Amplituhedron meets Jeffrey-Kirwan Residue

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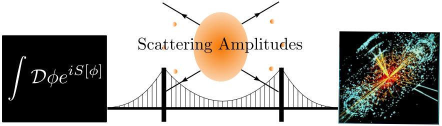
Introduction

- **Scattering Amplitudes** in Quantum Field Theory



  → Modern On-Shell Methods: Recursion Relations, Generalized Unitarity, ...

- $\mathcal{N} = 4$ super Yang-Mills (SYM)
  - → supersymmetric cousin of QCD
  - → "Hydrogen Atom of 21st century" - *Integrability*
  - → numerous dualities & correspondences

- Geometrization of Scattering Amplitudes: Grassmannian, **Amplituhedron**, ...

  "Amplitude = Volume of a Geometric Space"

- The **Jeffrey-Kirwan Residue** is a powerful concept in
  - → *Mathematics*: Localization of Group Actions in equivariant cohomology
  - → *Physics*: Supersymmetric Localization in Gauge Theories

Our work establishes: the connection between JK-Residue and the Amplituhedron

Outline

Amplituhedron: Geometry

The **Amplituhedron** $\mathcal{A}_{n,k}^{(m)}$ - a recently discovered mathematical object: [Arkani-H., Trnka, '13]

- lives in the space of k-planes in $(m+k)$-dimensions, i.e. the *Grassmannian*
- is a generalization of polytopes inside the Grassmannian
- has elements $Y \in \mathcal{A}_{n,k}^{(m)}$ of the form

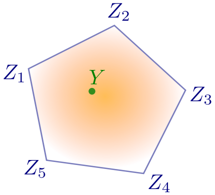
$$Y = C \cdot Z$$

Z: $n \times (m+k)$ matrix of fixed *External Data*

C: $k \times n$ matrix $/GL(k)$ spanning the *Interior*

+ Positivity: C, Z have all ordered maximal minors positive

- Examples
  - $\rightarrow (k=1, m=2)$: $n-$gon in two dimensions
  - $\rightarrow (k=1, \text{any } m)$: *cyclic polytopes* in m dimensions
  - $\rightarrow (k>1)$: previously *unknown* objects



Amplituhedron: Volume Function

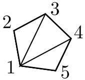
For each Amplituhedron $\mathcal{A}_{n,k}^{(m)}$ one defines a **volume function** $\Omega_{n,k}^{(m)}$ such that

> $\Omega_{n,k}^{(m)}$ has logarithmic *singularities* at all *boundaries* of $\mathcal{A}_{n,k}^{(m)}$

- Tree Amplitudes in $\mathcal{N}=4$ SYM can be extracted from $\Omega_{n,k}^{(4)}$ (4: space-time dimension; k: helicity sector; n: number of particles)
- How do we find it?

→ *Geometrically*: Triangulate $\mathcal{A}_{n,k}^{(m)}$ and sum over the known volumes of each triangle

e.g. $k=1, m=2, n=5$: $\quad \Omega_{n,1}^{(2)} = [123] + [134] + [145]$



→ *Analytically*: Evaluate the contour integral

$$\Omega_{n,k}^{(m)}(Y,Z) = \int_\gamma \frac{d^{k\cdot n}\, C}{(12\ldots k)\ldots(n1\ldots k-1)}\delta^{k(m+k)}(Y - C\cdot Z) = \int_\gamma \omega_{n,k}^{(m)}$$

> different contours ↔ different triangulations

Finding γ or triangulations is a *highly non-trivial* task!

Jeffrey-Kirwan Residue

The **Jeffrey-Kirwan Residue** is an operation on Differential Forms

Toy Example

[Jeffrey, Kirwan, '95]

$$\omega = \frac{dx_1 \wedge \ldots \wedge dx_r}{\beta_1(x)\ldots\beta_n(x)}, \qquad \beta_i(x) = \beta_i \cdot x + \alpha_i$$

$$\omega = \frac{dx_1 \wedge dx_2}{\beta_1(x)\ldots\beta_5(x)}$$

- $B = \{\beta_i\}$ is the set of r dimensional vectors: *charges*
- A *Cone* is the positive span of r charges

  e.g. $C_{25} = \mathrm{Span}_+\{\beta_2, \beta_5\}$
- For fixed $\eta \in \mathbb{R}^r$, the *Jeffrey-Kirwan Residue* is

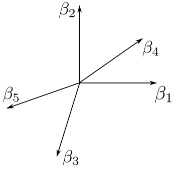
$$\mathrm{JKRes}^{B,\eta}\omega = \sum_{Cone \ni \eta} \mathrm{Res}_{Cone}\,\omega$$

