

# The Natural Structure of Scattering Amplitudes

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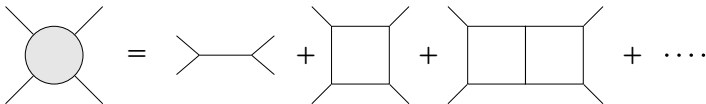
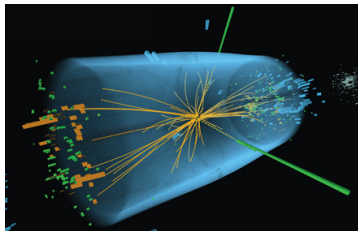
In collaboration with: P. Mastrolia and T. Peraro

# Outline

- Motivation
- Numerical tree-level amplitudes from recursive relations
- Functional reconstruction on  $\mathbb{Q}$
- Functional reconstruction on  $\mathbb{Z}_n$
- BCFW tree-level recursion and its relations to the maximum-cut
- Conclusions and outlook

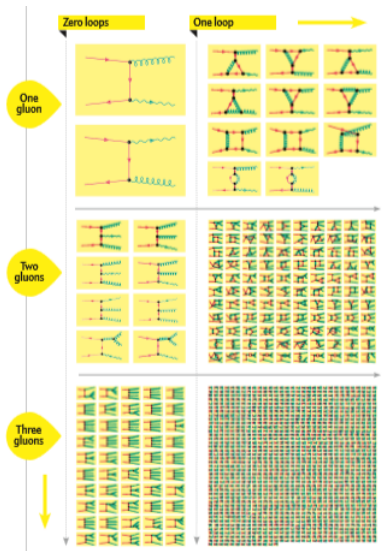
## Scattering amplitudes:

- connect theory and experiment
- confirm established models
- open the way for new physics
- infer general structure of a theory



High precision  $\xrightarrow{\text{requires}}$  loop corrections

High c.o.m. energy  $\xrightarrow{\text{leads to}}$  multi-particle interactions

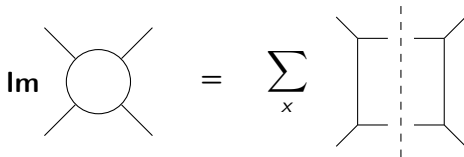


Complexity of the calculations increases quickly with the number of legs and the number of loops.

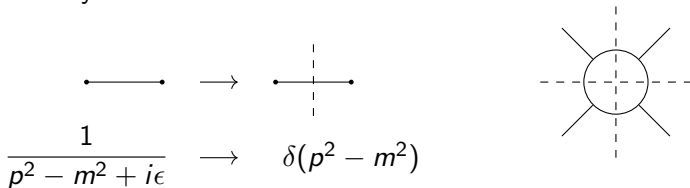
Efficient techniques needed

- Generalized unitarity  $\xrightarrow{\text{extends}}$  optical theorem:

$$\text{M}(i \rightarrow f) - \text{M}^*(i \rightarrow f) = \sum_x \text{M}(i \rightarrow x) \text{M}^*(x \rightarrow f)$$



- unitarity cut:



Tree-level amplitudes  $\xrightarrow{\text{building blocks}}$  multi-loop amplitudes


Simple  $n$ -gon amplitudes

$$iM(1^\pm, 2^\pm, \dots, n-1^\pm, n^\pm) = 0 \quad [\text{Parke, Taylor '86}]$$

$$iM(1^\mp, 2^\pm, \dots, n-1^\pm, n^\pm) = 0$$

$$iM(1^+, \dots, i^-, \dots, j^-, \dots, n^+) = i \frac{\langle ij \rangle^4}{\langle 1|2 \rangle \langle 2|3 \rangle \cdots \langle n-1|n \rangle \langle n|1 \rangle}$$

$$iM(1^-, \dots, i^+, \dots, j^+, \dots, n^-) = (-1)^n \frac{[ij]^4}{[1|2][2|3] \cdots [n-1|n][n|1]}$$



A diagram showing a tree-level amplitude. On the left, a wavy line labeled  $1^+$  enters a central circle labeled "Tree". On the right, a wavy line labeled  $8^+$  exits the circle. Above the circle, a series of dots forms an arc, representing internal propagators. To the right of the diagram, the text "= 34300 addenda = 0" is displayed.

