E \to$ gluons in Yang-Mills

$E \times E \to$ gravitons in pure gravity

Clues of an Amplituhedron - 1

$$\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0 \quad \Rightarrow \; \phi : \quad \begin{array}{c} \mathcal{K} \to \mathcal{M}_{0,n} \\ \partial\mathcal{K} \to \partial\mathcal{M}_{0,n} \sim \mathcal{M}_{0,n_L} \times \mathcal{M}_{0,n_R} \end{array}$$



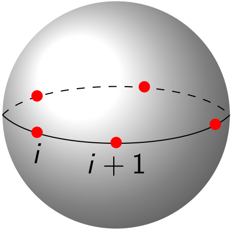
Clues of an Amplituhedron - 2

There are natural positive geometries living in moduli space: Associahedra (Kapranov, 1993)

$$\mathbb{R}(\mathcal{M}_{0,n}) = \cup_{\alpha \in S_n} \mathcal{A}_\alpha$$

$$\mathcal{A}_\alpha = \{\sigma_\alpha(i) - \sigma_\alpha(i+1) \geq 0\}$$

$$\omega_\alpha = \frac{d^n\sigma}{\text{Vol SL}(2,\mathbb{C})} \frac{1}{\prod \sigma_{\alpha(i)} - \sigma_{\alpha(i+1)}}$$



$m_n(\alpha|\beta) \sim$ Intersection number of $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ (Mizera, 2017)

Kinematical Associahedron arXiv:1711.09102

Kinematical space for scattering of n massless particles

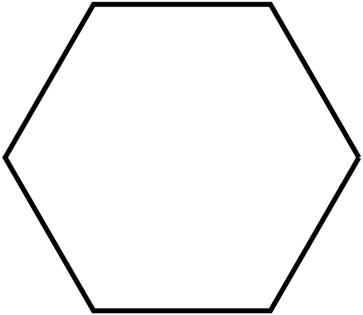
$$\mathcal{K}_n := \{k_{ab} = k_a \cdot k_b\}$$

$$\begin{cases} k_{ab} = k_{ba} \\ \sum_{b\neq a} k_{ab} = 0 \end{cases} \Rightarrow \dim(\mathcal{K}_n) = \frac{n(n-3)}{2} > n-3 = \dim(\mathcal{M}_{0,n})$$

- Unphysical Boundaries: e.g. $k_{13} = 0$ is not a pole of m_4.
- Cut them away: impose $k_{ij} = -c_{ij}$ if i, j non adjacent
- We are left with a $n-3$ dimensional subspace

Positive planar propagators $(k_i + k_{i+1} + \cdots + k_j)^2 > 0$ cut an Associahedron in $\mathcal{K}_n$.

The faces of the Associahedron



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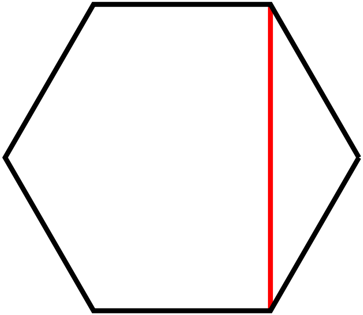

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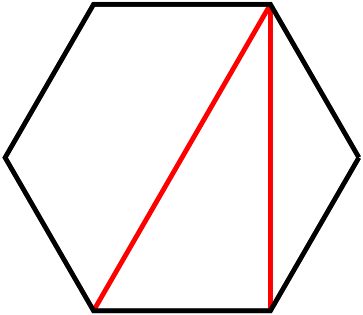

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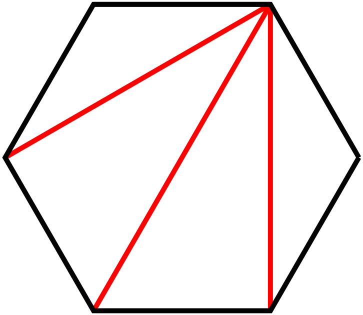

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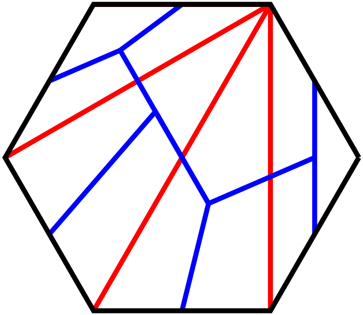

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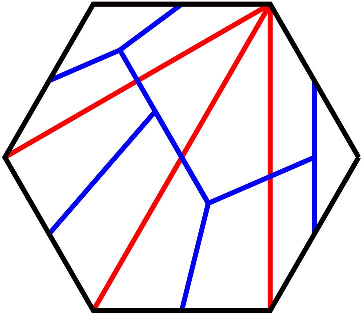

