The Natural Structure of Scattering Amplitudes

Manuel Accettulli Huber
Università degli studi di Padova

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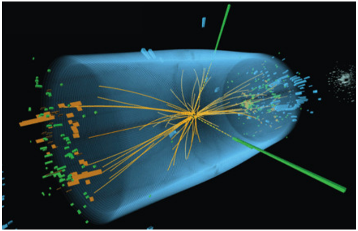
In collaboration with: P. Mastrolia and T. Peraro

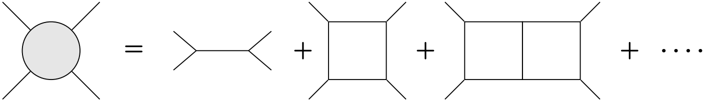
Outline

- Motivation
- Numerical tree-level amplitudes from recursive relations
- Functional reconstruction on $\mathbb{Q}$
- Functional reconstruction on $\mathbb{Z}_n$
- BCFW tree-level recursion and its relations to the maximum-cut
- Conclusions and outlook

Scattering amplitudes:

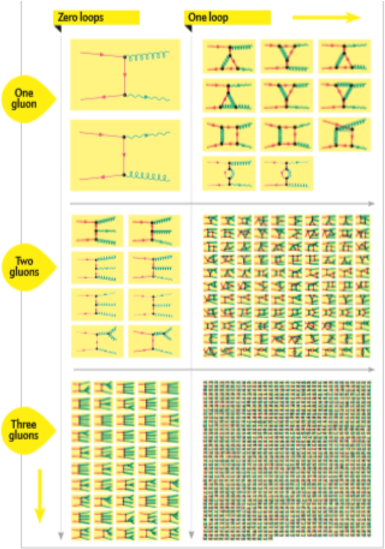
- connect theory and experiment
- confirm established models
- open the way for new physics
- infer general structure of a theory





High precision $\xrightarrow{\text{requires}}$ loop corrections

High c.o.m. energy $\xrightarrow{\text{leads to}}$ multi-particle interactions



Complexity of the calculations increases quickly with the number of legs and the number of loops.

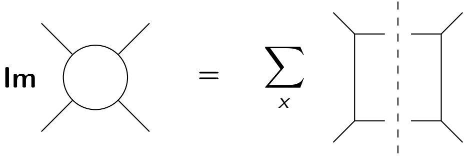
↓

Efficient techniques needed

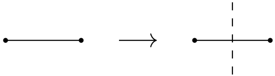
[Bern, Dixon, Kosover, Scientific American '12]

- Generalized unitarity $\xrightarrow{\text{extends}}$ optical theorem:

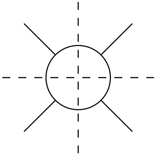
$$\mathbf{M}(i \to f) - \mathbf{M}^*(i \to f) = \sum_x \mathbf{M}(i \to x)\mathbf{M}^*(x \to f)$$



- unitarity cut:



$$\frac{1}{p^2 - m^2 + i\epsilon} \longrightarrow \delta(p^2 - m^2)$$



Tree-level amplitudes $\xrightarrow{\text{building blocks}}$ multi-loop amplitudes

Simple *n*-guon amplitudes

[Parke, Taylor '86]

$$iM(1^{\pm}, 2^{\pm}, \cdots, n-1^{\pm}, n^{\pm}) = 0$$

$$iM(1^{\mp}, 2^{\pm}, \cdots, n-1^{\pm}, n^{\pm}) = 0$$

$$iM(1^{+}, \cdots, i^{-}, \cdots, j^{-}, \cdots, n^{+}) = i\frac{\langle i|j\rangle^4}{\langle 1|2\rangle\langle 2|3\rangle\cdots\langle n-1|n\rangle\langle n|1\rangle}$$

$$iM(1^{-}, \cdots, i^{+}, \cdots, j^{+}, \cdots, n^{-}) = (-1)^n\frac{[i|j]^4}{[1|2][2|3]\cdots[n-1|n][n|1]}$$

= 34300 addenda =0

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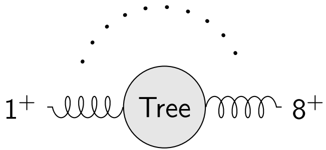

$=$ 34300 addenda $=0$

Solution:

Reconstruct the analytic expression of the tree-level amplitudes from numerical evaluations over finite fields

↓

Large intermediate expressions are replaced by natural numbers.