$$1^+ \text{--- Tree ---} 8^+ = 34300 \text{ addenda} = 0$$

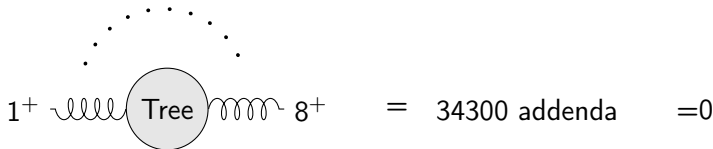
Simple  $n$ -gon amplitudes

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$$iM(1^-, \dots, i^+, \dots, j^+, \dots, n^-) = (-1)^n \frac{[ij]^4}{[1|2][2|3] \cdots [n-1|n][n|1]}$$



$$1^+ \text{---} \text{Tree} \text{---} 8^+ = 34300 \text{ addenda} = 0$$

## Solution:

Reconstruct the analytic expression of the tree-level amplitudes from numerical evaluations over finite fields



Large intermediate expressions are replaced by natural numbers.

## Important:

We are using functional reconstruction algorithms for rational functions, thus the amplitude must be written as rational function.

N.B. For simplicity, from here on we consider amplitudes involving only gluons.



# Spinor-helicity formalism

[Mangano,Parke]

Dirac equation in momentum space for massless particles:

$$i\not{p}\psi(x) = 0 \Rightarrow \begin{cases} \not{p}U_s(p) = 0 \\ \not{p}V_s(p) = 0 \end{cases} \quad s = L, R$$

- Spinors:  $U_R(p), U_L(p), \bar{U}_L(p), \bar{U}_R(p) \leftrightarrow |p\rangle, |p], \langle p|, [p|$
- Momenta:  $p^\mu = \frac{\langle p|\sigma^\mu|p\rangle}{2} \quad \not{p} = |p\rangle[p| + |p]\langle p|$
- Polarizations:  $\epsilon_R^\mu(k, r) = \frac{1}{\sqrt{2}} \frac{\langle r|\sigma^\mu|k\rangle}{\langle r|k\rangle} \quad \epsilon_L^\mu(k, r) = \frac{1}{\sqrt{2}} \frac{\langle k|\sigma^\mu|r\rangle}{[k|r]}$
- Invariants:  $s_{ij} = (p_i + p_j)^2 = \langle i|j\rangle[j|i]$

Tree-level scattering amplitudes are rational functions of the spinor components.

# Amplitude parametrization

$M(p_1, \dots, p_n) \rightarrow M(|p_1\rangle, |p_1], \dots, |p_n\rangle, |p_n])$  rational function

Suitable parametrization of  $|p\rangle, |p]$  needed: [Hodges '09; Badger '16; Peraro '15]

momentum-twistor variables

- $3n - 10$  independent variables  $\{x_1, \dots, x_{3n-10}\}$

$$3n - 10 = 4n - \underbrace{n}_{\text{onshell}} - \underbrace{4}_{\text{momentum conservation}} - \underbrace{6}_{\text{Lorentz invariance}}$$

- $M$  is rational in the  $x_i$
- $x_i$  allow arbitrary values

momentum conservation and on-shellness always satisfied

- $n = 5$ , a parametrization:

[Badger '16]

$$\begin{pmatrix} |1\rangle & |2\rangle & |3\rangle & |4\rangle & |5\rangle \\ |\mu_1\rangle & |\mu_2\rangle & |\mu_3\rangle & |\mu_4\rangle & |\mu_5\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1} + \frac{1}{x_1 x_2} & \frac{1}{x_1} + \frac{1}{x_1 x_2} + \frac{1}{x_1 x_2 x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & \frac{x_5}{x_4} - 1 \\ 0 & 0 & 0 & \frac{x_4}{x_2} & 1 \end{pmatrix}$$

$$s_{12} = x_1 \quad s_{23} = x_1 x_4 \quad s_{34} = \frac{x_1((x_5-1)x_2 x_3 + (x_3+1)x_4)}{x_2}$$

$$s_{45} = x_1 x_5 \quad s_{51} = x_1 x_3 (x_2 - x_4 + x_5)$$

- $n = 4$ , an amplitude:

$$M_4(1^+, 2^-, 3^+, 4^-) = -\frac{8(1+x_2)^4}{x_1^2 x_2^3}$$

- $n = 5$ , a parametrization:

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$$\begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1} + \frac{1}{x_1 x_2} & \frac{1}{x_1} + \frac{1}{x_1 x_2} + \frac{1}{x_1 x_2 x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & \frac{x_5}{x_4} - 1 \\ 0 & 0 & 0 & \frac{x_4}{x_2} & 1 \end{pmatrix}$$

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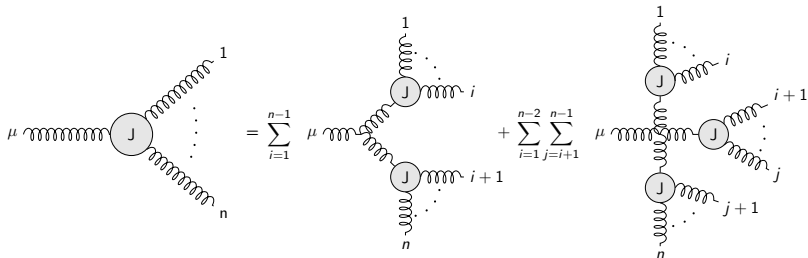
- $n = 4$ , an amplitude:

$$\mathcal{M}_4(1^+, 2^-, 3^+, 4^-) = -\frac{8(1+x_2)^4}{x_1^2 x_2^3}$$

## Off-shell recurrence relation

[Berends, Giele; Mangano, Parke]

$$M(1, \dots, n+1) \mapsto J^\mu(1, \dots, n)$$



$$M_{n+1}(1, \dots, n+1) = \lim_{P_{1,n} \rightarrow 0} \epsilon_\mu(n+1) J^\mu(1, \dots, n) P_{1,n}^2$$

# On-shell recurrence relation

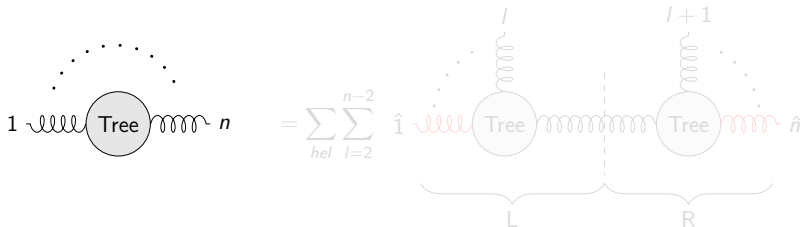
Complex  
momenta

+ residue theorem =

Amplitude  
factorization

$$p_i, p_j \rightarrow p_i(z), p_j(z) \quad z \in \mathbb{C} \quad \Rightarrow \quad M \rightarrow M(z)$$

$$\frac{1}{2\pi i} \oint_{C_\infty} \frac{iM(z)dz}{z} = \sum_{\text{all poles } \alpha} \text{Res}_{z=z_\alpha} \frac{iM(z)}{z} = 0$$



[Britto, Cachazo, Feng, Witten '05]

## On-shell recurrence relation

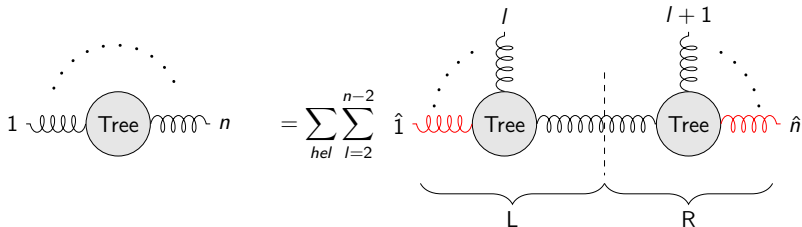
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# On-shell recurrence relation

$$= \sum_{hel} \sum_{l=2}^{n-2} \sum_{k=2}^{l-1} \sum_{j=l+1}^{n-2} \hat{1} \text{---} \underbrace{\text{Tree} \text{---} \text{Tree}}_L \text{---} \underbrace{\text{Tree} \text{---} \text{Tree}}_R \text{---} \hat{n}$$

