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Master equations for collision models

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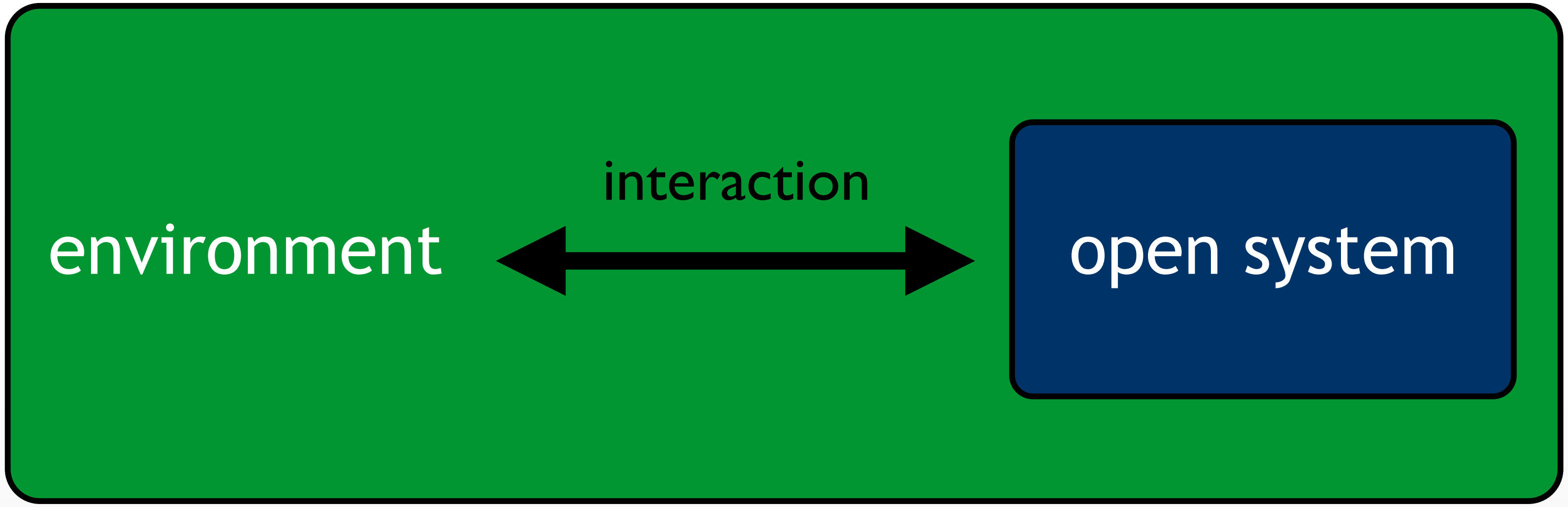
New frontiers in testing quantum mechanics from underground to the space
Frascati, November 2017

Outline

- Reduced dynamics
- Collision models
- Non-Markovian master equations

Open quantum systems



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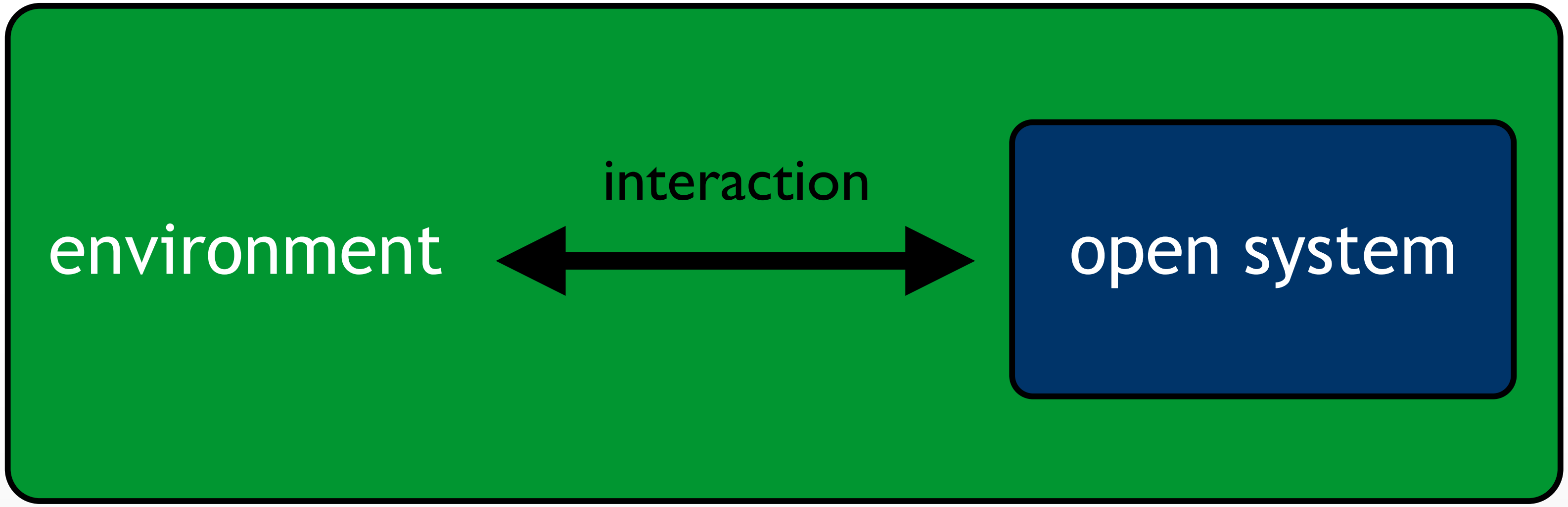
Open quantum systems



