The Thirring quantum cellular automaton

Authors:

Alessandro Bisio, Giacomo Mauro D'Ariano, Paolo Perinotti, and AT

arXiv:1711.03920

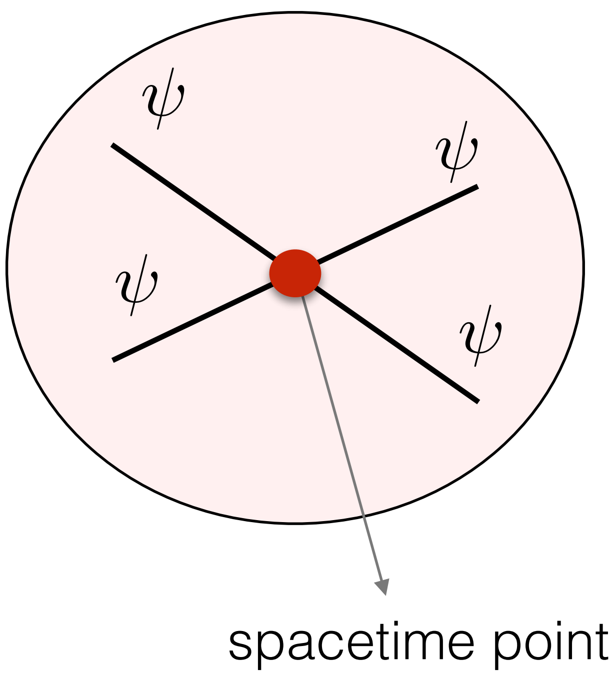
Alessandro Tosini, QUIT group, Pavia University

Frascati, November 30, 2017

quantum information theory group

John Templeton Foundation

Four fermion interaction: 4th power of fermion fields as interaction



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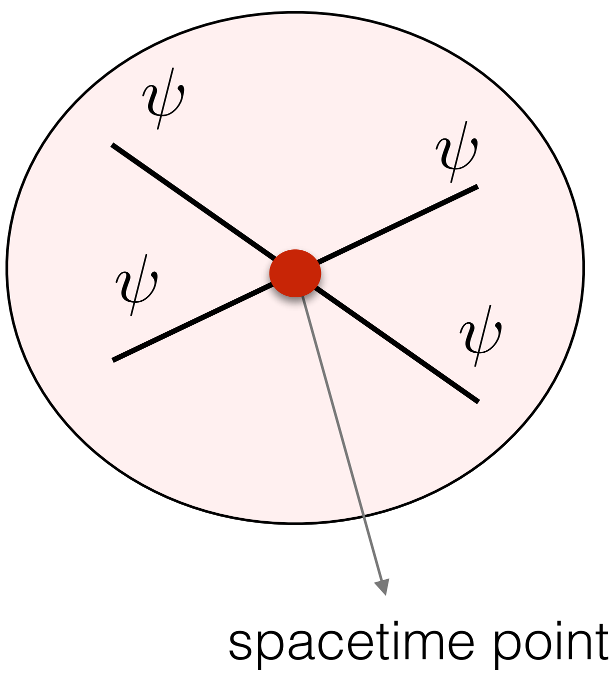

............non relativistic example

Hubbard model: 1968 integrable model in 1+1 dimension

Hubbard, J., Proceedings of the Royal Society of London. **276** (1365): 238 (1963)
E. H. Lieb and F. Y. Wu, Physical Review Letters **20**, 1445 (1968)

interaction

$$H = \text{kinetic term} + g\sum_x \psi^\dagger_{x\uparrow}\psi_{x\uparrow}\psi^\dagger_{x\downarrow}\psi_{x\downarrow}$$

switches on
when two fermions
are at the same site *x*

…………examples of relativistic QFTs with 4th power of fermion fields as interaction

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kinetic term

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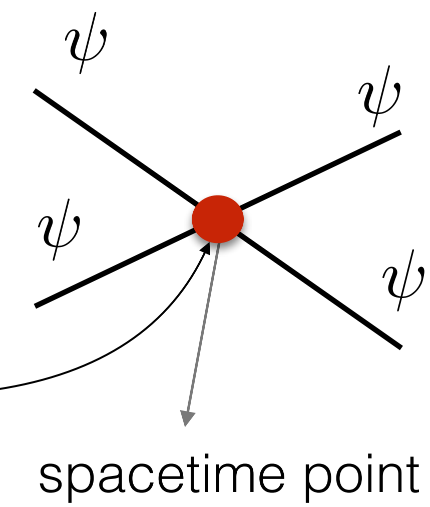
$$\mathcal{L} = \bar{\psi}(\partial\!\!\!/ - m)\psi + \sum_{\alpha} \frac{g_{\alpha}}{2}(\bar{\psi}\Gamma_{\alpha}\psi)^2$$



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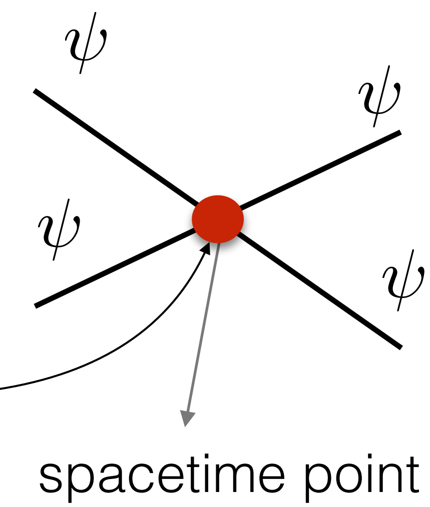


Thirring: 1958, integrable model in 1+1 dim

$$\Gamma_\alpha = \gamma_\mu$$

W. E. Thirring, Annals of Physics **3**, 91 (1958)
S. Coleman, Phys. Rev. D **11**, 2088 (1975)

Nambu & Jona-Lasinio: 1961, dynamical mass generation in 3+1 dim

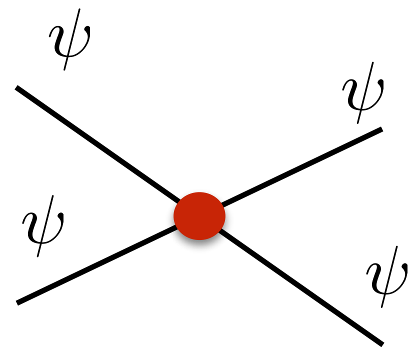
$$\Gamma_1 = I$$
$$\Gamma_2 = \gamma_5$$

Y. Nambu, and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961)
Y. Nambu, and G. Jona-Lasinio, Phys. Rev. **124**, 246 (1961)

Gross-Neveu: 1974, asymptotic freedom, dynamical symmetry breaking (1+1 dim)

$$\Gamma_\alpha = I$$

D. J. Gross, and A. Neveu, Phys. Rev. D. **10,** 3235 (1974)



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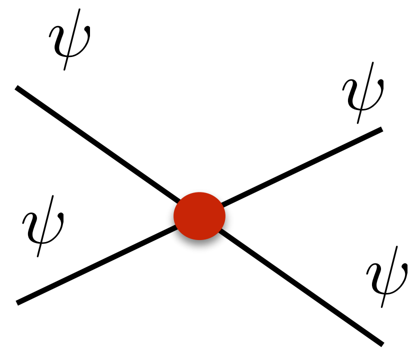


Four fermion interaction

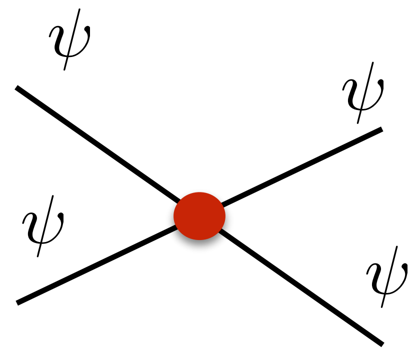
In the literature

Extensively studied in Lagrangian/Hamiltonian models:

1) continuous spacetime

2) on the lattice

TIME is continuous



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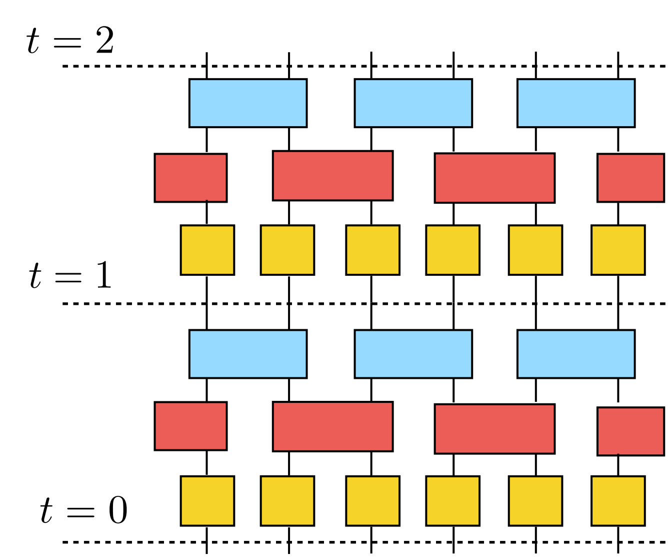
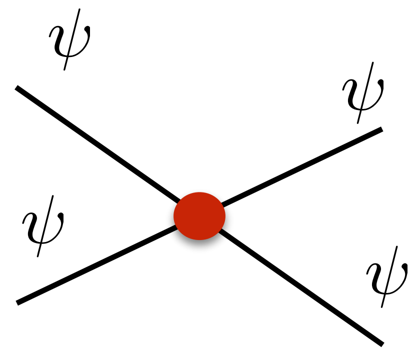


$$U^t = \underbrace{U \cdots U U}_{t \text{ times}}$$

discrete TIME



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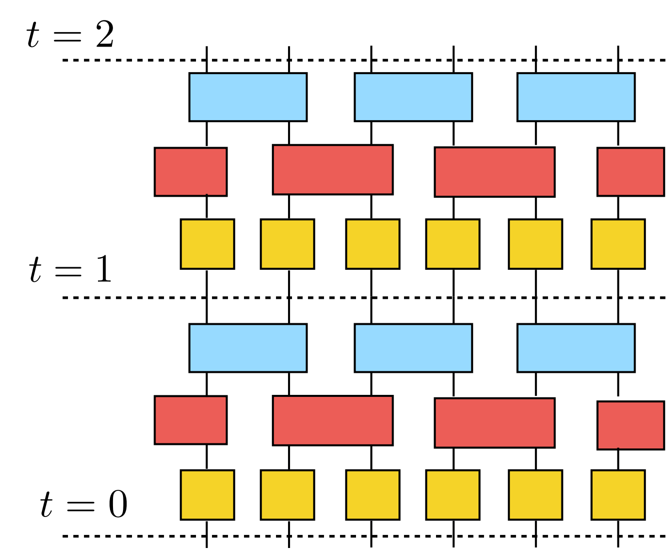

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quantum cellular automaton

Outline

1. Cellular automata and their quantum counterpart (QCA)

2. QCA model of four-fermion interaction: the Thirring automaton

3. The analytical solution in the two particles sector
    - set of possible scattering processes
    - bound sates

Cellular automata and their quantum counterpart

S. Ulam and J. von Neumann cellular automata (late 1940s)

Original idea: model ***complex behaviour*** based on a ***simple rule***

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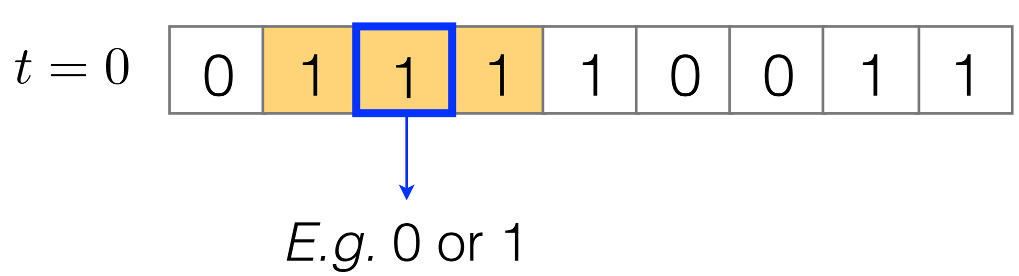