Building blocks:

$$1^- \text{---} 2^- \text{---} 3^+ = \frac{(12)^4}{(12)(23)(31)} = 1^+ \text{---} 2^+ \text{---} 3^- = - \frac{[12]^4}{[12][23][31]}$$

# On-shell recurrence relation

$$= \sum_{hel} \sum_{l=2}^{n-2} \sum_{k=2}^{l-1} \sum_{j=l+1}^{n-2} \hat{1} \text{---} \underbrace{\text{Tree}(k) \text{---} \text{Tree}(k+1)}_L \text{---} \underbrace{\text{Tree}(j) \text{---} \text{Tree}(j+1)}_R \text{---} \hat{n}$$

⋮

$$= \sum_{hel} \sum_{\sigma} S(\sigma) \hat{1} \text{---} \overset{2}{\text{wavy}} \text{---} \sigma(1) \text{---} \overset{3}{\text{wavy}} \text{---} \sigma(2) \text{---} \sigma(n-4) \text{---} \overset{n-2}{\text{wavy}} \text{---} \sigma(n-3) \text{---} \overset{n-1}{\text{wavy}} \text{---} \hat{n}$$

Building blocks:

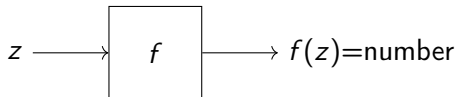
$$1^- \text{---} \overset{2^-}{\text{wavy}} \text{---} 3^+ = \frac{(12)^4}{(12)(23)(31)} \quad 1^+ \text{---} \overset{2^+}{\text{wavy}} \text{---} 3^- = -\frac{[12]^4}{[12][23][31]}$$

# On-shell recurrence relation

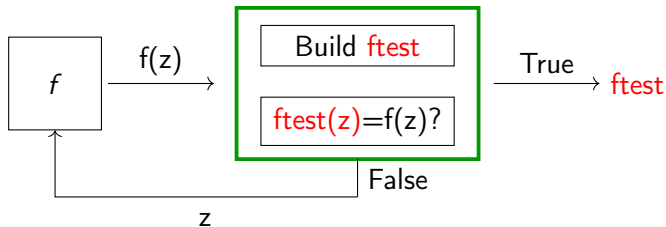
$$\begin{aligned}
 &= \sum_{hel} \sum_{l=2}^{n-2} \sum_{k=2}^{l-1} \sum_{j=l+1}^{n-2} \hat{1} \text{---} \text{Tree}_k \text{---} \text{Tree}_{k+1} \text{---} \text{Tree}_j \text{---} \text{Tree}_{j+1} \text{---} \hat{n} \\
 &\quad \underbrace{\hspace{10em}}_L \hspace{10em} \underbrace{\hspace{10em}}_R \\
 &\quad \vdots \\
 &= \sum_{hel} \sum_{\sigma} S(\sigma) \hat{1} \text{---} \text{Tree}_{\sigma(1)} \text{---} \text{Tree}_{\sigma(2)} \text{---} \dots \text{---} \text{Tree}_{\sigma(n-4)} \text{---} \text{Tree}_{\sigma(n-3)} \text{---} \text{Tree}_{\sigma(n-2)} \text{---} \text{Tree}_{\sigma(n-1)} \text{---} \hat{n} \\
 \\
 \text{Building blocks: } & \text{1}^- \text{---} \text{Tree}_{2^-} \text{---} \text{3}^+ = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \text{1}^+ \text{---} \text{Tree}_{2^+} \text{---} \text{3}^- = -\frac{[12]^4}{[12][23][31]}
 \end{aligned}$$

# Black-box interpolation

Black-box algorithm:



Reconstruction of the analytic expression of  $f$ :



# Univariate polynomials and rational functions

- Newton's polynomial form:

$$f(z) = a_0 + (z - y_0) \left( a_1 + (z - y_1) (\cdots + (z - y_{r-1}) a_r) \right)$$

- Thiele's interpolation formula for rational functions:

$$f(z) = a_0 + \frac{z - y_0}{a_1 + \frac{z - y_1}{a_2 + \frac{z - y_2}{\vdots \frac{z - y_{N-1}}{a_{N-1} + \frac{z - y_{N-1}}{a_N}}}}$$

Example, a polynomial:

- 1  $f(y_0) \xrightarrow{\text{Yields}} a_0 \Rightarrow f_{\text{test}}(x) = a_0$
- 2  $f(y_1) \xrightarrow{\text{Yields}} a_1 \Rightarrow f_{\text{test}}(x) = a_0 + (x - y_0)a_1$
- 3 ...

```
In[67]:= rationalrec[F, z, 4]
```

```
Out[67]:= 3 + (-1 + z) /
```

$$\left( -\frac{741}{580} + (-3 + z) / \left( -\frac{17400}{1637} + (-4 + z) / \left( \frac{140600293}{657044880} + (-5 + z) / \left( \frac{2467158210960}{327086106887} + \right. \right. \right. \right. \\ \left. \left. \left. (-6 + z) / \left( \frac{615564163571358245}{5995432842109296} + (-7 + z) / \left( \frac{128795531544061251840}{1474536242413473730979} + \right. \right. \right. \right. \\ \left. \left. \left. (-8 + z) / \left( -\frac{30717868632058486438134999}{3713653815285726138460} + \right. \right. \right. \right. \\ \left. \left. \left. (-9 + z) / \left( -\frac{119452926980939870886866}{120679006462873659811540965} + \right. \right. \right. \right. \\ \left. \left. \left. (-10 + z) / \left( \frac{638117589910825534294864815}{217541179828791209956} + \right. \right. \right. \right. \\ \left. \left. \left. (-11 + z) / \left( \frac{17994283078074524}{5602084863015834085935} + \right. \right. \right. \right. \\ \left. \left. \left. \frac{-12 + z}{1550113748} - \frac{342578411}{71} (-13 + z) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right)$$

```
In[68]:= % // Simplify
```

```
Out[68]:= \frac{1 + z^2 + 13 z^4}{3 + z^2 + z^6}
```

Many rationals are involved in the computation



they often present numerators and denominators described by a huge number of digits



extensive use of arbitrary-precision arithmetic is needed



the computation is strongly slowed down

# From rational to natural numbers and back

Finite Field:

[Wang '82, Peraro '16]

$$\mathbb{Z}_n = \{0, \dots, n-1\} \quad n = \text{prime}$$

- From  $\mathbb{Z}$  to  $\mathbb{Z}_n$ :  $\mathbb{Z} \ni a = b \pmod n \iff a = b + mn$

Example:  $15 = 1 \pmod 7$  since  $15 = 1 + 2 \times 7$

- From  $\mathbb{Q}$  to  $\mathbb{Z}_n$ :  $q = \frac{a}{b} \mapsto (a \times (b^{-1} \pmod n)) \pmod n$
- From  $\mathbb{Z}_n$  to  $\mathbb{Q}$ :

$$\frac{a}{b} = c \pmod n$$

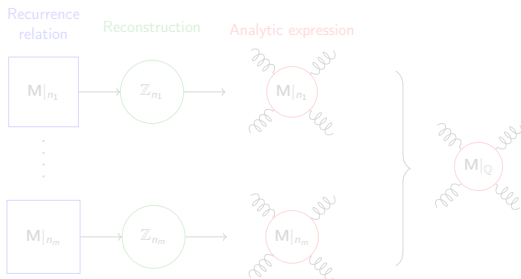
Conditions	Possible pairs $(a, b)$	
none	Infinitely many	$\longrightarrow n$ big enough $\Rightarrow$ unique inverse $\downarrow$ Machine-size limit!
$a, b < n$	Finitely many	
$a^2, b^2 < \frac{n}{2}$	<b>Only one</b>	



# Scattering amplitudes on finite fields

Chinese remainder theorem:

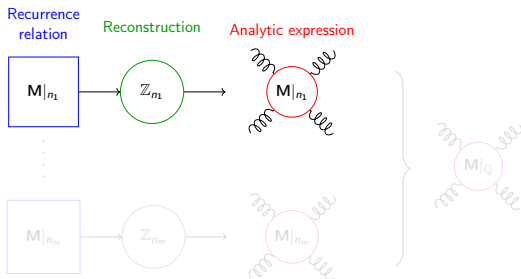
$$\begin{cases} X = x_1 \pmod{n_1} \\ \vdots \\ X = x_m \pmod{n_m} \end{cases} \xrightarrow{\text{Combine into}} X = \tilde{X} \pmod{n_1 \cdots n_m}$$



- Off-shell Berends-Giele recurrence [Peraro '16]
- On-shell BCFW recurrence [A.M.]