Important:

We are using functional reconstruction algorithms for rational functions, thus the amplitude must be written as rational function.

N.B. For simplicity, from here on we consider amplitudes involving only gluons.

Spinor-helicity formalism

[Mangano,Parke]

Dirac equation in momentum space for massless particles:

$$i\partial\!\!\!/\psi(x) = 0 \Rightarrow \begin{cases} p\!\!\!/ U_s(p) = 0 \\ p\!\!\!/ V_s(p) = 0 \qquad s = L, R \end{cases}$$

- Spinors: $U_R(p), U_L(p), \overline{U}_L(p), \overline{U}_R(p) \quad \leftrightarrow \quad |p\rangle, |p], \langle p|, [p|$
- Momenta: $p^\mu = \frac{\langle p|\sigma^\mu|p]}{2} \qquad p\!\!\!/ = |p\rangle[p| + |p]\langle p|$
- Polarizations: $\epsilon^\mu_R(k,r) = \frac{1}{\sqrt{2}}\frac{\langle r|\sigma^\mu|k]}{\langle r|k\rangle} \qquad \epsilon^\mu_L(k,r) = \frac{1}{\sqrt{2}}\frac{\langle k|\sigma^\mu|r]}{[k|r]}$
- Invariants: $s_{ij} = (p_i + p_j)^2 = \langle i|j\rangle[j|i]$

Tree-level scattering amplitudes are rational functions of the spinor components.

Amplitude parametrization

$\mathbf{M}(p_1, \cdots, p_n) \rightarrow \mathbf{M}(|p_1\rangle, |p_1], \cdots, |p_n\rangle, |p_n])$ <u>rational function</u>

Suitable parametrization of $|p\rangle, |p]$ needed: [Hodges '09;Badger '16;Peraro '15]

momentum-twistor variables

- $3n - 10$ independent variables $\{x_1, \cdots, x_{3n-10}\}$

$$3n - 10 = 4n - \underbrace{n}_{\text{onshell}} - \underbrace{4}_{\text{momentum conservation}} - \underbrace{6}_{\text{Lorentz invariance}}$$

- $\mathbf{M}$ is rational in the x_i
- x_i allow arbitrary values →

momentum conservation and on-shellness always satisfied

- $n = 5$, a parametrization: [Badger '16]

$$\begin{pmatrix} |1\rangle & |2\rangle & |3\rangle & |4\rangle & |5\rangle \\ |\mu_1] & |\mu_2] & |\mu_3] & |\mu_4] & |\mu_5] \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1} + \frac{1}{x_1 x_2} & \frac{1}{x_1} + \frac{1}{x_1 x_2} + \frac{1}{x_1 x_2 x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & \frac{x_5}{x_4} - 1 \\ 0 & 0 & 0 & \frac{x_4}{x_2} & 1 \end{pmatrix}$$

$$s_{12} = x_1 \qquad s_{23} = x_1 x_4 \qquad s_{34} = \frac{x_1((x_5-1)x_2x_3+(x_3+1)x_4)}{x_2}$$

$$s_{45} = x_1 x_5 \qquad s_{51} = x_1 x_3 (x_2 - x_4 + x_5)$$

- $n = 4$, an amplitude:

$$M_4(1^+, 2^-, 3^+, 4^-) = -\frac{8(1+x_2)^4}{x_1^2 x_2^3}$$

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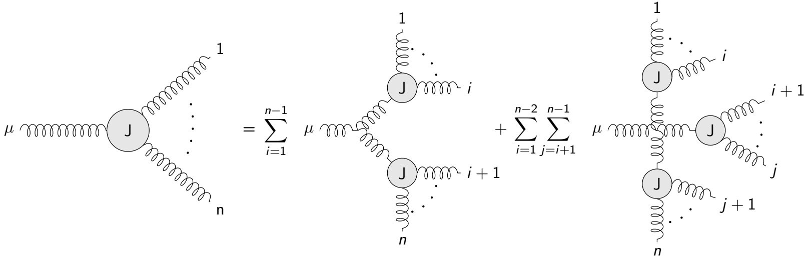
Off-shell recurrence relation