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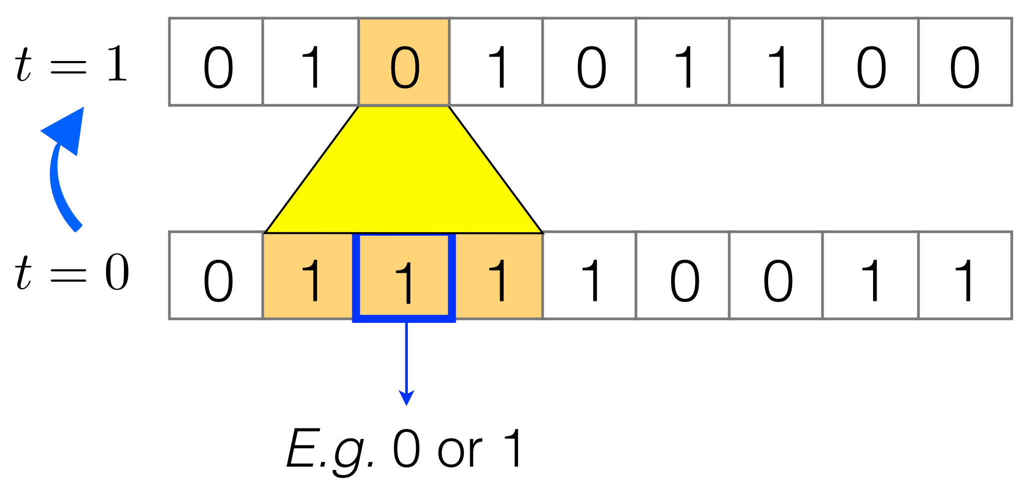
- LOCAL update rule: *the same for each cell*



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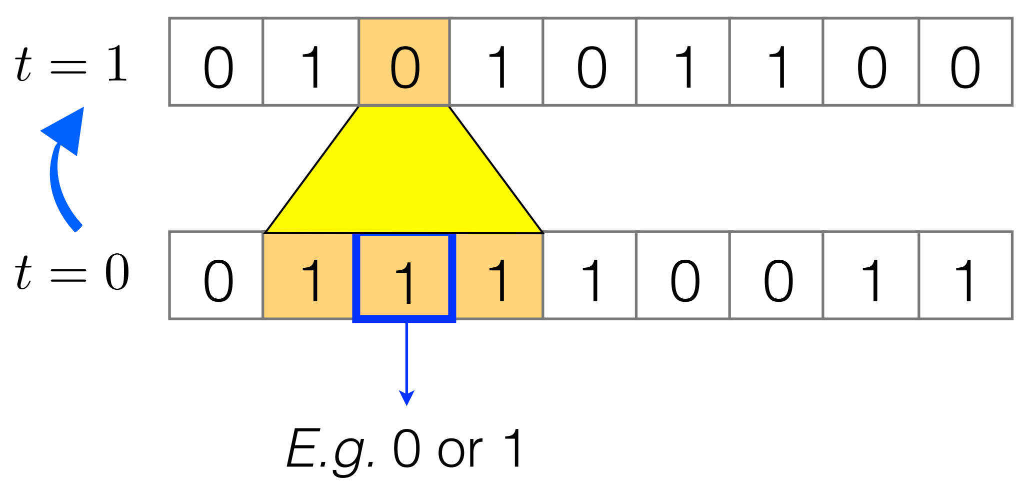
1d — first-neighbourhood — binary



Steven Wolfram book "*A new kind of science*"

$2^8 = 256$ possible LOCAL rules

The rules generate 4 classes of phenomena

Class 1: Static
Class 2: Periodic
Class 3: Chaotic
Class 4: "Mixed"

R.P. Feynman (1985)

Extend the idea to the quantum world: *Universal quantum simulator*

R. Feynman, International journal of theoretical physics **21**, 467 (1982)

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- Systems are finite dimensional
- Discrete time evolution which is:

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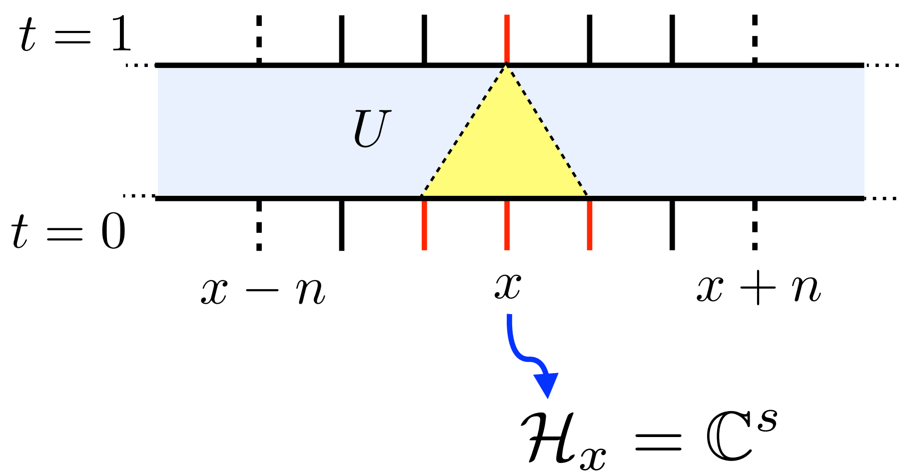
Translation-invariant



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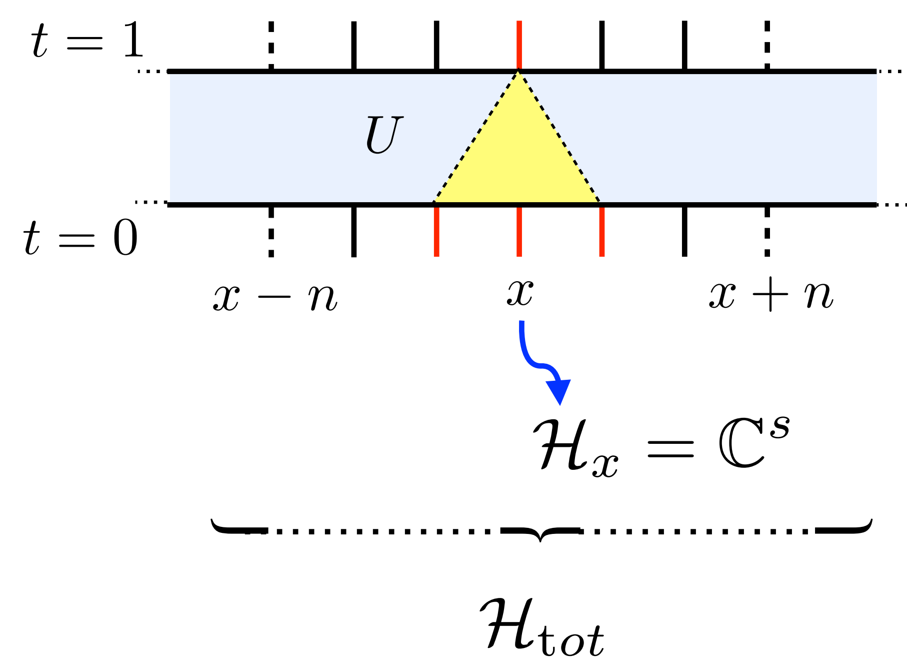

Quantum cellular automaton

$$U : \mathcal{H}_{tot} \to \mathcal{H}_{tot}$$

QCA model of four-fermion interaction

Alessandro Bisio, Giacomo Mauro D'Ariano, Paolo Perinotti, and AT, arXiv:1711.03920 (2017)

The automaton unitary operator must describe

- Free massive Dirac field
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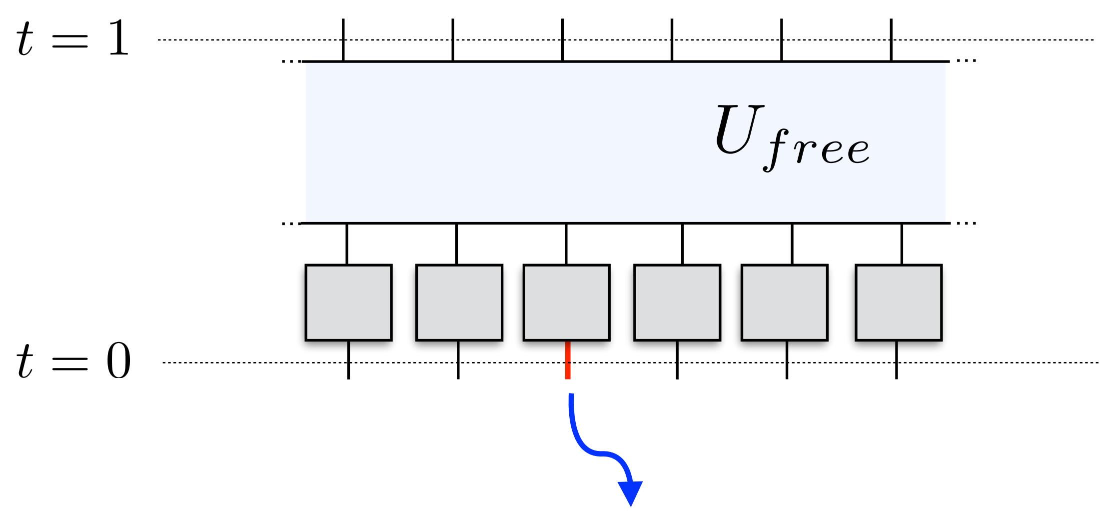
$$U = V_{int} U_{free}$$



$$\mathcal{H}_x = \mathbb{C}^{2N}$$

$\mathbb{C}^2$ for each Fermion with spin $\begin{pmatrix} \psi_{x,\uparrow} \\ \psi_{x,\downarrow} \end{pmatrix}$

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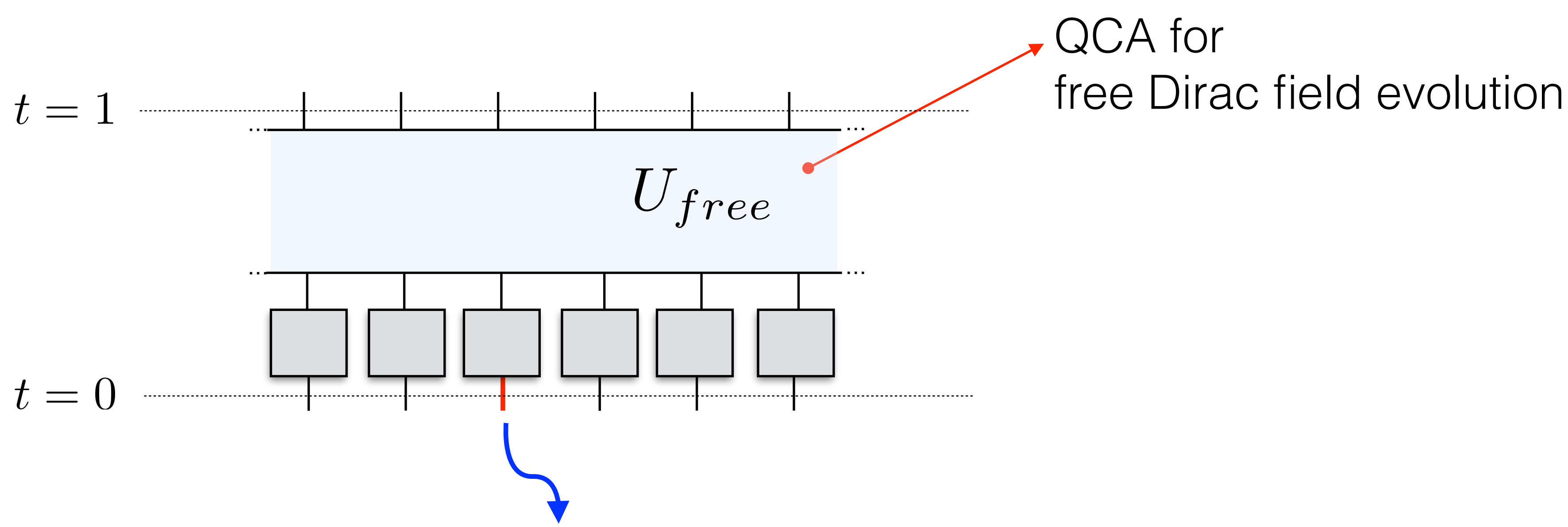