Bipartite setting

$$H = H_S + H_E + H_I$$

$$H \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_E) \qquad \rho_{SE} \in \mathcal{T}(\mathcal{H}_S \otimes \mathcal{H}_E)$$

Reduced dynamics

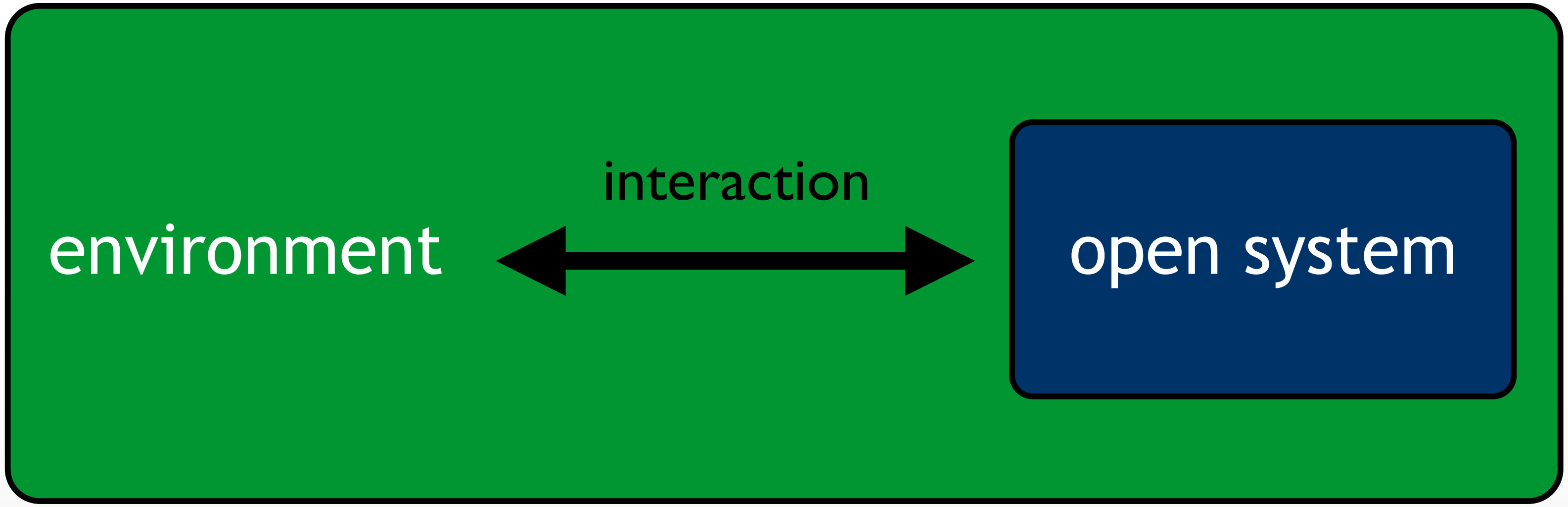
$$\rho_S(0) \mapsto \rho_S(t) = \Phi(t)\rho_S(0)$$