[Berends, Giele; Mangano,Parke]

$$\mathbf{M}(1,\cdots,n+1) \mapsto J^{\mu}(1,\cdots,n)$$



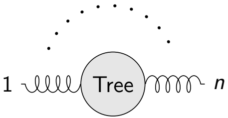
$$\mathbf{M}_{n+1}(1,\cdots,n+1) = \lim_{P_{1,n}\to 0} \epsilon_{\mu}(n+1) J^{\mu}(1,\cdots,n) P_{1,n}^2$$

On-shell recurrence relation

Complex momenta + residue theorem = Amplitude factorization

$$p_i, p_j \to p_i(z), p_j(z) \qquad z \in \mathbb{C} \qquad \Rightarrow \qquad M \to M(z)$$

$$\frac{1}{2\pi i}\oint_{C_\infty} \frac{iM(z)dz}{z} = \sum_{all\ poles\ \alpha} \underset{z=z_\alpha}{\textbf{Res}} \frac{iM(z)}{z} = 0$$



[Britto,Cachazo,Feng,Witten '05]

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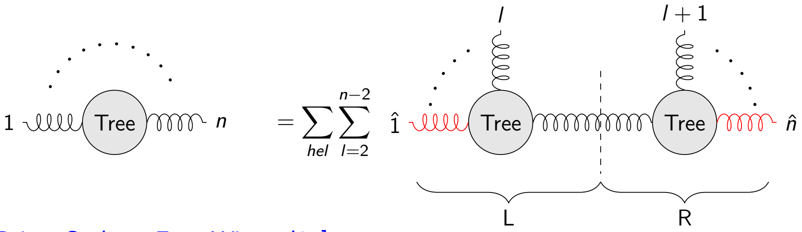
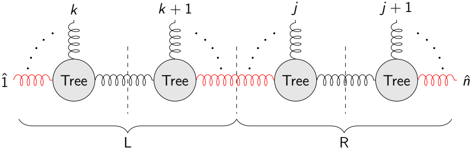


[Britto,Cachazo,Feng,Witten '05]

On-shell recurrence relation

$$= \sum_{hel} \sum_{l=2}^{n-2} \sum_{k=2}^{l-1} \sum_{j=l+1}^{n-2}$$



$$\vdots$$

$$= \sum_{hel} \sum_{\sigma} S(\sigma)$$

Building blocks:

$$= \frac{\langle 1|2\rangle^4}{\langle 1|2\rangle\langle 2|3\rangle\langle 3|1\rangle}$$

$$= -\frac{[1|2]^4}{[1|2][2|3][3|1]}$$

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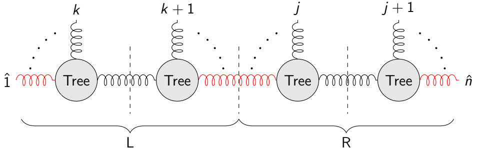

⋮

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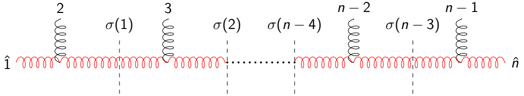

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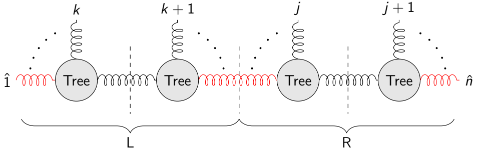

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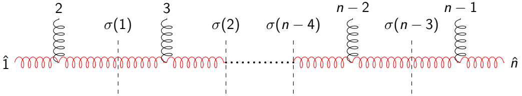

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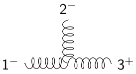