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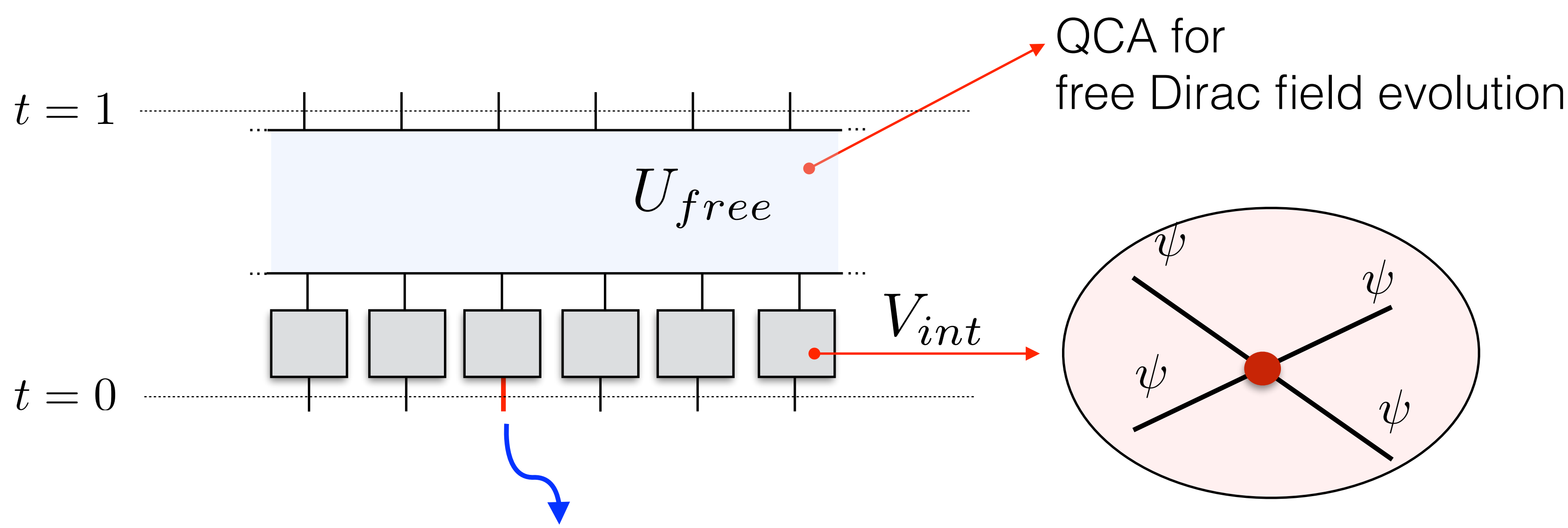

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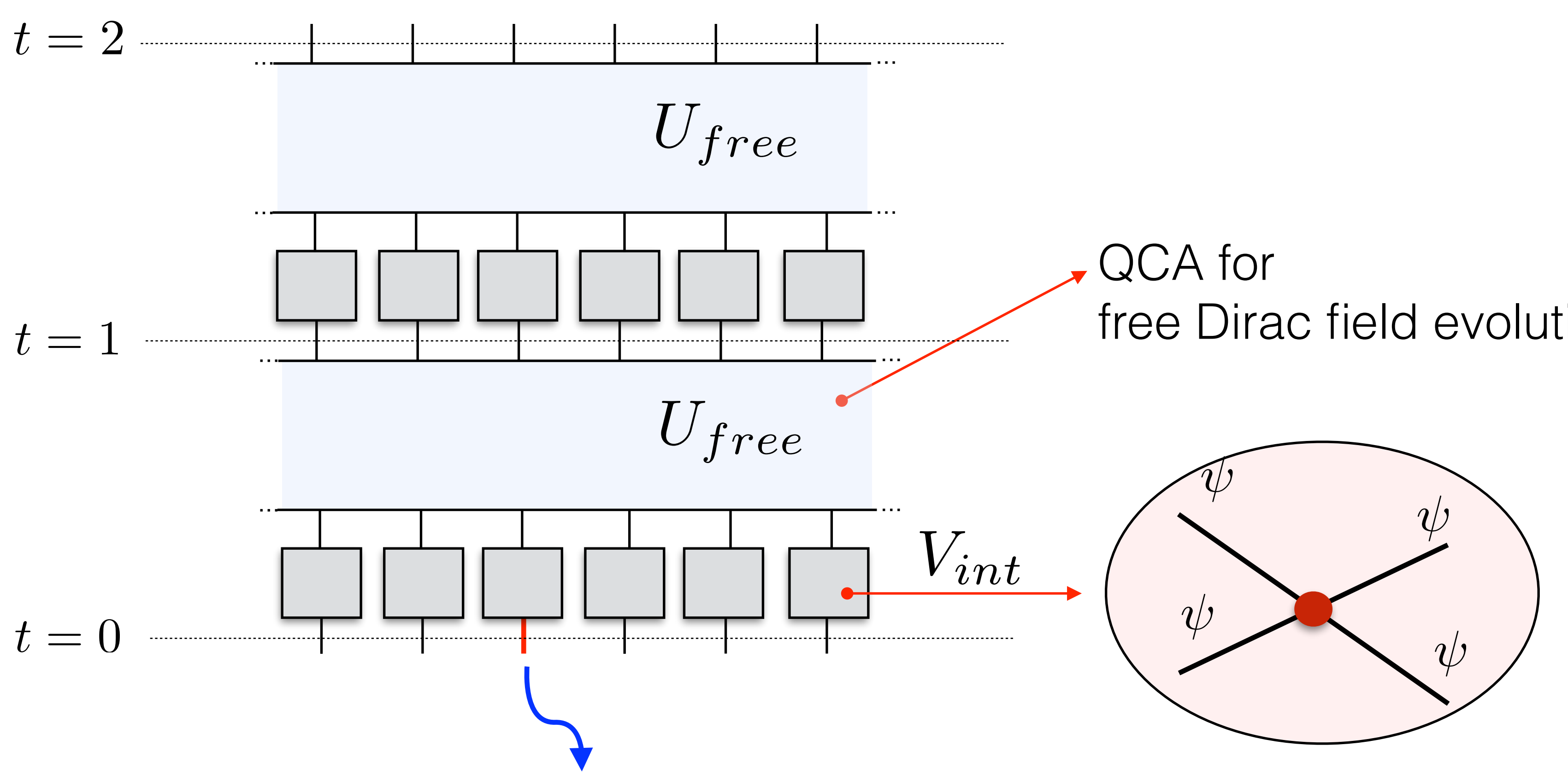

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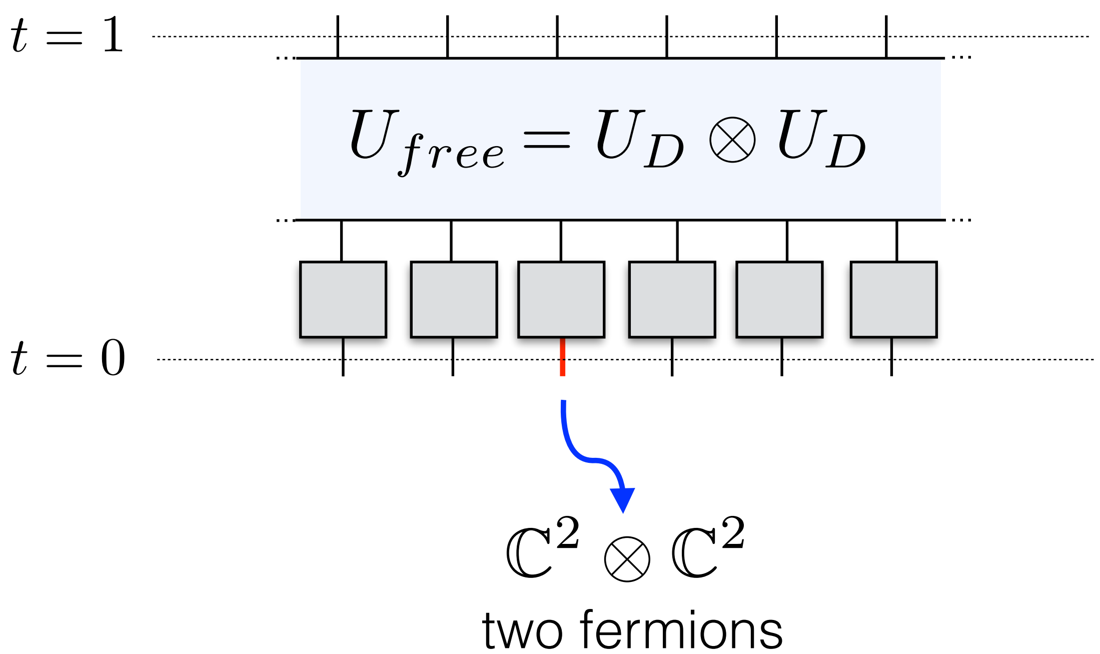
Two-massive fermions sector



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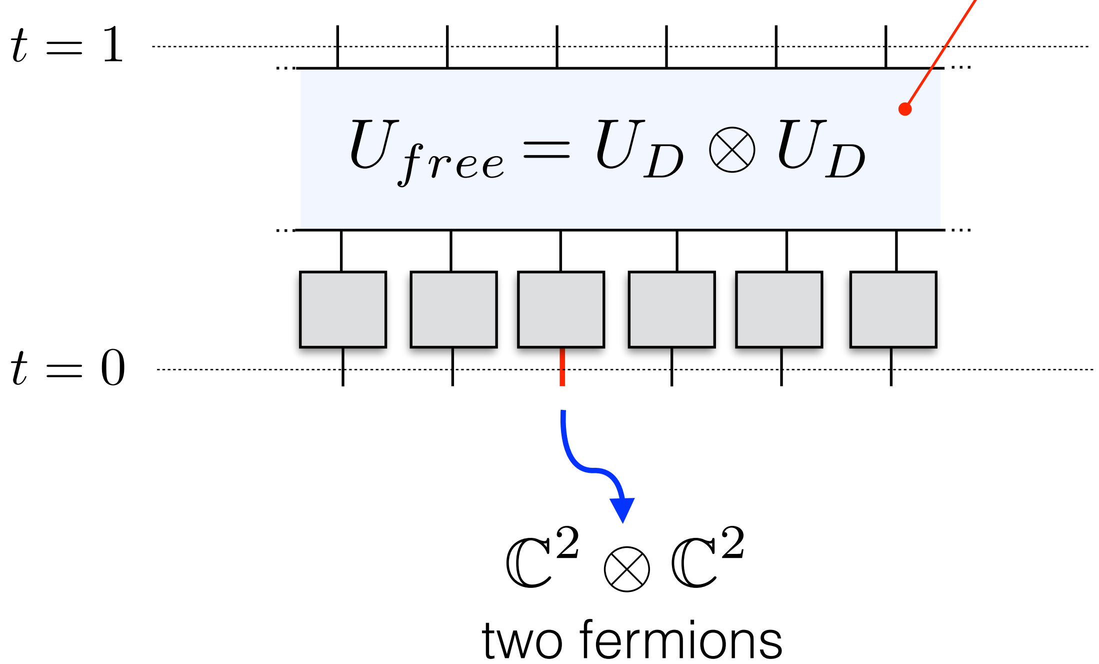

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QCA for free Dirac field evolution

$$U_D \begin{pmatrix} \psi_{x,\uparrow} \\ \psi_{x,\downarrow} \end{pmatrix} = \begin{pmatrix} nT & -im \\ -im & nT^\dagger \end{pmatrix} \begin{pmatrix} \psi_{x,\uparrow} \\ \psi_{x,\downarrow} \end{pmatrix}$$

m: particle mass

T: shift operator $T\psi(x) = \psi(x+1)$

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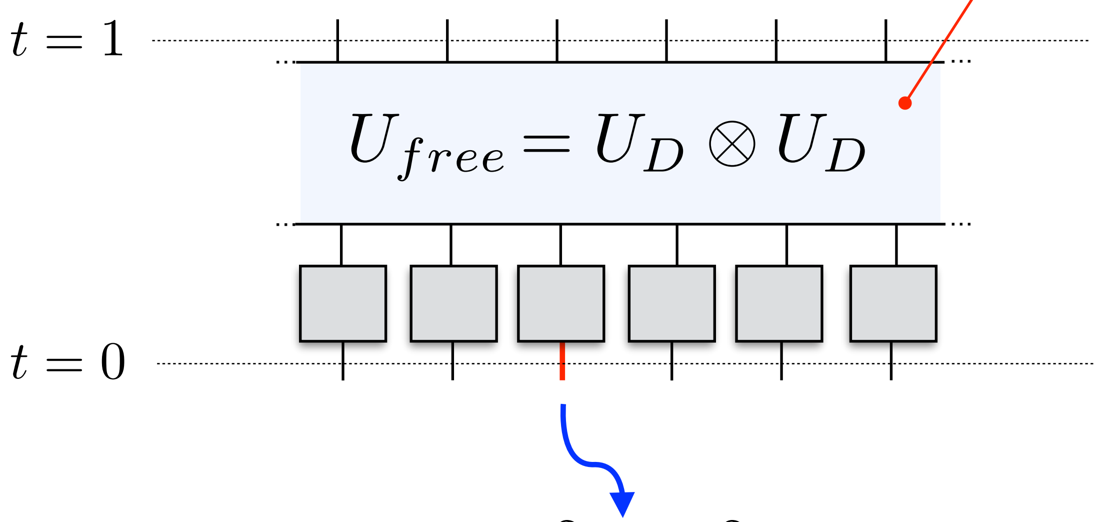