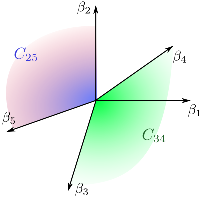
$\rightarrow$ Res_{Cone} is the multivariate residue, up to a sign

e.g. $\mathrm{Res}_{C_{25}}\omega$: residue at $\{\beta_2(x) = \beta_5(x) = 0\}$

$\rightarrow$ signs determined by the orientation of cones

e.g. $\det(\beta_2\beta_5) > 0$ for the cone C_{25}

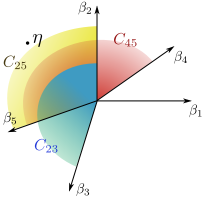




Jeffrey-Kirwan Residue

$$\mathrm{JKRes}^{B,\eta}\omega = \sum_{Cone \ni \eta} \mathrm{Res}_{Cone}\omega$$

Toy Example

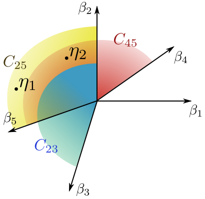


$$\mathrm{JKRes}^{B,\eta} = \mathrm{Res}_{C_{25}} + \mathrm{Res}_{C_{45}} + \mathrm{Res}_{C_{23}}$$

Jeffrey-Kirwan Residue

$$\mathrm{JKRes}^{B,\eta}\omega = \sum_{Cone \ni \eta} \mathrm{Res}_{Cone}\omega$$

Toy Example

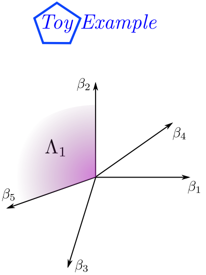


$$\mathrm{JKRes}^{B,\eta_1} = \mathrm{Res}_{C_{25}} + \mathrm{Res}_{C_{45}} + \mathrm{Res}_{C_{23}} = \mathrm{JKRes}^{B,\eta_2}$$

same form of the answer!

Jeffrey-Kirwan Residue

$$\text{JKRes}^{B,\eta}\omega = \sum_{Cone \ni \eta} \text{Res}_{Cone}\omega$$



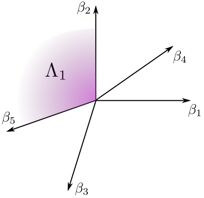
- A *Chamber* is a (non-empty) maximal intersection of cones
  $\rightarrow$ JKRes gives the *same form* of the answer

$$\text{JKRes}^{B,\eta} = \text{Res}_{C_{25}} + \text{Res}_{C_{45}} + \text{Res}_{C_{23}}, \; \forall \eta \in \Lambda_1$$

Jeffrey-Kirwan Residue

$$\mathrm{JKRes}^{B,\eta}\omega = \sum_{Cone \ni \eta} \mathrm{Res}_{Cone}\,\omega$$

Toy Example



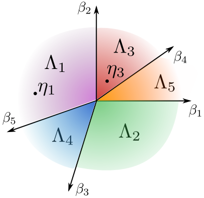
- A *Chamber* is a (non-empty) maximal intersection of cones
  → JKRes gives the *same form* of the answer

$$\mathrm{JKRes}^{B,\eta} = \mathrm{Res}_{C_{25}} + \mathrm{Res}_{C_{45}} + \mathrm{Res}_{C_{23}},\ \forall \eta \in \Lambda_1$$

- <u>Remarkable Property</u>

  JKRes is *independent* from the chamber

$$\mathrm{JKRes}^{B,\eta_1} = \mathrm{Res}_{C_{25}} + \mathrm{Res}_{C_{45}} + \mathrm{Res}_{C_{23}}$$
$$||$$
$$\mathrm{JKRes}^{B,\eta_3} = \mathrm{Res}_{C_{45}} + \mathrm{Res}_{C_{12}} + \mathrm{Res}_{C_{42}}$$

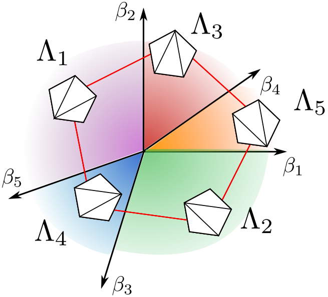


The JK-Residue provides a particular contour on which we integrate ω

Amplituhedron meets Jeffrey-Kirwan Residue

→ Recall $\Omega^{(m)}_{n,k}(Y,Z) = \int_\gamma \omega^{(m)}_{n,k}$

Toy Amplituhedron



- Apply JK-Residue to Amplituhedron

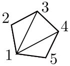
$$\Omega^{(2)}_{5,1} = \mathrm{JKRes}^{B,\eta}\omega^{(2)}_{5,1}$$

- Res_{Cone} computes the volume of triangles

$$\mathrm{Res}_{C_{25}} = [134]$$

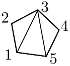
- Each *Chamber* corresponds to a *Triangulation*

$$\mathrm{JKRes}^{B,\eta_1}\omega^{(2)}_{5,1} = [134] + [123] + [145]$$



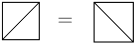
$$||$$

$$\mathrm{JKRes}^{B,\eta_3}\omega^{(2)}_{5,1} = [345] + [351] + [312]$$



- Adjacent chambers correspond to triangulations related by *bistellar flip*

→ *Geometrically*:



→ *Analytically*: Global Residue Theorem

General Statements: Cyclic Polytopes

For **Cyclic Polytopes** ($k = 1$, any m, any n):

$$\Omega_{n,1}^{(m)} = \mathrm{JKRes}^{B,\eta}\omega_{n,1}^{(m)}$$

[Ferro, Łukowski, MP, '18]

- *Positivity* of Amplituhedron determines the configuration of *Chambers*
- For each Chamber
  - → *Geometrically*: different triangulation of $\mathcal{A}_{n,1}^{(m)}$
  - → *Analytically*: different representation of $\Omega_{n,1}^{(m)}$
- For adjacent Chambers
  - → *Geometrically*: triangulations related by bistellar flip
  - → *Analytically*: representations related by Global Residue Theorem
- **Secondary Polytope** $\Sigma(\mathcal{P})$ - vertices are triangulations of a given polytope $\mathcal{P}$

[Gelfand, Zelevinsky, Kapranov, '90]

→ For an n-gon it is the *Associahedron*

→ In general it is very complicated

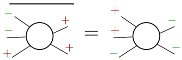
e.g. *physical case*, 8 particles: 3-dimensional and 40 vertices

→ It is fully captured by the Jeffrey-Kirwan Residue!

General Statements: Conjugates to Polytopes

Parity Conjugation is an operation which in

→ *Amplitudes*: flips the helicities of the particles

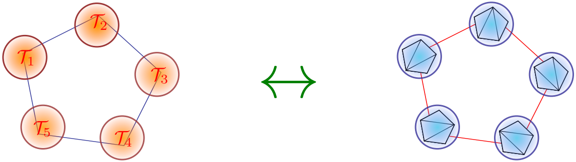
e.g. 

→ *Amplituhedron*: is realized by replacing k with $n-m-k$

e.g. $\overline{\mathcal{A}^{(2)}_{5,1}} = \mathcal{A}^{(2)}_{5,2}$

Conjugates to Polytopes $\mathcal{A}^{(m)}_{n,n-m-1}$ are *not Polytopes*! Nevertheless:

- The Jeffrey-Kirwan Residue computes $\Omega^{(m)}_{n,n-m-1}$, for *even* m
- Triangles and triangulations of $\mathcal{A}^{(m)}_{n,n-m-1}$ are in direct correspondence with the ones of the conjugate polytope $\mathcal{A}^{(m)}_{n,1}$
- The **Secondary Amplituhedron** is the same as the *Secondary Polytope* of the conjugate



Conclusions and Outlook

The Jeffrey-Kirwan Residue

- strenghten the connection between

  → The complex *Analytic-Algebraic* nature of Scattering Amplitudes

  → The rich *Geometric-Combinatorial* structure of the Amplituhedron

- naturally leads to the study of Secondary Amplituhedron

  → encodes *all triangulations* of the Amplituhedron

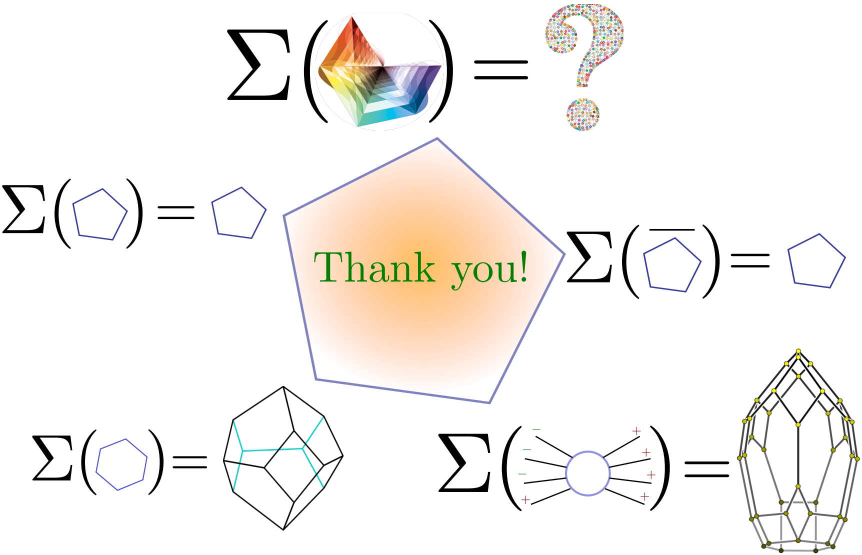
  → generalises the notion of *Secondary Polytope*

We showed this for polytopes and their conjugates

Open Questions

- What is the generalisation of the Jeffrey-Kirwan Residue for all other Amplituhedra, i.e. other helicity sectors and loops?
- Can we find the Secondary Amplituhedron in all these cases?

  → many new representations of Scattering Amplitudes!


Thank you!