Correlations

$$\rho_{SE}(t) \neq \rho_S(t) \otimes \rho_E(t)$$

[Davies, 1976; Alicki & Lendi, 1987; Breuer & Petruccione, 2002; Rivas & Huelga, 2012]

Open quantum systems



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Reduced quantum dynamical map

$$\rho_S(0) \otimes \rho_E \xrightarrow{\text{unitary}} \rho_{SE}(t) = U(t)\rho_S(0) \otimes \rho_E U(t)^\dagger$$

Evolution equation

$$\begin{cases} \frac{d}{dt}\rho_{SE}(t) = -\frac{i}{\hbar}[H, \rho_{SE}(t)] \\ \rho_{SE}(0) \end{cases}$$

Reduced dynamics

Reduced quantum dynamical map

$$
\begin{array}{ccc}
\rho_S(0) \otimes \rho_E & \xrightarrow{\text{unitary}} & \rho_{SE}(t) = U(t)\rho_S(0) \otimes \rho_E U(t)^\dagger \\
\Big\downarrow \mathrm{Tr}_E & & \Big\downarrow \mathrm{Tr}_E \\
\rho_S(0) & \xrightarrow{\Phi(t)} & \rho_S(t)
\end{array}
$$

Evolution equation

$$
\begin{cases} \frac{d}{dt}\rho_{SE}(t) = -\frac{i}{\hbar}[H, \rho_{SE}(t)] \\ \rho_{SE}(0) \end{cases} \Longrightarrow \begin{cases} \frac{d}{dt}\rho_S(t) = ? \\ \rho_S(0) \end{cases}
$$

Open quantum system dynamics

Quantum Markov process

$$\Phi(t)\Phi(s) = \Phi(t+s) \qquad t, s \geq 0$$

leading to

$$\Phi(t) = \exp(\mathcal{L}t)$$

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}\rho_S(t)$$

break reversibility but retain CP

$$\mathcal{L}\rho = -i[H, \rho] + \sum_k \gamma_k \left[A_k \rho A_k^\dagger - \frac{1}{2}\{A_k^\dagger A_k, \rho\} \right]$$

GKLS generator also known as Lindblad form
Workhorse for open quantum systems by 40 years now

[Kossakowski, RMP 1972; Gorini, Kossakowski & Sudarshan, JMP 1976; Lindblad, CMP 1976]

Quantum divisibility and contractivity

Composition law

$$\Phi(t,\tau)\Phi(\tau,s) = \Phi(t,s) \qquad t \geq \tau \geq s \geq 0$$

divisibility property of quantum dynamical map

CP-divisibility in that $\Phi(t,s)$ is CP $\forall\, t \geq s \geq 0$

[Breuer, Laine & Piilo, PRL 2009; Rivas, Huelga & Plenio, PRL 2010]
[Luo & al., PRA 2012; Lorenzo & al., PRA 2013; Wolf et al. PRL 2008]

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Contractivity under trace distance

$$D(\rho_1(t+s),\rho_2(t+s)) \leq D(\rho_1(t),\rho_2(t))$$

monotonic decrease of trace distance between different initial states with elapsing time

[Breuer, Laine & Piilo, PRL 2009; Rivas, Huelga & Plenio, PRL 2010]
[Luo & al., PRA 2012; Lorenzo & al., PRA 2013; Wolf et al. PRL 2008]