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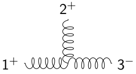

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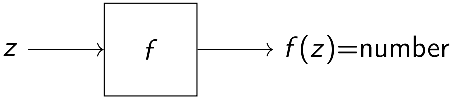
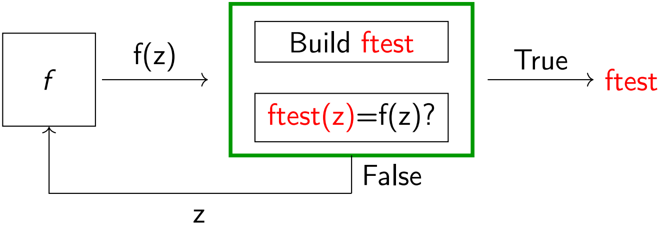
Black-box interpolation

Black-box algorithm:



Reconstruction of the analytic expression of f:



Univariate polynomials and rational functions

- Newton's polynomial form:

$$f(z) = a_0 + (z - y_0)\Big(a_1 + (z - y_1)\big(\cdots + (z - y_{r-1})a_r\big)\Big)$$

- Thiele's interpolation formula for rational functions:

$$f(z) = a_0 + \cfrac{z - y_0}{a_1 + \cfrac{z - y_1}{a_2 + \cfrac{z - y_2}{\cfrac{\vdots}{a_{N-1} + \cfrac{z - y_{N-1}}{a_N}}}}}$$

Example, a polynomial:

1. $f(y_0) \xrightarrow{Yields} a_0 \quad \Rightarrow \quad ftest(x) = a_0$
2. $f(y_1) \xrightarrow{Yields} a_1 \quad \Rightarrow \quad ftest(x) = a_0 + (x - y_0)a_1$
3. $\cdots$

In[67]:= `rationalrec[F, z, 4]`

Out[67]=

$$3 + (-1+z) \Big/ \left(-\frac{741}{580} + (-3+z) \Big/ \left(-\frac{17400}{1637} + (-4+z) \Big/ \left(\frac{140600293}{657044880} + (-5+z) \Big/ \left(\frac{2467158210960}{327086106887} + (-6+z) \Big/ \left(\frac{615564163571358245}{5995432842109296} + (-7+z) \Big/ \left(\frac{128795531544061251840}{1474536242413473730979} + (-8+z) \Big/ \left(-\frac{30717868632058486438134999}{3713653815285726138460} + (-9+z) \Big/ \left(-\frac{119452926980939870886866}{120679006462873659811540965} + (-10+z) \Big/ \left(\frac{638117589910825534294864815}{217541179828791209956} + (-11+z) \Big/ \left(\frac{17994283078074524}{5602084863015834085935} + \frac{-12+z}{-\frac{124839340988342589}{1550113748} - \frac{342578411}{71}(-13+z)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)$$

In[68]:= `% // Simplify`

Out[68]=

$$\frac{1 + z^2 + 13 z^4}{3 + z^2 + z^6}$$

Many rationals are involved in the computation

$\downarrow$

they often present numerators and denominators described by a huge number of digits

$\downarrow$

extensive use of arbitrary-precision arithmetic is needed

$\downarrow$

the computation is strongly slowed down

From rational to natural numbers and back

Finite Field: [Wang '82,Peraro '16]

$$\mathbb{Z}_n = \{0, \cdots, n-1\} \qquad n = \text{prime}$$

- From $\mathbb{Z}$ to $\mathbb{Z}_n$: $\mathbb{Z} \ni a = b \mod n \quad \Leftrightarrow \quad a = b + mn$

  Example: $15 = 1 \mod 7$ since $15 = 1 + 2 \times 7$

- From $\mathbb{Q}$ to $\mathbb{Z}_n$: $q = \frac{a}{b} \mapsto (a\, x\, (b^{-1} \mod n)) \mod n$
- From $\mathbb{Z}_n$ to $\mathbb{Q}$:

$$\frac{a}{b} = c \mod n$$

Conditions	Possible pairs (a, b)
none	Infinitely many
$a, b < n$	Finitely many
$a^2, b^2 < \frac{n}{2}$	Only one

$\longrightarrow$ n big enough $\Rightarrow$ unique inverse

$\downarrow$

Machine-size limit!