# Scattering amplitudes on finite fields

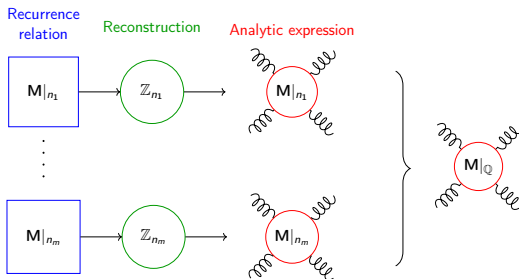
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# Scattering amplitudes on finite fields

$$\text{Chinese remainder theorem: } \begin{cases} X = x_1 \pmod{n_1} \\ \vdots \\ X = x_m \pmod{n_m} \end{cases} \xrightarrow{\text{Combine into}} X = \tilde{X} \pmod{n_1 \cdots n_m}$$



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# Maximum-cut

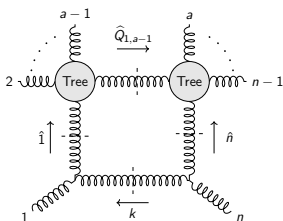
- 1 unitarity-cut = one constraint [Mastrolia, Mirabella, Ossola, Peraro '12]
- $\ell$  loops  $\Rightarrow 4\ell$  parameters  $\Rightarrow$  at most  $4\ell$  constraints

$$\text{Maximum-cut: } D_1 = D_2 = \dots = D_{4\ell} = 0$$

Ex: 1-loop  $\rightarrow$  quadruple cut:

$$k = x_1 p_1 + x_2 p_4 + x_3 \epsilon_{14} + x_4 \epsilon_{41}$$

$$\text{span}(\{p_1, p_4\}) \perp \text{span}(\{\epsilon_{14}, \epsilon_{41}\})$$

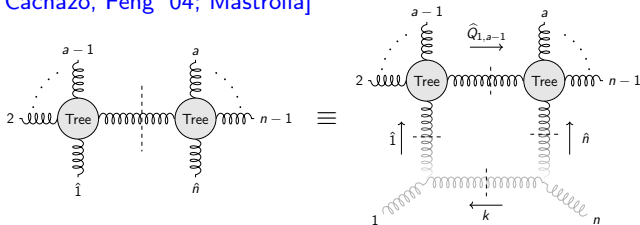


$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= -\frac{Q_{1,a-1}^2}{Q_{1,a-1} \cdot \epsilon_{1n}} \wedge x_4 = 0 \\ x_3 &= 0 \wedge x_4 = -\frac{Q_{1,a-1}^2}{Q_{1,a-1} \cdot \epsilon_{n1}} \end{aligned}$$

[Britto, Cachazo, Feng '04]

## BCFW vs maximum-cut

[Britto, Cachazo, Feng '04; Mastrolia]



Conditions defining  
BCFW pole

=

Conditions defining  
the 4-ple cut

Systematic extension to higher-loop [A.M.]:

$n$ -gluon tree-level amplitude  $\xrightarrow{\text{BCFW recursion}}$   $(n-3)$ -loop maximum-cut

## 5-gluons vs 2-loop

The diagram illustrates the functional reconstruction of a 5-gluon tree-level amplitude. On the left, a central grey circle is connected to five external lines labeled 1, 2, 3, 4, and 5. This is equated to a sum of two terms:

- The first term is a sum of two diagrams:
  - A diagram with a horizontal line from 1(1) to a vertex, a vertical line from that vertex to 2, and a horizontal line from the vertex to a grey circle labeled  $Q_{1,2}(1)$ .
  - A diagram with a vertical dashed line from a circled 1 to a vertex, a horizontal line from that vertex to a grey circle labeled  $Q_{1,2}(1)$ , and a vertical line from the grey circle to 5(1).
- The second term is a sum of two diagrams:
  - A diagram with a vertical line from 3 to a grey circle, a horizontal line from the grey circle to a vertex, a horizontal line from the vertex to a grey circle labeled  $Q_{1,3}(1)$ , and a vertical line from the vertex to (1).
  - A diagram with a vertical dashed line from a circled 1 to a vertex, a horizontal line from that vertex to a grey circle labeled  $Q_{1,3}(1)$ , and a vertical line from the grey circle to a vertex, which is then connected to 4 and 5(1).