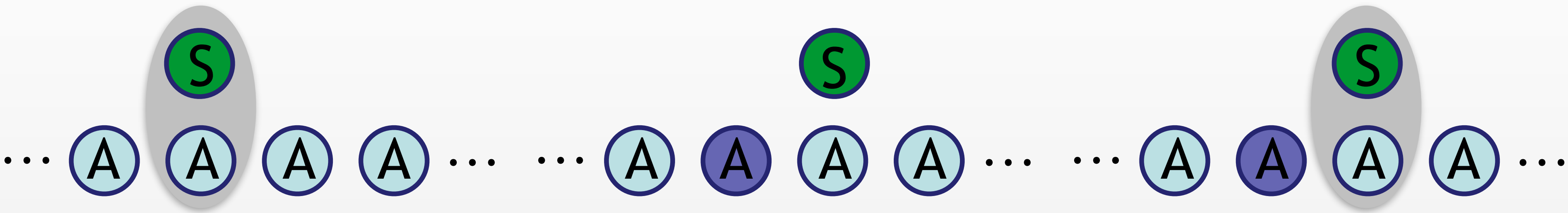
Outline

- Reduced dynamics
- **Collision models**
- Non-Markovian master equations

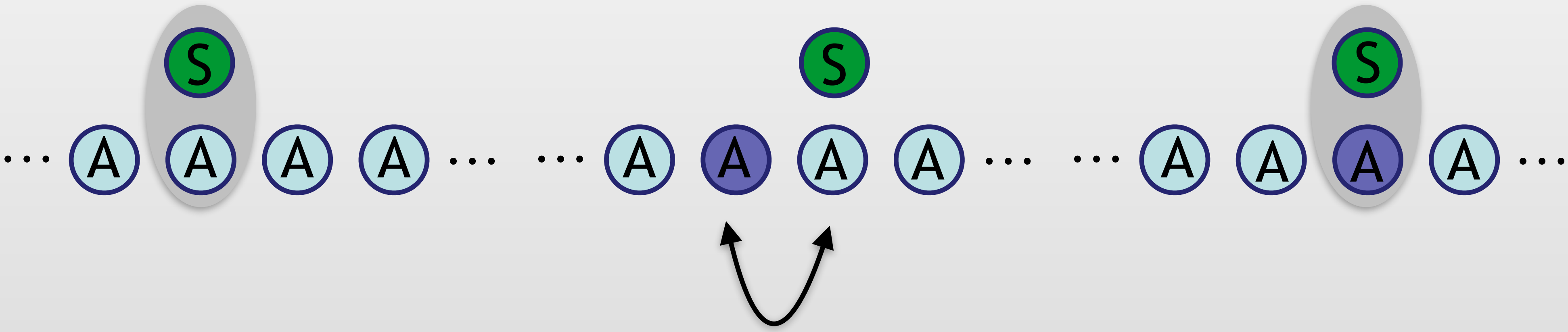
Collision models

Model environment as collection of ancillas
&
interaction as sequence of collisions

In-dependent



Inter-dependent



Collision models

In-dependent collisions → Markov

System collides with ancillas
identical, initially independent and non interacting

$$\Phi(n)[\rho_S] = \mathcal{E}^n[\rho_S]$$

with

$$\mathcal{E}[\rho_S] = \mathrm{Tr}_A \,\mathcal{U}_{SA}[\rho_S \otimes \rho_A]$$

so that

$$\Phi(n+m) = \Phi(n)\Phi(m)$$

Continuous time limit $\Longrightarrow$ Lindblad master equation

[Rybar & al., JPB 2012]

Collision models

Inter-dependent collisions
& additional layer of ancillas
& step dependent swap probability

→ non Markov

System interacts with ancillas both directly
and via memory
memory itself interacts with ancillas

Continuous time limit ⟹
general memory kernel master equation

[Lorenzo, Ciccarello, Palma, PRA 2016; Lorenzo, Ciccarello, Palma, Vacchini, OSID 2017]