Scattering amplitudes on finite fields

Chinese remainder theorem:

$$\begin{cases} X &= x_1 \mod n_1 \\ & \vdots \\ X &= x_m \mod n_m \end{cases} \xrightarrow{\text{Combine into}} X = \widetilde{X} \mod n_1 \cdots n_m$$

- Off-shell Berends-Giele recurrence [Peraro '16]
- On-shell BCFW recurrence [A.M.]

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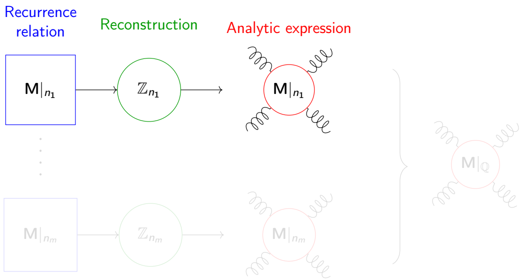

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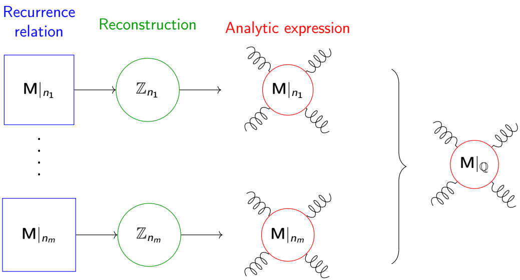

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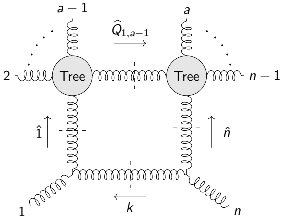
Maximum-cut

- 1 unitarity-cut = one constraint [Mastrolia, Mirabella, Ossola, Peraro '12]
- ℓ loops $\Rightarrow$ 4ℓ parameters $\Rightarrow$ at most 4ℓ constraints

$$\underline{\text{Maximum-cut:}}\ D_1 = D_2 = \cdots = D_{4\ell} = 0$$

Ex: 1-loop $\rightarrow$ quadruple cut:

$k = x_1 p_1 + x_2 p_4 + x_3 \epsilon_{14} + x_4 \epsilon_{41}$ $\quad$ $\text{span}(\{p_1, p_4\}) \perp \text{span}(\{\epsilon_{14}, \epsilon_{41}\})$



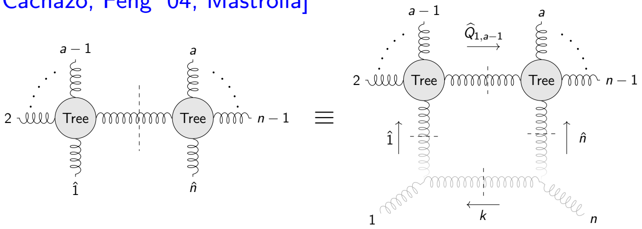
[Britto, Cachazo, Feng '04]

$\longrightarrow$

$$x_1 = 0$$
$$x_2 = 0$$
$$x_3 = -\frac{Q_{1,a-1}^2}{Q_{1,a-1}\cdot\epsilon_{1n}} \ \wedge\ x_4 = 0$$
$$x_3 = 0 \ \wedge\ x_4 = -\frac{Q_{1,a-1}^2}{Q_{1,a-1}\cdot\epsilon_{n1}}$$

BCFW vs maximum-cut

[Britto, Cachazo, Feng '04; Mastrolia]