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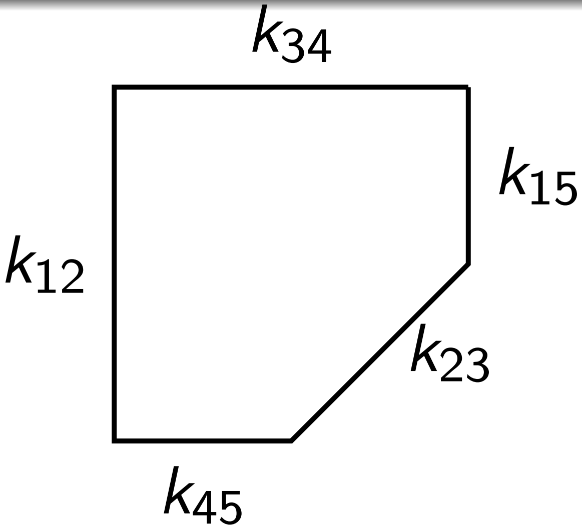
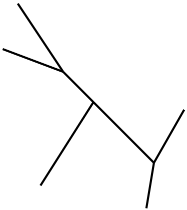
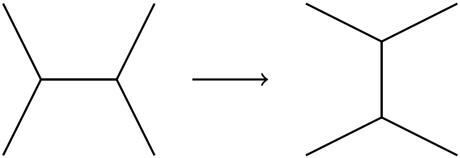


Planar Scattering Form



$$\Rightarrow \quad \omega_g = sgn(g) \bigwedge_{I \in g} \frac{dS_I}{S_I}$$

Planar scattering form $\Omega = \sum_g \omega_g$
Invariance under $S_I \to \alpha(S_I) S_I$ requires $sgn(g)$ to satisfy the "mutation rule".



$$sgn(g) \to -sgn(g)$$

Canonical Forms and Amplitudes

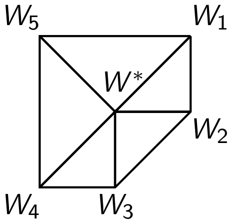
$\Omega|_{\{k_{ij}=-c_{ij}\}}$ is the canonical form of the kinematical Associahedron.

The mutation rule and the costraints imply that $\Omega = m_n \bigwedge_{I\in g} dS_I$

The Associahedron is the Amplituhedron for planar ϕ^3

Triangulations of the Associahedron

$$Y = (1, k_{12}, k_{45}) \Rightarrow f_I = W_I \cdot Y \text{ with } \begin{cases} W_1 = (0,1,0) \\ W_2 = (c_{14}+c_{24}, 0, -1) \\ W_3 = (c_{13}+c_{14}, -1, 0) \\ W_4 = (c_{13}, -1, 1) \\ W_5 = (0,0,1) \end{cases}$$



$$\mathrm{Vol}(\mathcal{A}^*) = \sum_I \frac{sign(I)\langle W^* W_{v1} W_{v2}\rangle}{W^* \cdot Y \prod_{v\in I} W_v \cdot Y}$$

$$= \frac{1}{k_{12}k_{34}} + \frac{1}{k_{12}k_{45}} + \ldots \; (W^* = (1,0,0))$$

$$= \frac{k_{12}+k_{234}}{k_{12}k_{34}k_{234}} + \frac{k_{12}+k_{234}}{k_{12}k_{234}k_{23}} + \frac{k_{12}-k_{123}+k_{23}}{k_{12}k_{23}k_{123}}$$

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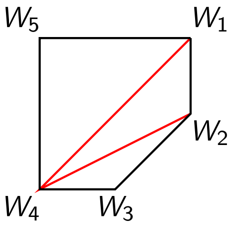

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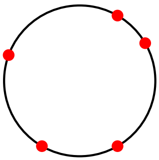
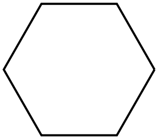
Scattering Equations as a Pushforward





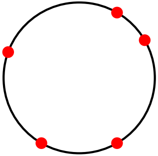
$$\omega_\alpha = \frac{d^n\sigma}{\mathrm{Vol}\ \mathrm{SL}(2,\mathbb{C})} \frac{1}{\prod \sigma_{\alpha(i)} - \sigma_{\alpha(i+1)}}$$

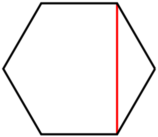
$$\Omega = m_n \bigwedge_I dS_I$$

ϕ is a $(n-3)! - 1$ map from the Moduli space Associahedron to the kinematical one.

$\phi_*(\omega) = \Omega \Rightarrow$ derivation of CHY formulae

Scattering Equations as a Pushforward





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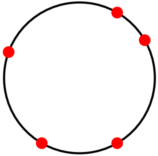

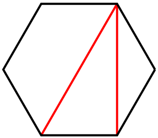

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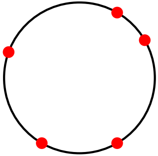

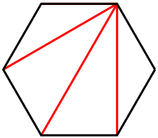

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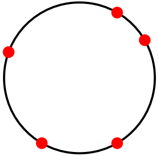

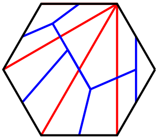

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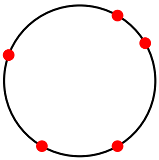