$\mathbb{C}^2 \otimes \mathbb{C}^2$

two fermions

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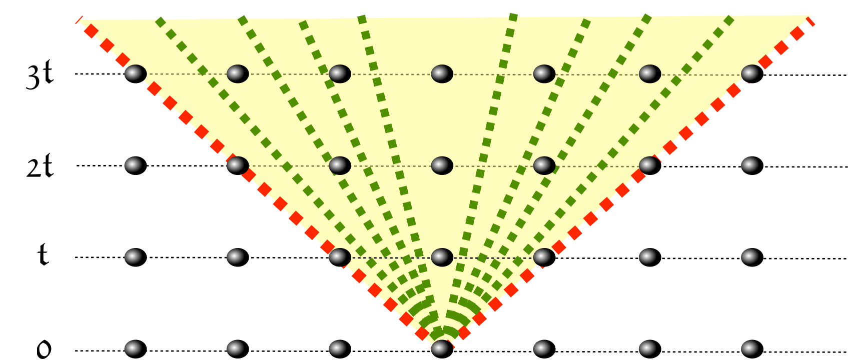

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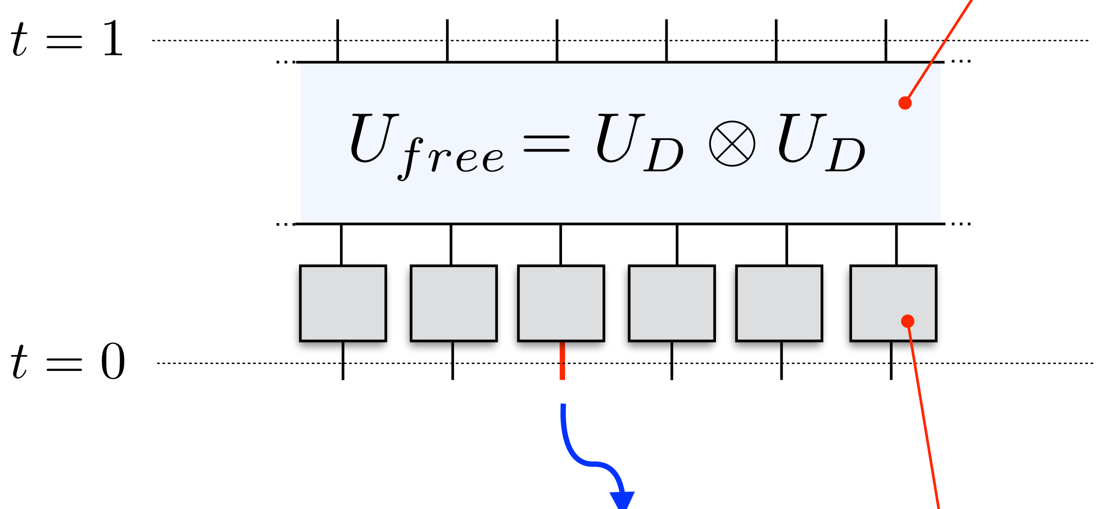

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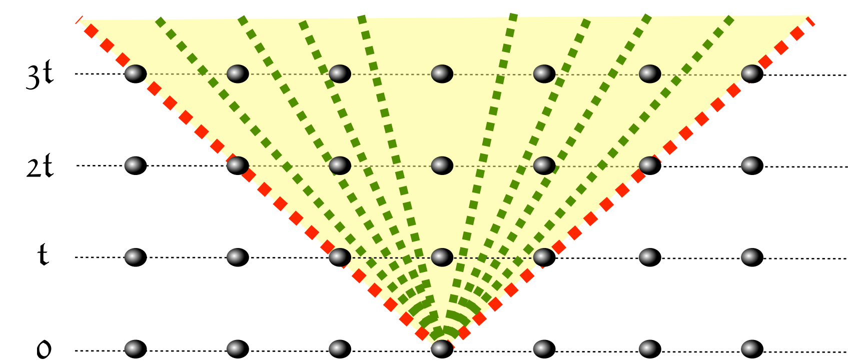
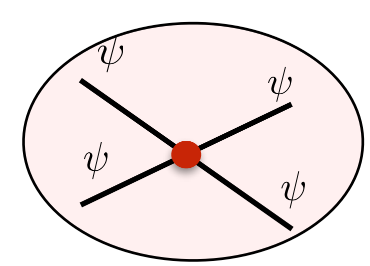


$$V_g = e^{ig \sum_x \psi^\dagger_{x\uparrow}\psi_{x\uparrow}\psi^\dagger_{x\downarrow}\psi_{x\downarrow}}$$

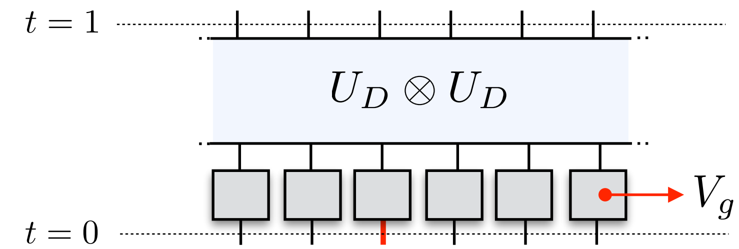


$$V_g = \begin{cases} e^{ig}, & x_1 = x_2, \\ 1, & x_1 \neq x_2 \end{cases}$$

x_1 : position of particle 1

x_2 : position of particle 2

Analytical solution



$$U_g = (U_D \otimes U_D) V_g$$

how the two particles evolve?

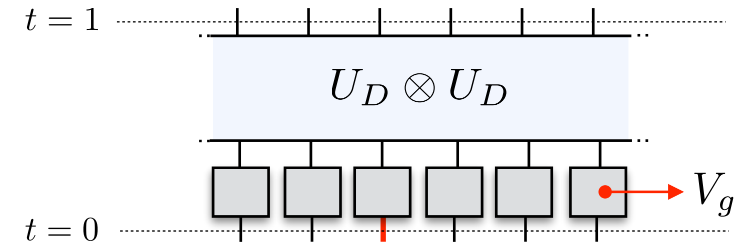
non linear op. => non-trivial

Analytical solution

Conserver quantities

Total energy: $\omega = \omega_1 + \omega_2$

Total momentum: $p = \frac{1}{2}(p_1 + p_2)$



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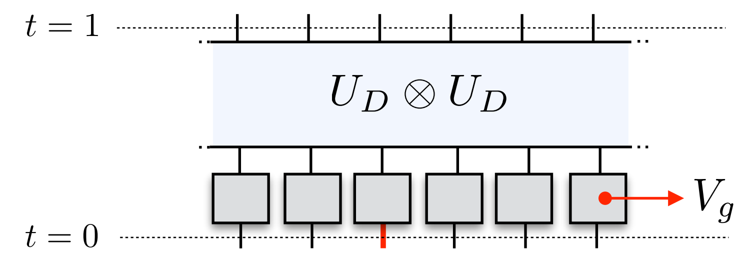

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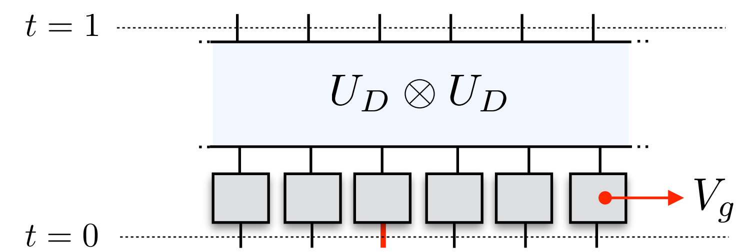
$$U_g^{(p)} |\Phi^\omega\rangle = e^{i\omega} |\Phi^\omega\rangle$$

solution of energy ω

g_3

Analytical solution



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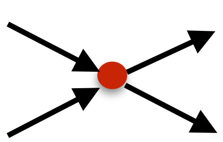
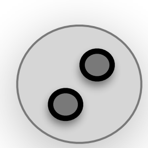
Spectrum $\Omega := \Omega_{cont} \cup \Omega_{disc}$

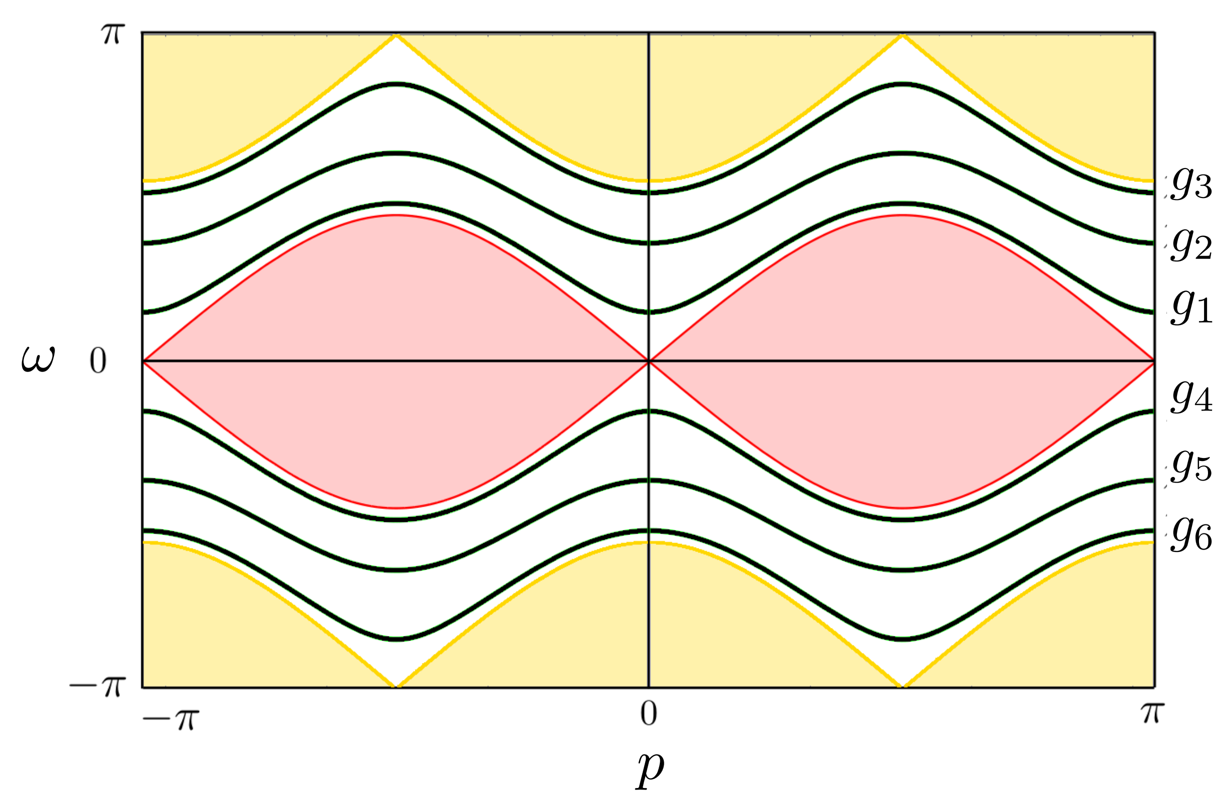
scattering solutions



bound states



We find the following spectrum

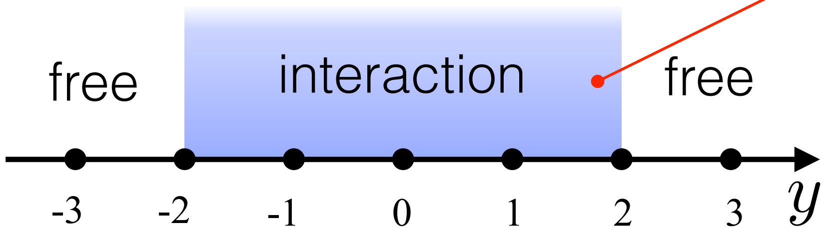


The method: modified Bethe ansatz

H. Bethe, Zeitschrift für Physik 71:205–226 (1931)

$y = x_1 - x_2$ particles distance

Due to local interaction



outside this region

$$|\Phi^{\omega}\rangle \equiv |\Phi^{\omega}_{free}\rangle$$

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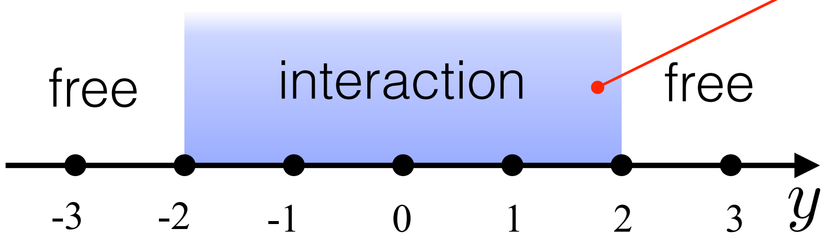