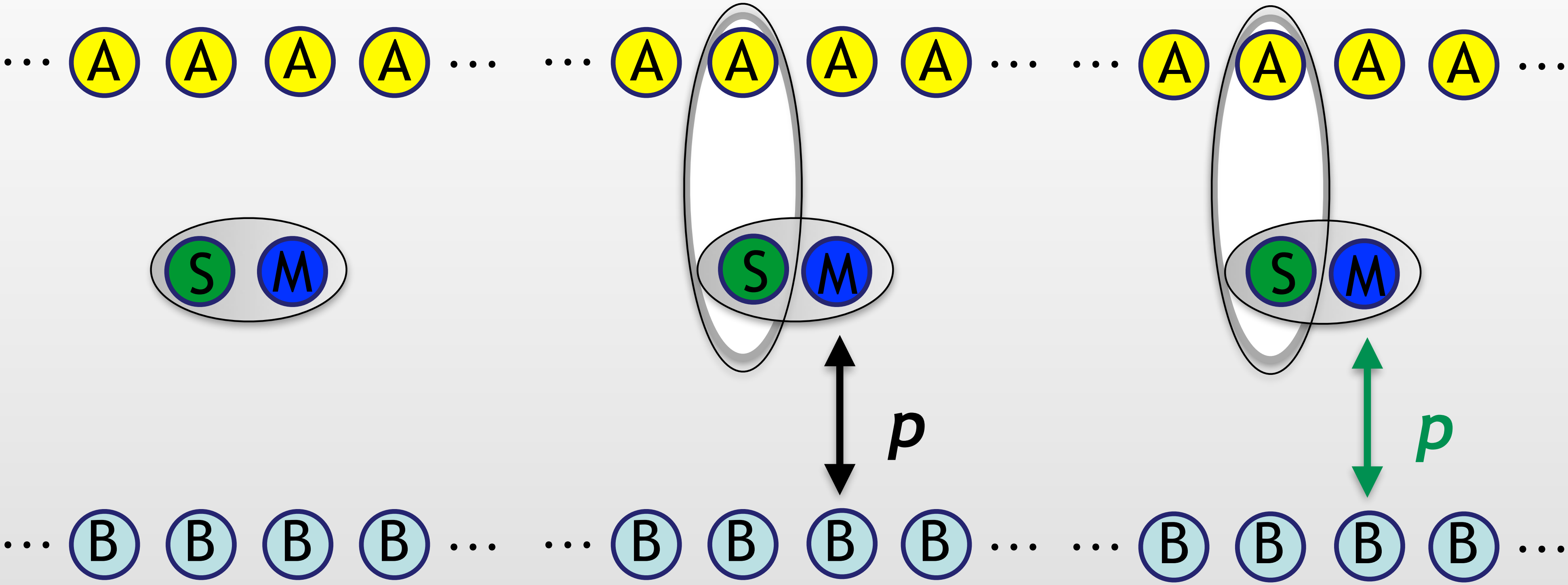
Collision models

Model environment as double layer of ancillas

System interacts with memory M

Memory swaps with B ancillas

System interacts with A ancillas directly



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Model environment as double layer of ancillas

System interacts with memory

$$\rho_S \to \mathcal{G}[\rho_S] = \mathrm{Tr}_M \mathcal{U}_{SM}[\rho_{SM}]$$

Memory swaps with B ancillas

$$\rho_{SM} \to p_n \rho_{SM} + (1 - p_n)\rho_{SB}$$

System interacts with A ancillas

$$\rho_S \to \mathcal{E}[\rho_S] = \mathrm{Tr}_A \mathcal{V}_{SA}[\rho_{SA}]$$

Probabilistic swap to be connected to jump distribution

$$p \to f$$

System interacts with memory again

$$\rho_S \to \mathcal{F}[\rho_S] = \mathrm{Tr}_B \mathcal{U}_{SB}[\rho_{SB}]$$

Collision models

Evolution of statistical operator

System+memory transformed stepwise

$$
\begin{aligned}
\rho^{(1)}_{SM} &= \mathcal{U}_{SM}[\rho^{(0)}_{SM}] \\
\rho^{(2)}_{SM} &= p_1\, \mathcal{U}_{SM}[\rho^{(1)}_{SM}] + q_1\, \mathcal{U}_{SM}\tilde{\mathcal{E}}[\rho^{(1)}_{SM}] \\
\rho^{(3)}_{SM} &= p_2\, \mathcal{U}_{SM}[\rho^{(2)}_{SM}] + q_2\, \mathcal{U}_{SM}\widetilde{\mathcal{Z}}[\rho^{(2)}_{SM}] \\
&= p_2 p_1 \mathcal{U}^3_{SM}[\rho^{(0)}_{SM}] + q_2 q_1 \mathcal{U}_{SM}\widetilde{\mathcal{E}}\mathcal{U}_{SM}\widetilde{\mathcal{E}}\mathcal{U}_{SM}[\rho^{(0)}_{SM}] \\
&\quad + (p_2 q_1 \mathcal{U}^2_{SM}\widetilde{\mathcal{E}}\mathcal{U}_{SM} + q_2 p_1 \mathcal{U}_{SM}\widetilde{\mathcal{E}}\mathcal{U}^2_{SM})[\rho^{(0)}_{SM}]
\end{aligned}
$$