The final result is the sum of two 2-loop diagrams, each featuring a square loop with a circled 1 in the center and a grey circle on one of the top edges:

- The first 2-loop diagram has external lines 1, 2, 3, 4, and 5.
- The second 2-loop diagram has external lines 1, 2, 3, 4, and 5.

Straight lines still represent gluons and all propagators are considered to be cut.

## 5-gluons vs 2-loop

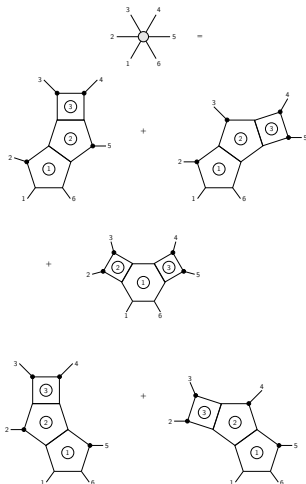
The diagram illustrates the reconstruction of a 5-gluon tree-level amplitude as a sum of two 2-loop diagrams. At the top, a tree-level vertex with five external legs labeled 1, 2, 3, 4, and 5 is shown. Below it, the amplitude is expressed as a sum of two terms:

- The first term consists of three tree-level diagrams connected by dashed lines representing cut propagators. The first diagram has legs 1(1) and 2, with a cut propagator  $Q_{1,2}(1)$  between them. The second diagram has legs 3 and 4, with a cut propagator  $Q_{1,3}(1,2)$  between them. The third diagram has legs 5(1,2) and 4, with a cut propagator  $Q_{1,3}(1,2)$  between them. The cut propagators are labeled with circled numbers 1 and 2.
- The second term is similar, but the cut propagators are  $Q_{1,2}(1,2)$  and  $Q_{1,3}(1)$ , and the external legs are labeled 1(1,2), 3, 4, and 5(1).

The sum of these two terms is equated to the sum of two 2-loop diagrams. Each 2-loop diagram is a pentagon with two internal lines forming a loop. The first 2-loop diagram has legs 1, 2, 3, 4, and 5, with a cut propagator labeled 1 and another labeled 2. The second 2-loop diagram is similar, with legs 1, 2, 3, 4, and 5, and cut propagators labeled 1 and 2.

Straight lines still represent gluons and all propagators are considered to be cut.

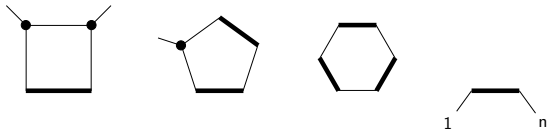
# 6-gluons vs 3-loop



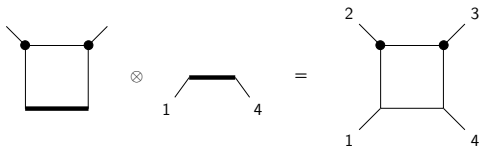


# Multi-loop building blocks

Maximum-cut graphs from polygons:



Example,  $n = 4$ :



# Multi-loop building blocks

The final-factorization diagrams can be obtained directly according to the following rules:

- 1 Compute the number of iterations needed for complete factorization:  $T = n - 3$
- 2 Build sets of  $T$  polygons among those represented in the previous slide that satisfy  $N = 5T - 1$ , with  $N$  sum of all the edges of the figures in the set. Each set must contain at least one square.
- 3 For each set combine the polygons among them and with the base line in all possible ways, considering that:
  - The polygons are to be connected to one another only on the thick lines
  - Polygons with the same number of edges are to be considered indistinguishable
  - Only connected diagrams are admitted

# Conclusions

## Summary

Any function which can be implemented as a sequence of rational operations is suited for rational reconstruction over finite fields.  
Example: tree-level scattering amplitudes.

## Outlook

- Implementation of multivariate polynomial/rational reconstruction algorithm.
- Extension to theories with fermions and scalars.
- Extension to massive particles.
- Application of the rational reconstruction to other techniques
- Thorough study of the relation between tree-level BCFW recurrence and multi-loop diagrams.