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sum over all free eigenstates with energy ω

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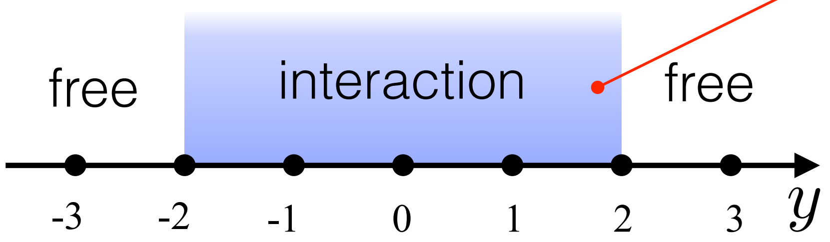

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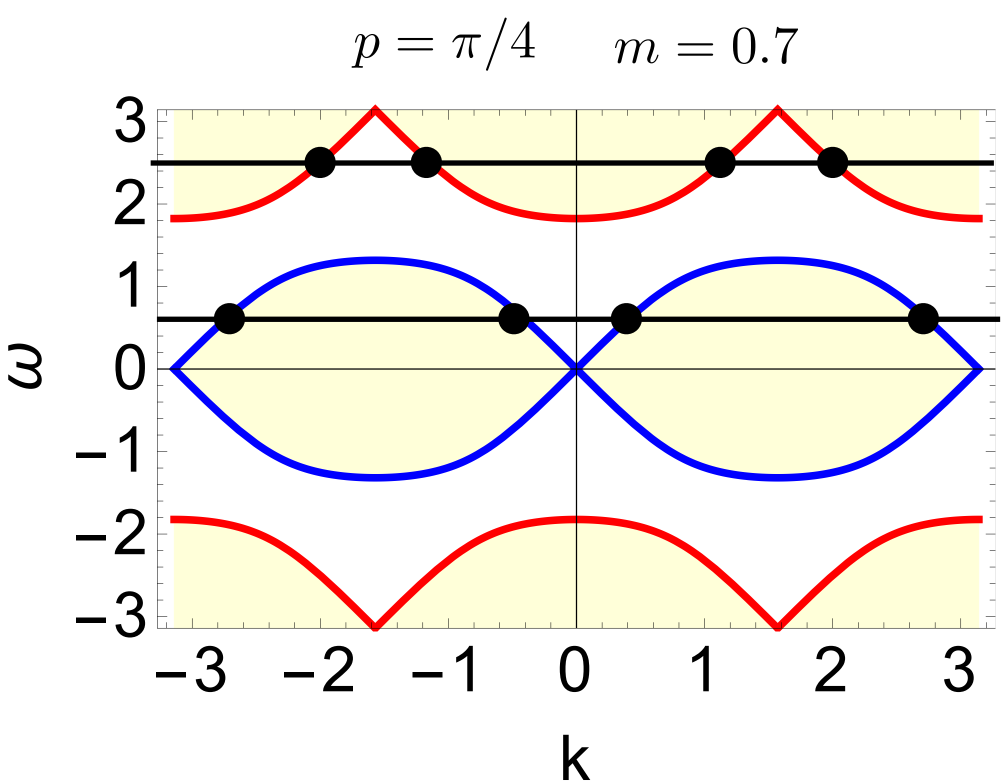
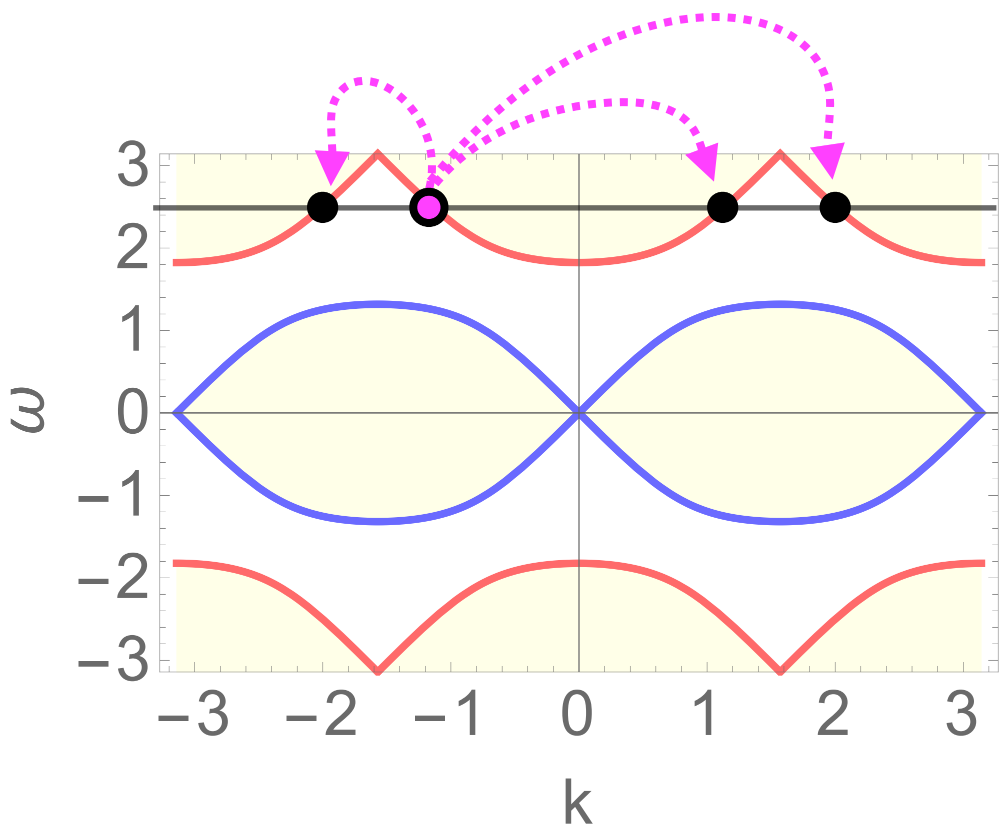
Continuous spectrum: **degeneracy 4**



Even: $|++\rangle_k$ $|++\rangle_{-k}$ $|--\rangle_{k-\pi}$ $|--\rangle_{-(k-\pi)}$

Odd: $|+-\rangle_k$ $|-+\rangle_{-k}$ $|+-\rangle_{k-\pi}$ $|-+\rangle_{-(k-\pi)}$

…….physical content



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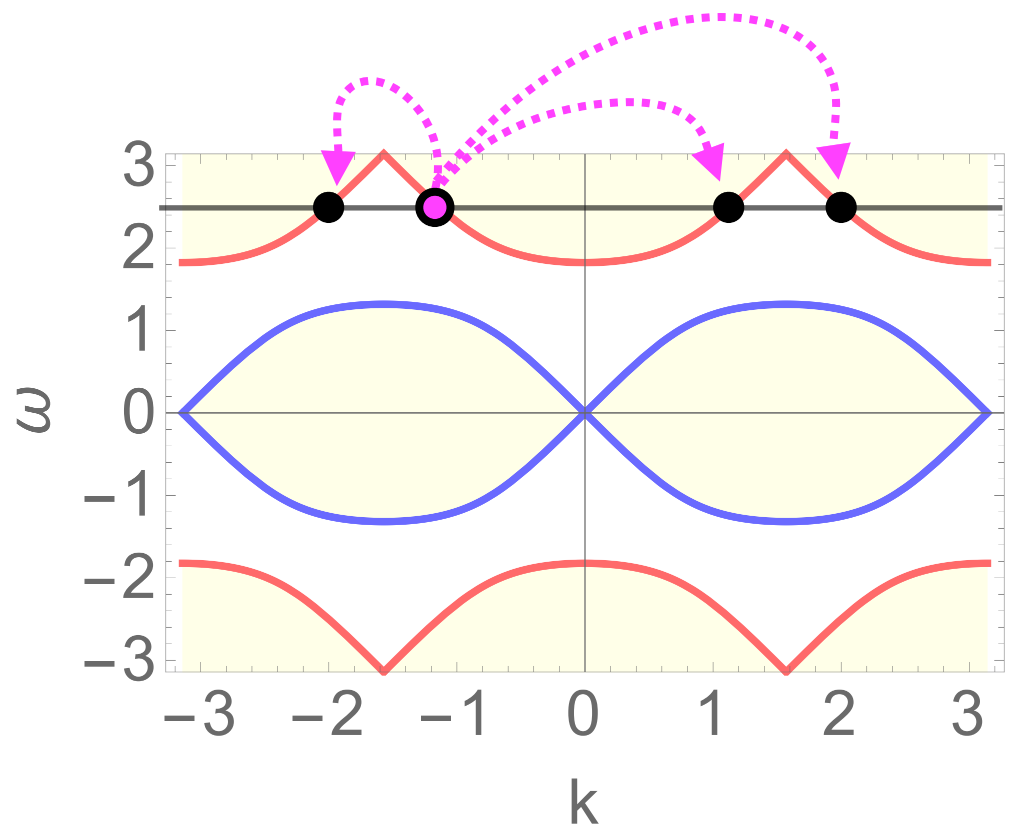
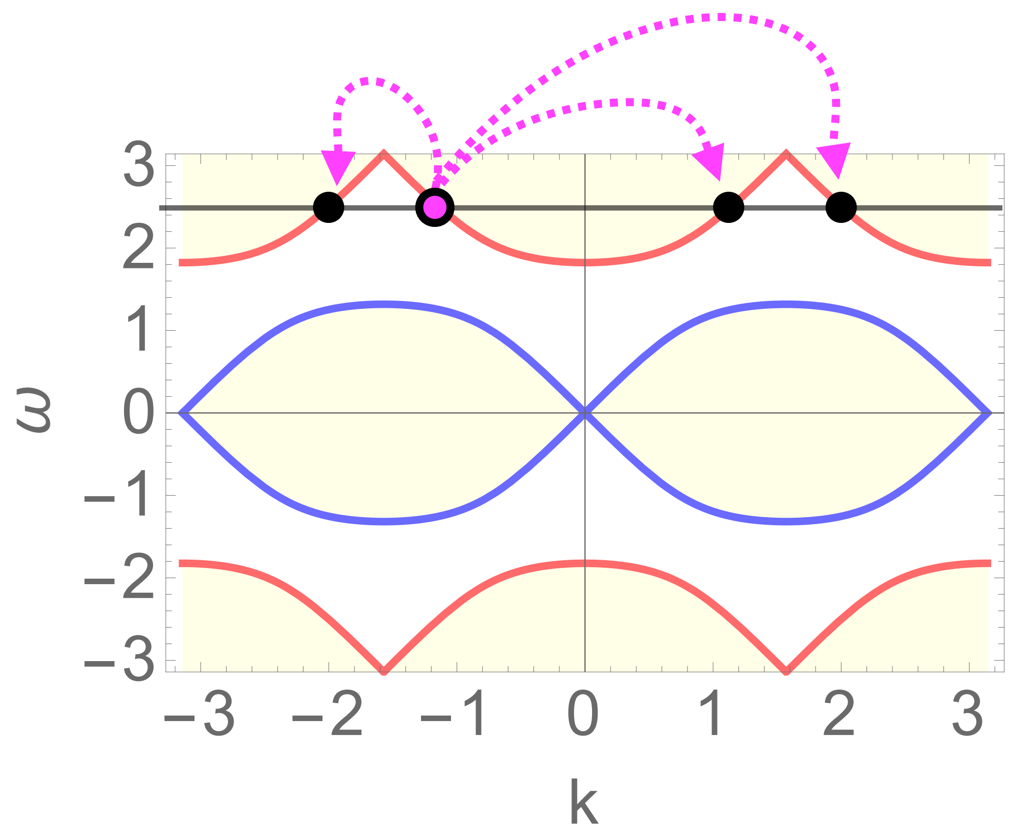


First difference with the Hamiltonian case which has degeneracy 2

The cause is *discrete time*: periodic energy spectrum

…….physical content



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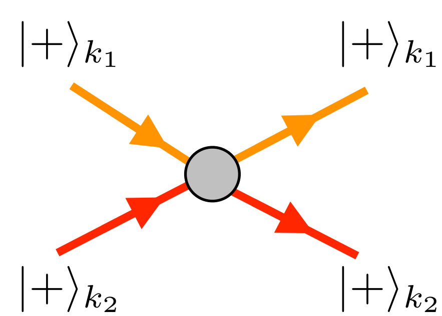
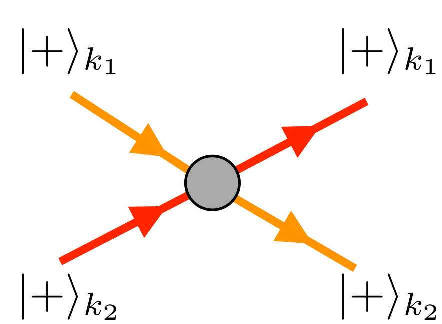
Hamiltonian processes