$\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots$

with

$$
\tilde{\mathcal{E}}[\rho_{SM}] = \mathcal{E}[\mathrm{Tr}_M \rho_{SM}] \otimes \eta_M
$$

partial trace + continuous limit
lead to closed evolution equation

Outline

- Reduced dynamics
- Collision models
- **Non-Markovian master equations**

Memory kernels

Reconsider Lindblad equation in view of "trajectories"

$$\mathcal{L}\rho = -i[H,\rho] + \sum_k \gamma_k \left[L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right]$$

$$\mathcal{L}\rho = R\rho + \rho R^\dagger + \sum_k \gamma_k L_k \rho L_k^\dagger \qquad R = -iH - \frac{1}{2}\sum_k \gamma_k L_k^\dagger L_k$$

$$\mathcal{R}(t)\rho = \exp(t\,R)\rho\exp(t\,R^\dagger) \qquad \mathcal{J}\rho = \sum_k \gamma_k L_k \rho L_k^\dagger$$

Memory kernels

Reconsider Lindblad equation in view of "trajectories"

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$$\mathcal{R}(t)\rho = \exp(t\,R)\rho\exp(t\,R^\dagger) \qquad \mathcal{J}\rho = \sum_k \gamma_k L_k \rho L_k^\dagger$$

Dyson expansion of exact solution

$$\Phi(t)\rho = \rho(t) = \mathcal{R}(t)\rho(0) + \sum_{n=1}^{\infty} \int_0^t dt_n \ldots \int_0^{t_2} dt_1$$

$$\times \mathcal{R}(t-t_n)\mathcal{J}\mathcal{R}(t_n - t_{n-1}) \ldots \mathcal{J}\mathcal{R}(t_1)\rho(0)$$

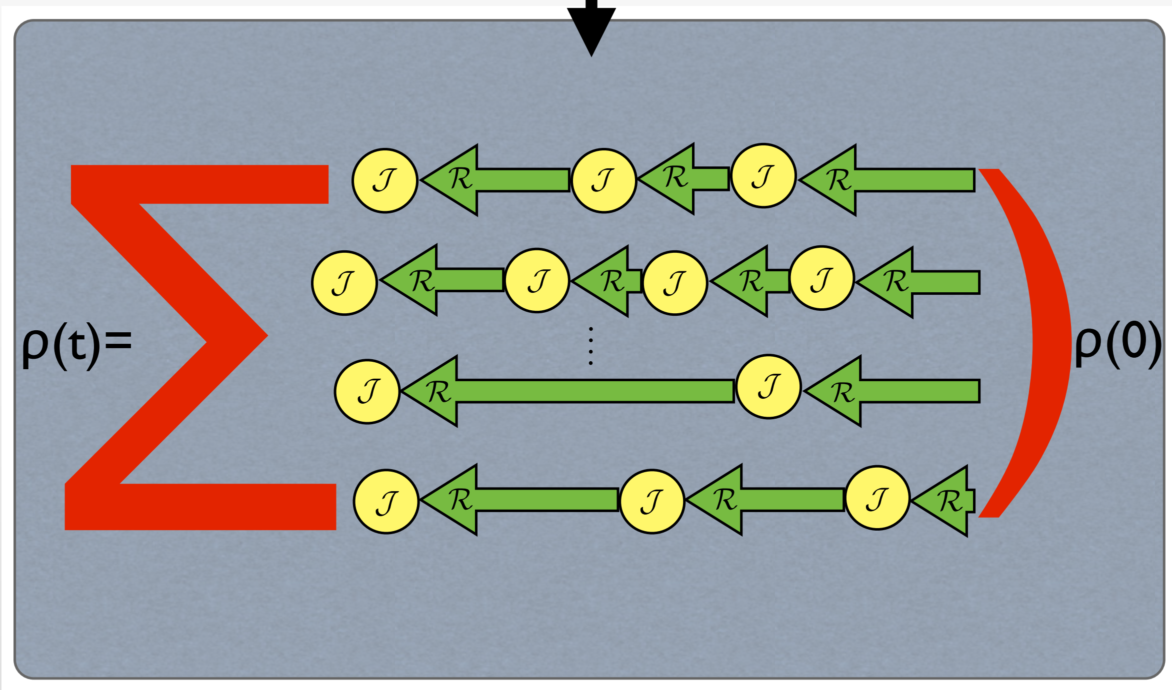
Memory kernels

$$\frac{d}{dt}\rho = -i[H,\rho] + \sum_k \gamma_k \left[L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right]$$