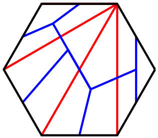

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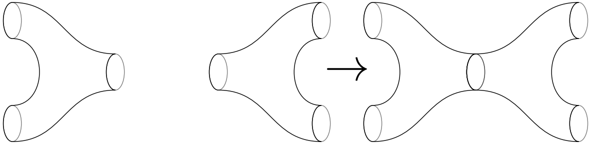
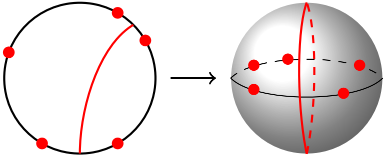
Positive Structures in $\mathcal{M}_{1,n}$

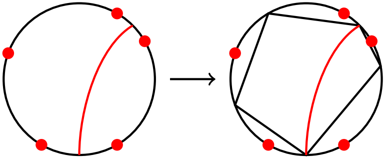
- Hyperbolic geometry: The moduli of a Riemann surface are given by geodesic lengths (and gluing angles)



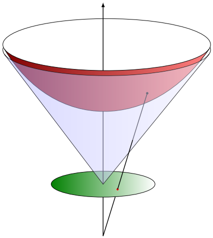
- For a surface with boundaries we take the Schottky double:



Hyperbolic Plane and the Associahedron



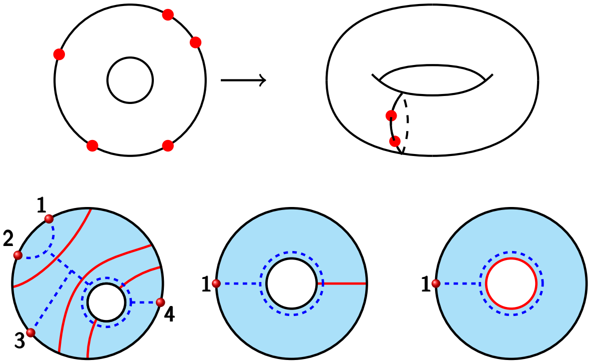
Geodesic Arcs $\leftrightarrow$ Diagonals $\Rightarrow$
Moduli space Associahedron



Configurations of light rays $\leftrightarrow$
Kinematical Associahedron

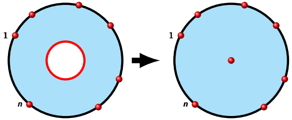
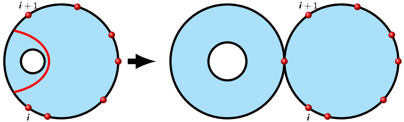
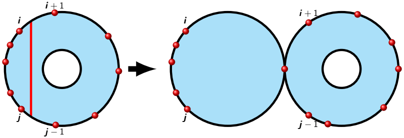
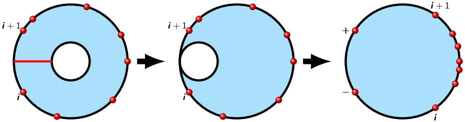
The map $k^\mu \to \sigma$ yields a solution of the scattering equations:
diffeomorphism

Annulus and the Halohedron

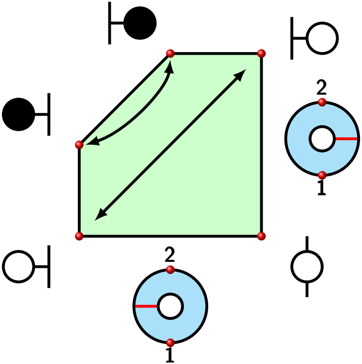


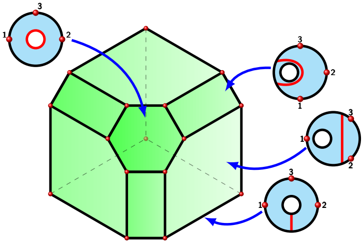
Geodesic Arcs on an Annulus ⇒ Halohedron (Sevadoss, 2010)

Faces of the Halohedron

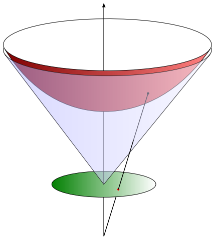
	Cyclohedron W_n
	Tadpole Factorisation $K_n \times H_1$
	Factorisation $K_m \times H_{n-m+1}$
	Cut Associahedron K_{n+1}

Some examples of Halohedra





The Kinematical Halohedron



l^μ, $l^2 \leq 1 \rightarrow$ Hyperbolic circle
$C_l = \{w \in H | l \cdot w = 1\}$
If we cut the circle away from the Hyperbolic plane we get an annulus

- Kinematical Halohedron: Configuration of light rays and a loop momentum
- Moduli Halohedron: Moduli of annuli with ordered markings
- Hyperboloid model: Map between the two Halohedra

The 1-loop Amplituhedron?

- Faces reproduce cuts of 1-loop integrand
- Vertices are given by Feynman diagrams
- Tadpoles appear paired as in 1-loop CHY formulae (He, 2015)
- Gram determinant relations makes difficult to reproduce even tree amplitudes
- The integrand has double poles due to bubble topologies

It is possible to define a 1-loop planar scattering form. The mutation rule is generalised and now mutate bubbles into pair of tadpoles. How to cut an Halohedron in mandelstam space? What should be the new $k_{i,j} = const$? Work in Progress!

Thanks for your attention!