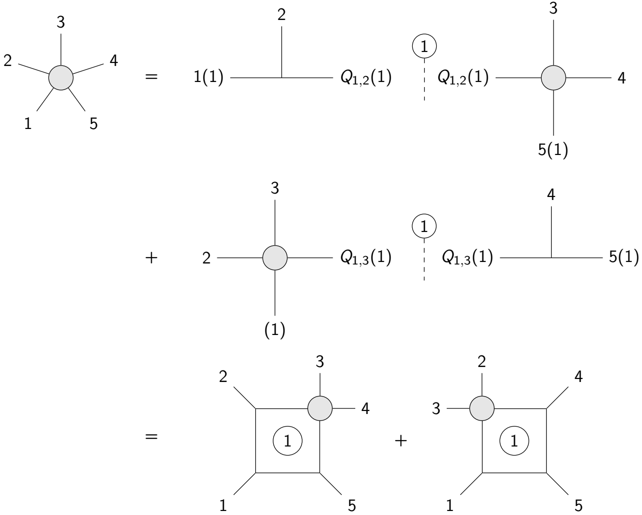


Conditions defining BCFW pole	=	Conditions defining the 4-ple cut

Systematic extension to higher-loop [A.M.]:

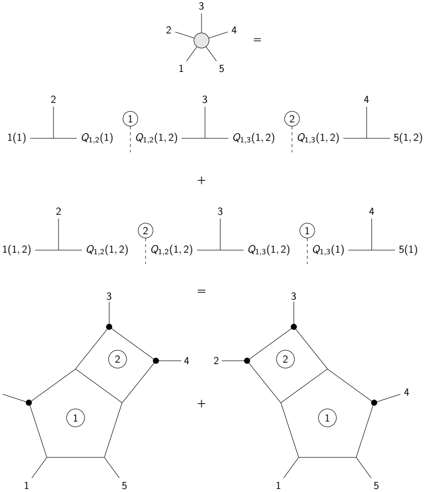
n-gluon tree-level amplitude $\xrightarrow{\text{BCFW recursion}}$ (n-3)-loop maximum-cut

5-gluons vs 2-loop



Straight lines still represent gluons and all propagators are considered to be cut.

5-gluons vs 2-loop



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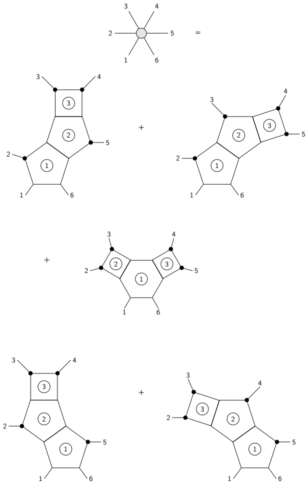
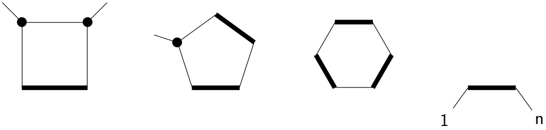
6-gluons vs 3-loop

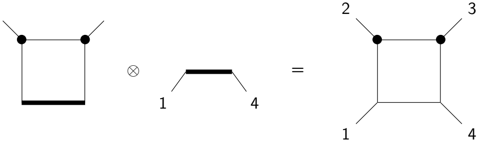


Multi-loop building blocks

Maximum-cut graphs from polygons:



Example, $n = 4$:



[Mathematica imp.]

Multi-loop building blocks

The final-factorization diagrams can be obtained directly according to the following rules:

1. Compute the number of iterations needed for complete factorization: $T = n - 3$
2. Build sets of T polygons among those represented in the previous slide that satisfy $N = 5T - 1$, with N sum of all the edges of the figures in the set. Each set must contain at least one square.
3. For each set combine the polygons among them and with the base line in all possible ways, considering that:
   - The polygons are to be connected to one another only on the thick lines
   - Polygons with the same number of edges are to be considered indistinguishable
   - Only connected diagrams are admitted

[Mathematica imp.]

Conclusions

Summary

Any function which can be implemented as a sequence of rational operations is suited for rational reconstruction over finite fields. Example: tree-level scattering amplitudes.

Outlook

- Implementation of multivariate polynomial/rational reconstruction algorithm.
- Extension to theories with fermions and scalars.
- Extension to massive particles.
- Application of the rational reconstruction to other techniques
- Thorough study of the relation between tree-level BCFW recurrence and multi-loop diagrams.