↓

$$\Phi(t)\rho = \mathcal{R}(t)\rho + \sum_{n=1}^{\infty} \int_0^t dt_n \ldots \int_0^{t_2} dt_1 \mathcal{R}(t-t_n)\ldots\mathcal{J}\mathcal{R}(t_1)\rho$$

↓



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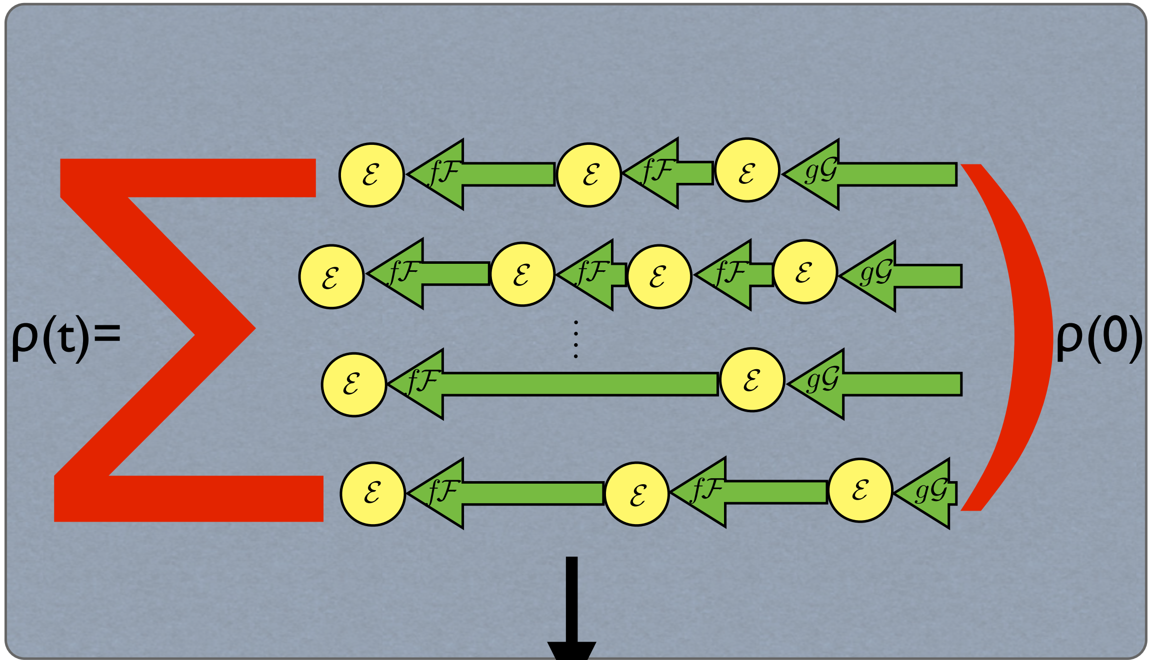
Memory kernels



$$\Phi(t)\rho = g(t)\mathcal{G}(t)\rho + \sum_{n=1}^{\infty} \int_0^t dt_n \ldots \int_0^{t_2} dt_1$$

$$\times f(t-t_n)\mathcal{F}(t-t_n)\mathcal{E}\ldots\mathcal{E}g(t_1)\mathcal{G}(t_1)\rho$$

$$\frac{d}{dt}\rho(t) = ?$$

[Budini, PRA 2004, B.V.,PRA(R) 2013, Chuscinski & Kossakowski, PRA 2016]

Memory kernels

$$\Phi(t)\rho = g(t)\mathcal{G}(t)\rho + \sum_{n=1}^{\infty} \int_0^t dt_n \ldots \int_0^{t_2} dt_1$$

$$\times f(t-t_n)\mathcal{F}(t-t_n)\mathcal{E}\ldots\mathcal{E}g(t_1)\mathcal{G}(t_1)\rho$$

$$\Downarrow$$

$$\frac{d}{dt}\rho(t) = \int_0^t d\tau \mathcal{K}(t-\tau)\rho(\tau)$$

memory kernel master equation

$$\widehat{\mathcal{K}}(u) = \frac{1}{\widehat{g\mathcal{G}}(u)}\widehat{f\mathcal{F}}(u)\mathcal{E} - \left(\frac{1}{\widehat{g\mathcal{G}}(u)} - u\right)$$