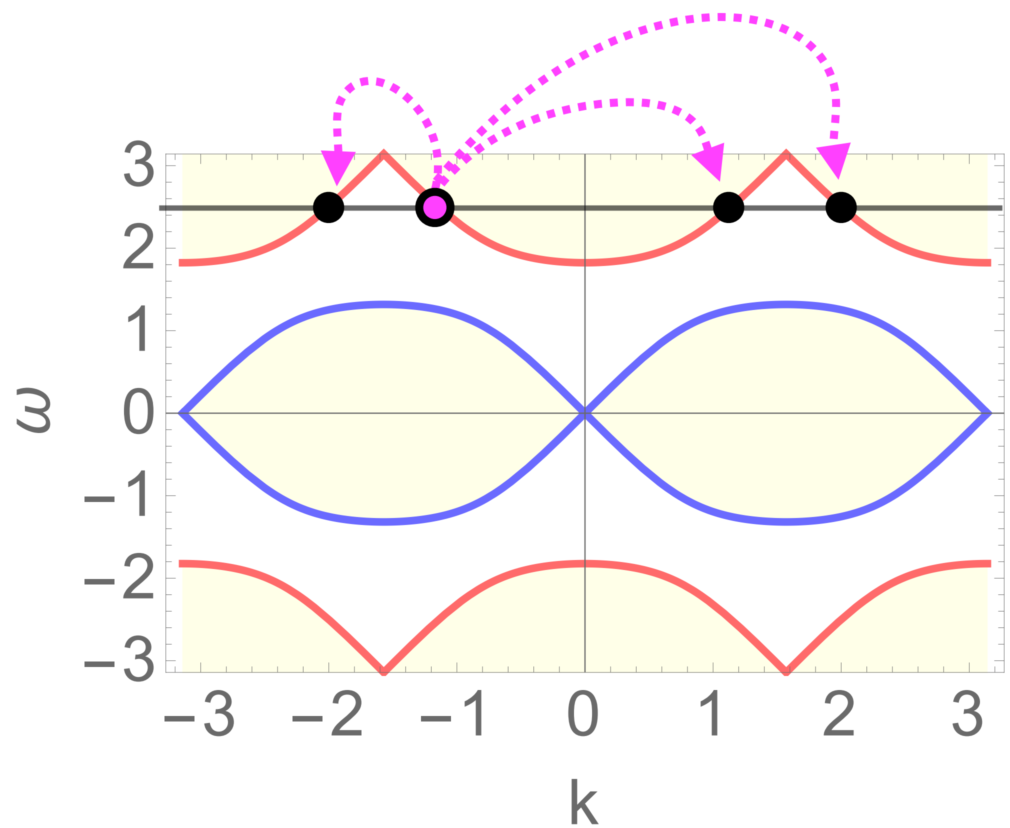
“elastic bouncing”



“tunnelling”



…….physical content



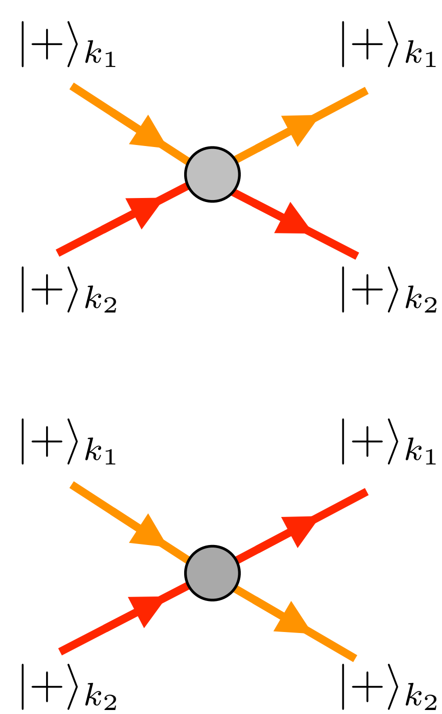
First difference with the Hamiltonian case which has degeneracy 2

The cause is *discrete time*: periodic energy spectrum

Hamiltonian processes

"elastic bouncing"

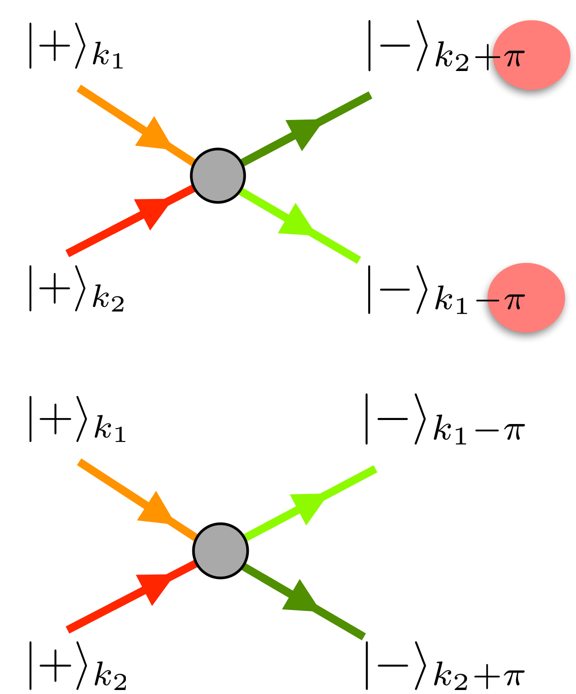
"tunnelling"



+

New processes due to discrete time

which momentum exchange



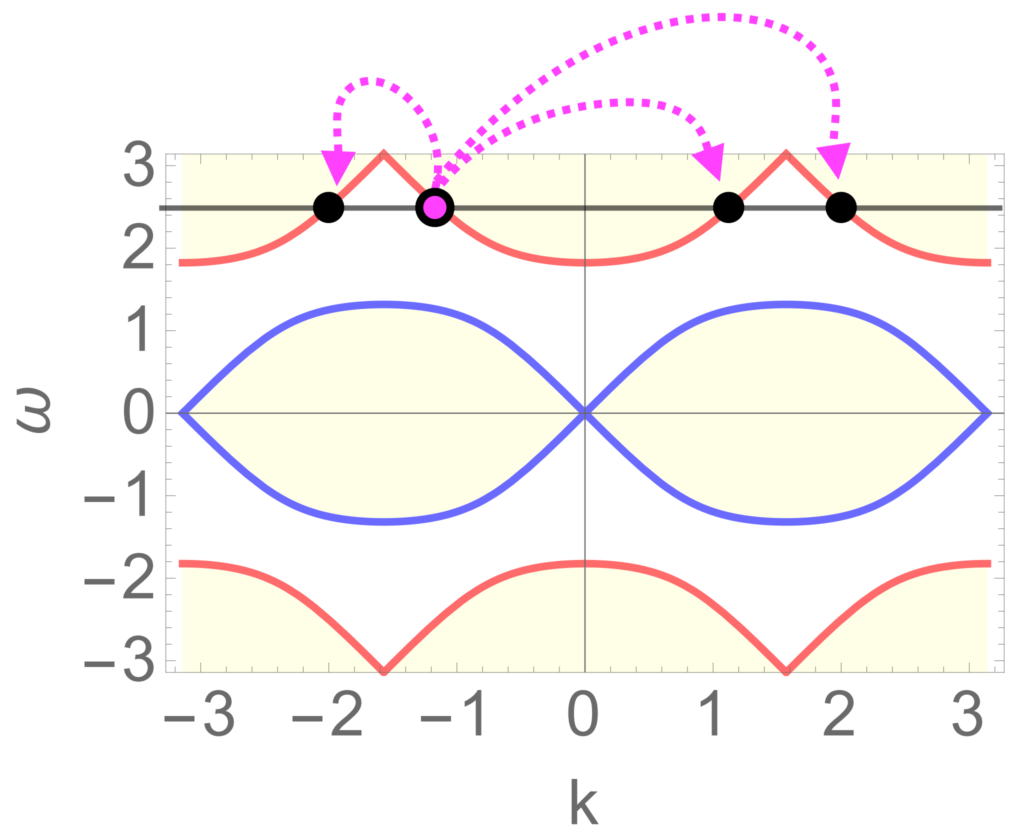
Suggestive parallel with fermion doubling

Known: Free fermions on lattice => “double particles”

Susskind, Leonard, Lattice fermions, Phys. Rev. D 16, 3031 (1977)

New: Interacting fermions in discrete time:
=> “double scattering processes”

…….physical content



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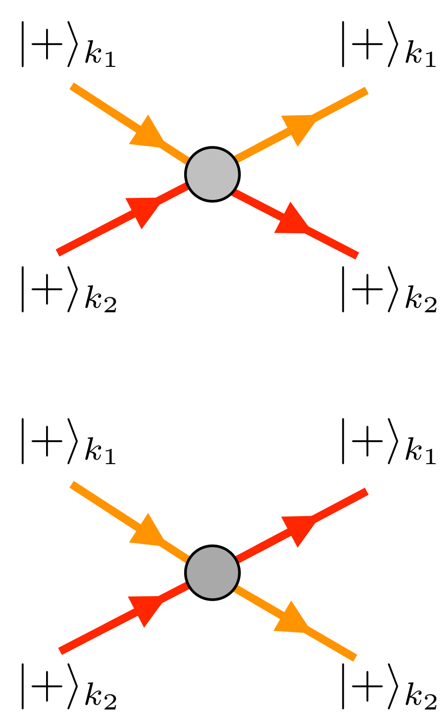

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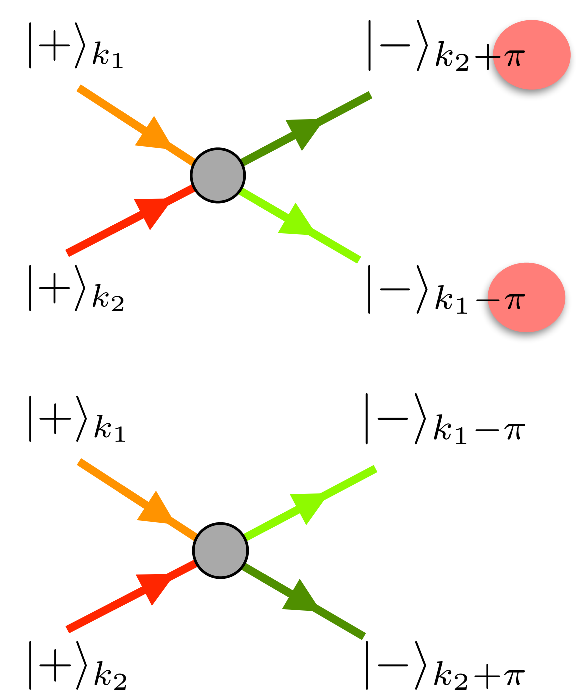
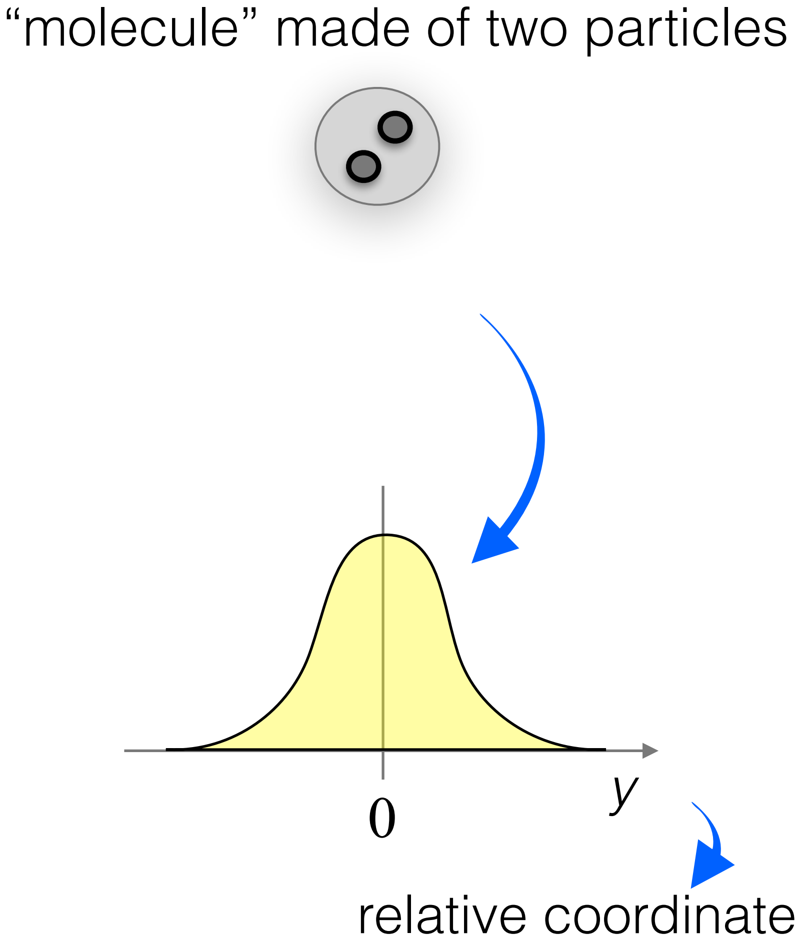
New processes due to discrete time which momentum exchange



Discrete spectrum and bound states

Bound states: configuration with a final state with vanishing probability distribution for large relative coordinate y

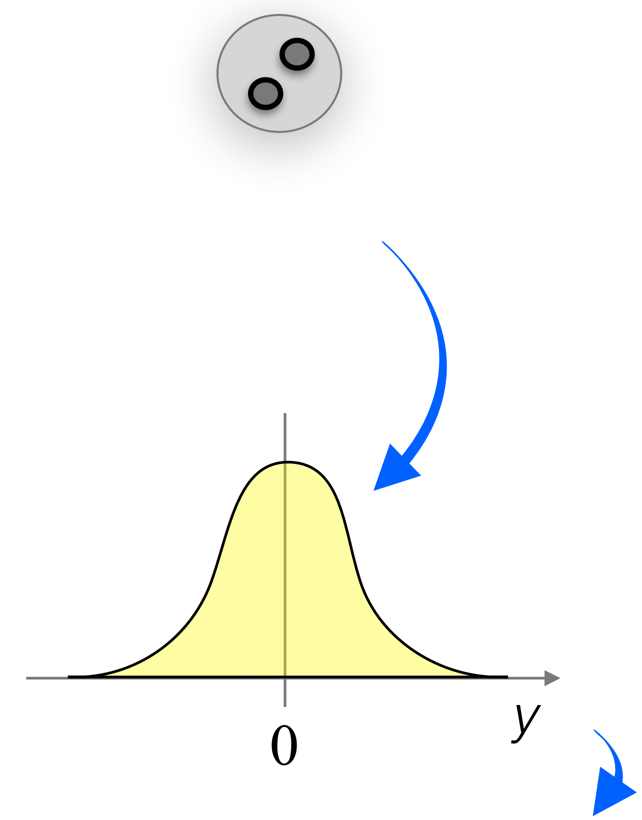


Alessandro Bisio, Giacomo Mauro D'Ariano, Paolo Perinotti, and AT, arXiv:1711.03920 (2017)

Discrete spectrum and bound states

"molecule" made of two particles

Bound states: configuration with a final state with vanishing probability distribution for large relative coordinate *y*



relative coordinate

$$|\Phi^{\omega}(y)\rangle = \begin{cases} e^{-iky}(\cdots) - e^{iky}T(\cdots) & y \geq 0 \\ \text{antisymm} & y < 0 \end{cases}$$

1) k real => scattering solutions (no bound states)

Alessandro Bisio, Giacomo Mauro D'Ariano, Paolo Perinotti, and AT, arXiv:1711.03920 (2017)

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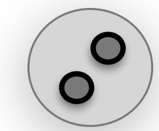

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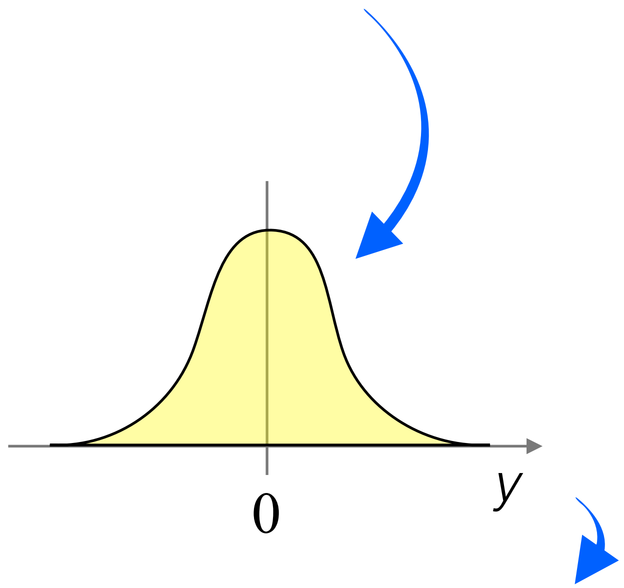

1) *k* real => scattering solutions (no bound states)

2) *k* has an imaginary part

rel. mom: $k = \frac{1}{2}(k_1' - k_2') - i\tilde{k}$

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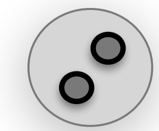

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$$e^{-\tilde{k}n} \qquad\qquad e^{\tilde{k}n}$$

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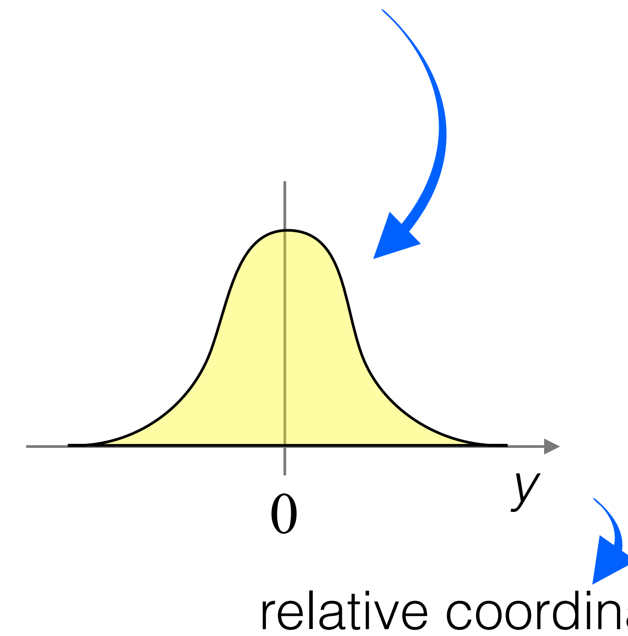

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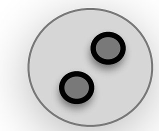

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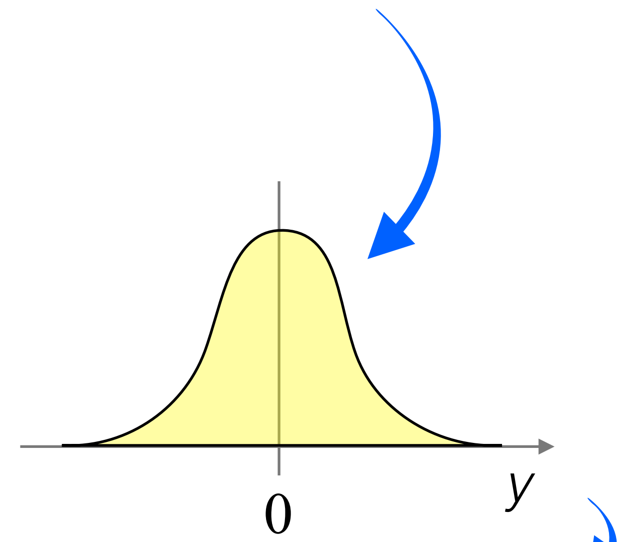

relative coordinate

1) *k* real => scattering solutions (no bound states)

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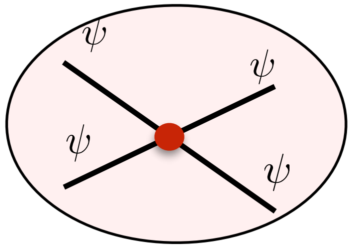
rel. mom: $k = \frac{1}{2}(k_1' - k_2') - i\tilde{k}$

Condition for bound state formation

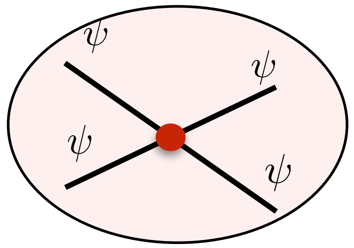
$$\tilde{k} > 0$$
$$T(p, k, g) = 0$$

Alessandro Bisio, Giacomo Mauro D'Ariano, Paolo Perinotti, and AT, arXiv:1711.03920 (2017)



$$V_g = e^{ig \sum_x \psi^\dagger_{x\uparrow}\psi_{x\uparrow}\psi^\dagger_{x\downarrow}\psi_{x\downarrow}}$$

Result: for any value of the coupling ***g*** and total momentum ***p*** there exists a unique bound state



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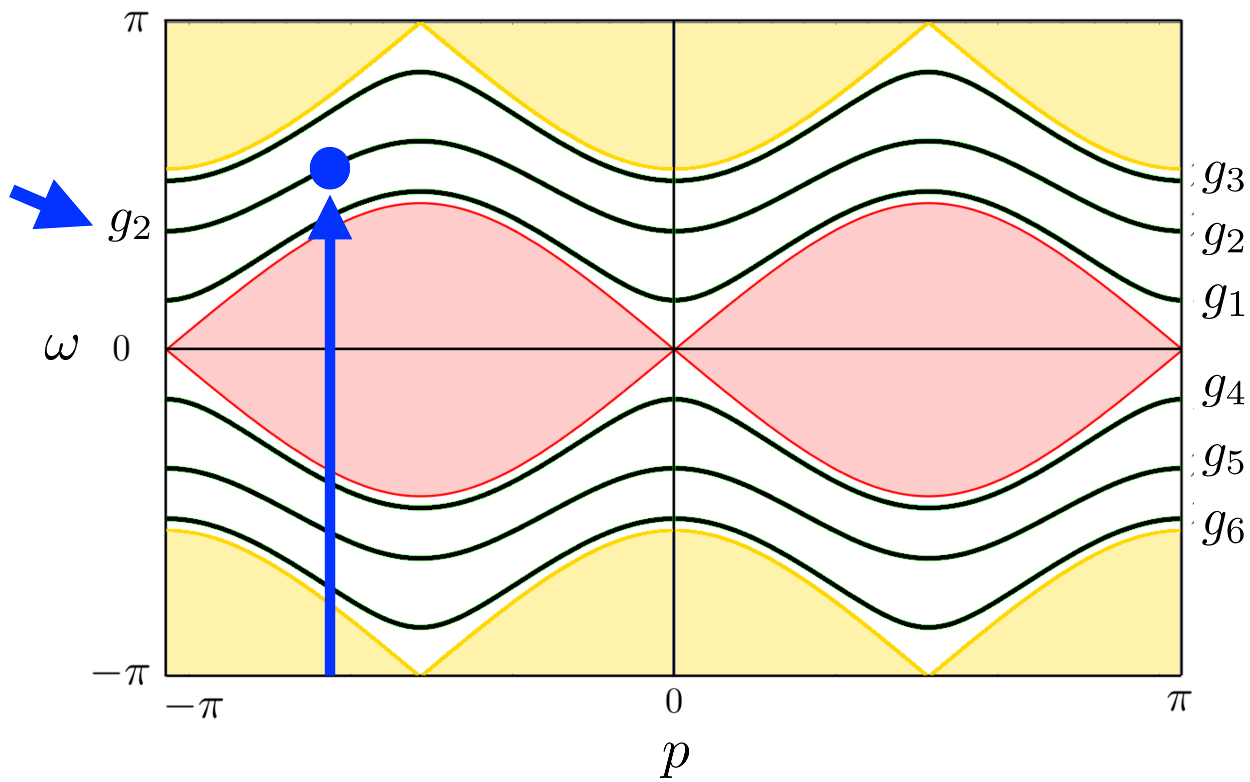