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Memory kernels

Connection to classical memory kernel

$$\frac{d}{dt}T_{nm}(t) = \int_0^t d\tau \sum_k \left[W_{nk}(\tau)T_{km}(t-\tau) - W_{kn}(\tau)T_{nm}(t-\tau) \right]$$

$$\hat{W}_{nk}(u) = \pi_{nk}\hat{f}_k(u)/\hat{g}_k(u)$$

Natural correspondence

waiting time distribution $\Longrightarrow$ collection of time evolutions

stochastic matrix $\Longrightarrow$ CPT map

$\pi \to \mathcal{E}$	CPT map
$f(t) \to f(t)\mathcal{F}(t)$	$\mathcal{F}(t)$CPT maps
$g(t) \to g(t)\mathcal{G}(t)$	$\mathcal{G}(t)$CPT maps, s.t. $\mathcal{G}(0) = \mathbf{1}$

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Relevance of operator ordering

$$u\hat{\rho}(u) - \rho(0) = \left\{ \mathcal{O}\left[\pi \frac{\hat{f}(u)}{\hat{g}(u)}\right] - \mathcal{O}\left[\frac{1}{\hat{g}(u)} - u\right] \right\} \hat{\rho}(u)$$

Different possible choices of operator kernel leading to different possible dynamics

$$\mathcal{O}\left[\pi \frac{\hat{f}(u)}{\hat{g}(u)}\right] \to \frac{1}{\widehat{g\mathcal{G}}(u)} \widehat{f\mathcal{F}}(u)\mathcal{E}$$

$$\downarrow$$

$$\widehat{\mathcal{K}}(u) = \frac{1}{\widehat{g\mathcal{G}}(u)} \widehat{f\mathcal{F}}(u)\mathcal{E} - \left(\frac{1}{\widehat{g\mathcal{G}}(u)} - u\right)$$

[B.V.,PRL 2016]

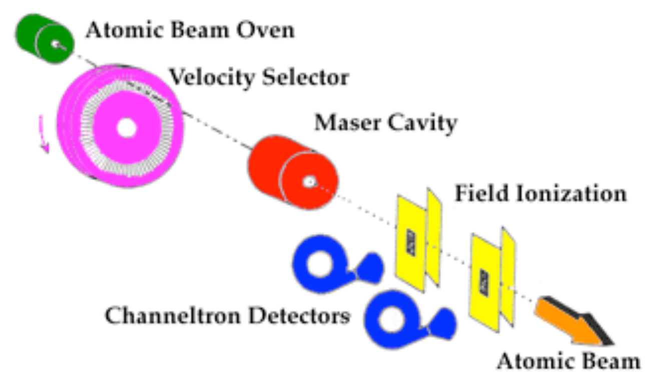
Memory kernels

Equations appearing in collision model & micromaser dynamics

$$\mathcal{O}\left[\pi\frac{\hat{f}(u)}{\hat{g}(u)}\right] \to \frac{1}{\widehat{g\mathcal{G}}(u)}\widehat{f\mathcal{F}}(u)\mathcal{E} \longrightarrow$$

versus

$$\mathcal{O}\left[\pi\frac{\hat{f}(u)}{\hat{g}(u)}\right] \to \mathcal{E}\widehat{f\mathcal{F}}(u)\frac{1}{\widehat{g\mathcal{G}}(u)} \longrightarrow$$



[Cresser, PRA 1992; Herzog, PRA 1995; Cresser, QS Optics 1996; B.V.,PRL 2016]

Conclusions & outlook

- Reduced open quantum system dynamics
- Collision model formulation of dynamics
- Continuous limit and memory kernel master equation

Conclusions & outlook

- Reduced open quantum system dynamics
- Collision model formulation of dynamics
- Continuous limit and memory kernel master equation

- What is the generality of collision models?
- What is their use in describing memory effects?
- What is their use in describing thermodynamic effects?

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mindpower

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other kind of power

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