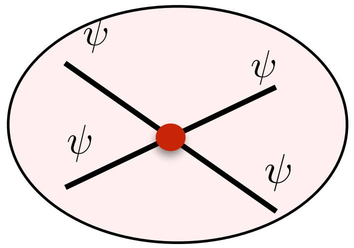

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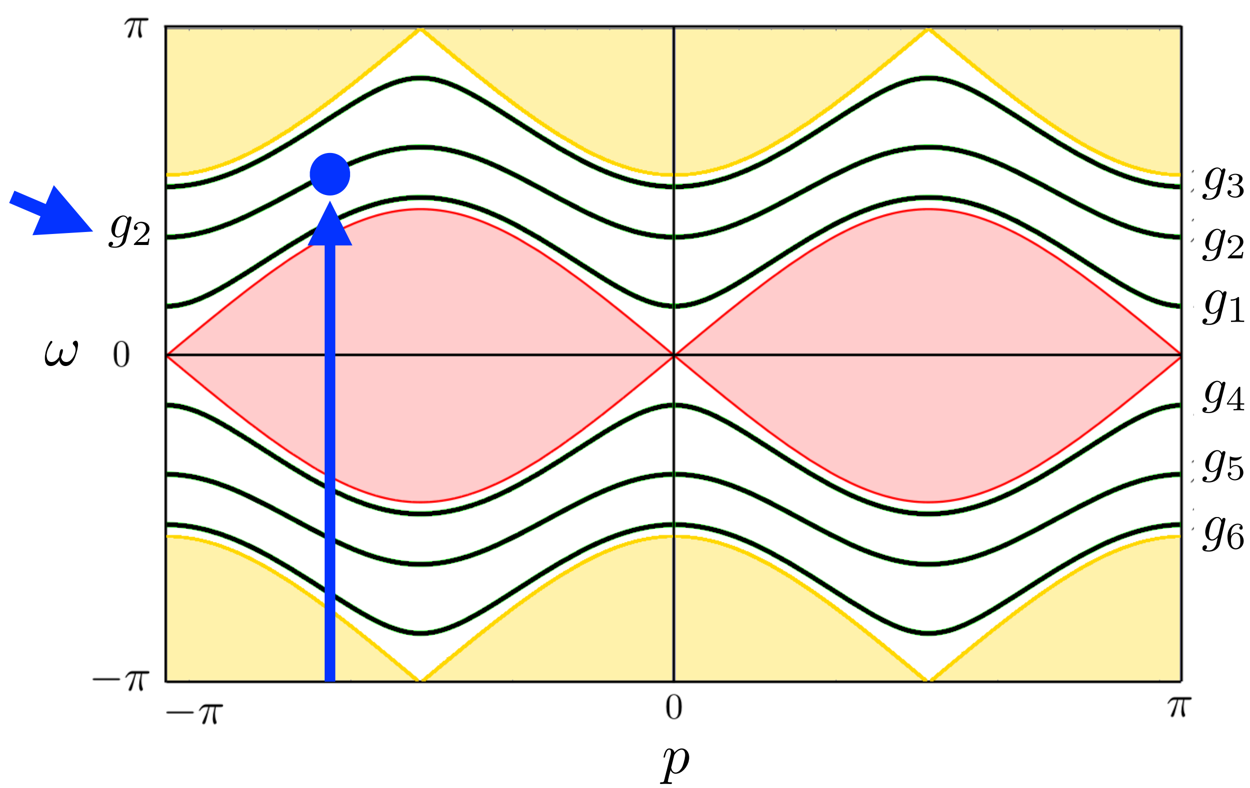
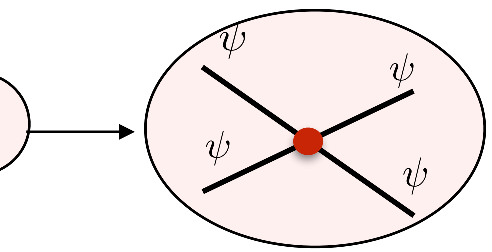


Second difference with the Hamiltonian case
where for some total momenta **p**
there are no *bound states*

Two observations on bound states

$$U = U_{free} V_g$$



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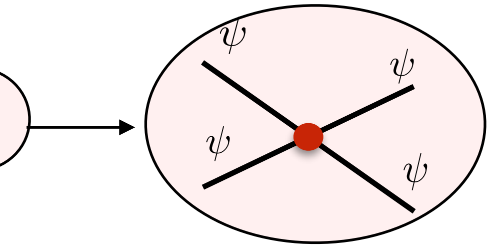


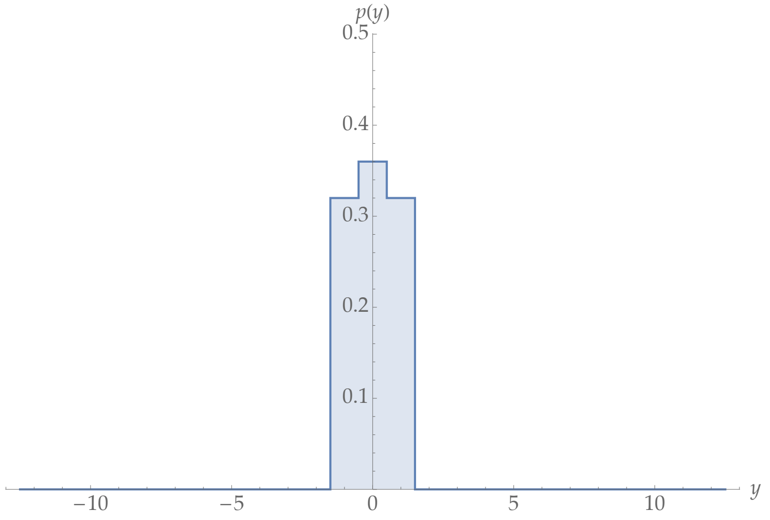
Perfectly localized bound states

when

$$g = \pm 2p$$

g → coupling

p → total momentum



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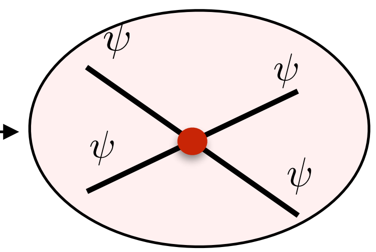

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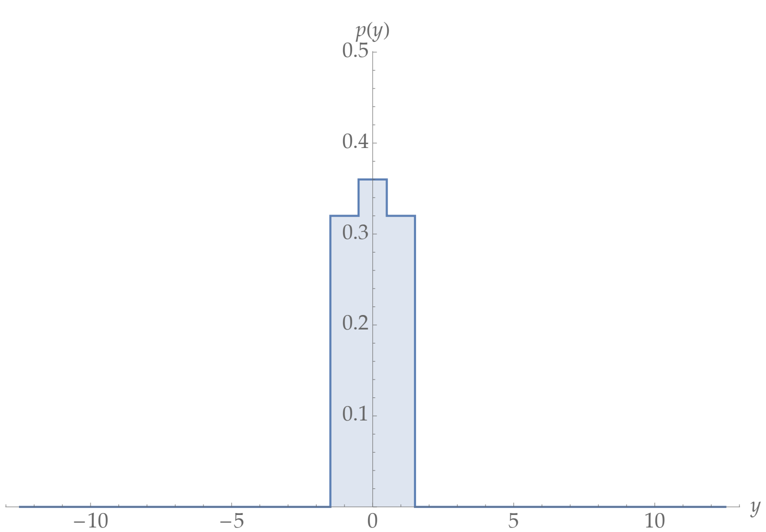
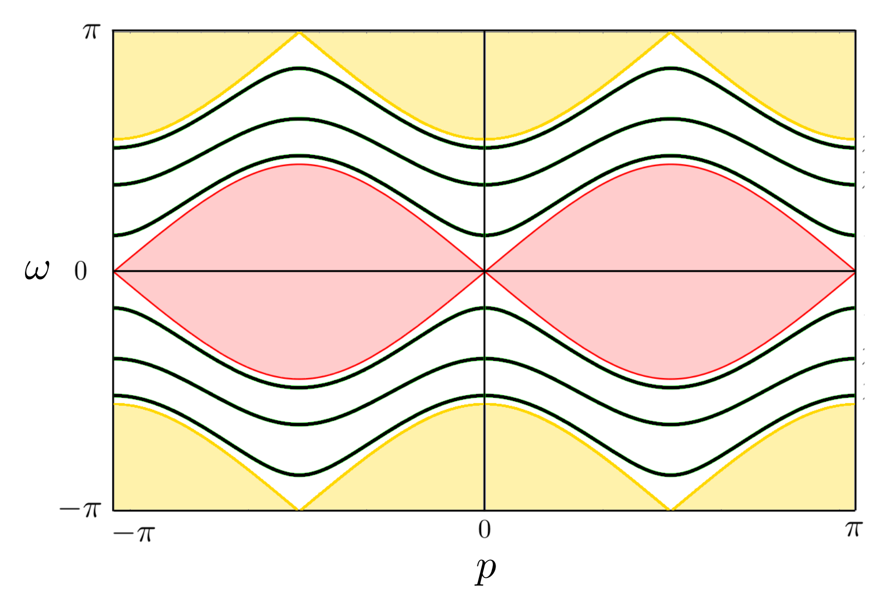
coupling   total momentum



“Bound” states with null coupling **g**

$$g = 0 \;\Rightarrow U = U_{free}$$



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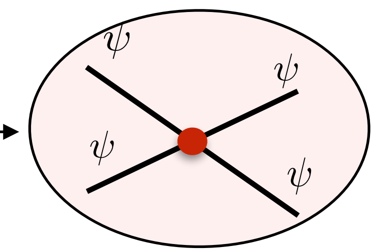

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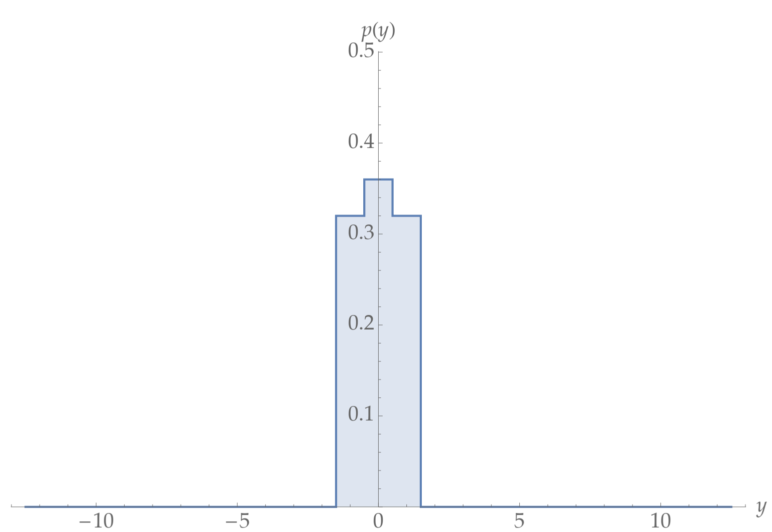
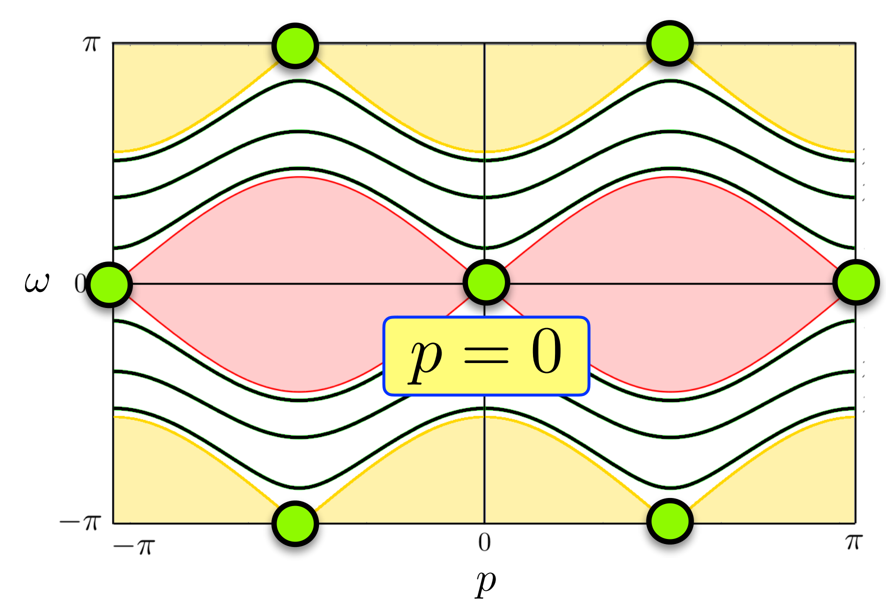


"Bound" states with null coupling **g**

$$g = 0 \;\Rightarrow U = U_{free}$$

.....still exist "bound" states

$$p = 0, \pm\tfrac{\pi}{2}, \pi$$



Third difference with Hamiltonian models

Again due to *discrete time*

Final comments

Result: discrete-time model of four Fermion interaction *solved* for two-particles

Proves the **effects** of **discrete time** in a many-body system

- more scattering processes than in the continuous-time case
- bound states with arbitrary total momentum
- stationary “bound” states even in the non-interacting case

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R.P. Feynman: *Universal* ***quantum simulator***

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Today:

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Thank you