

REMARKS ON LANDAU LEVELS, BRAID GROUPS AND LAUGHLIN WAVE FUNCTIONS

Mauro SPERA

Dipartimento di Matematica e Fisica "Niccolò Tartaglia"

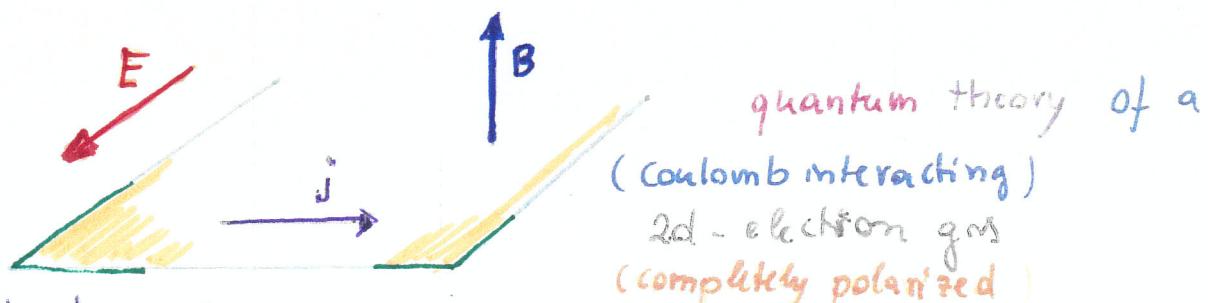
via dei Musei 41, 25121 Brescia, Italia

Outline

- The FQHE & Laughlin wave functions
introductory remarks
- Laughlin wave functions via
geometric quantization (joint work with
A. Besana, JKTR '06)
- RS braid groups and their
simplest unitary representations:
a geometric approach (M.S. J&P '15)
- Geometric quantization of Landau
levels revisited (joint work with A. Galasso,
IJGMMP '16)

* FQHE & Laughlin wave functions

(A.Besana, M.S '06 ; N.S. '15)



low temperature
strong magnetic fields

quantum theory of a
(Coulomb interacting)
2d - electron gas
(completely polarized)

ground state approximately
described by the Laughlin wave function

$$\Psi(z_1, \dots, z_n) = \prod_{i < j} (z_i - z_j)^m e^{-\sum |z_i|^2}$$

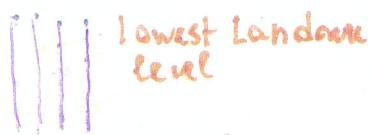
$$m = \frac{1}{\nu} \quad \nu: \text{filling factor}$$

ground state
of a quantum
harmonic oscillator

$$\Omega_H = \nu \frac{e^2}{h}$$

* Hall conductance

for a torus sample



ν = slope of a holomorphic vector bundle over a torus (Brillouin manifold)
parametrising boundary conditions (Varnhagen '95)

elementary excitations: quasi-particles / holes

with charge $e^* = \pm e\nu$ (fractional!)

and anyon statistics $(-1)^{\nu n}$ the braid group

generalised Laughlin
wave functions

Thus

ν = statistical parameter = slope

Task (M.S. 2015)

analyse the above coincidence from
a **RS braid group** theoretical
perspective

sample: closed RS of genus
 $g \geq 1$



Brillouin (or spectral)

manifold: $J(\mathbb{Z}_g)$

Jacobian

quantum computing

unitary representations
of $B(\mathbb{Z}_g, n)$
RS braid group

Weyl-
Heisenberg

Projectively flat (HE)
vector bundles

$E \rightarrow J(\mathbb{Z}_g)$

Matsushima

$$\frac{\text{degree}}{\text{rank}} = M = \nu g!$$

slope

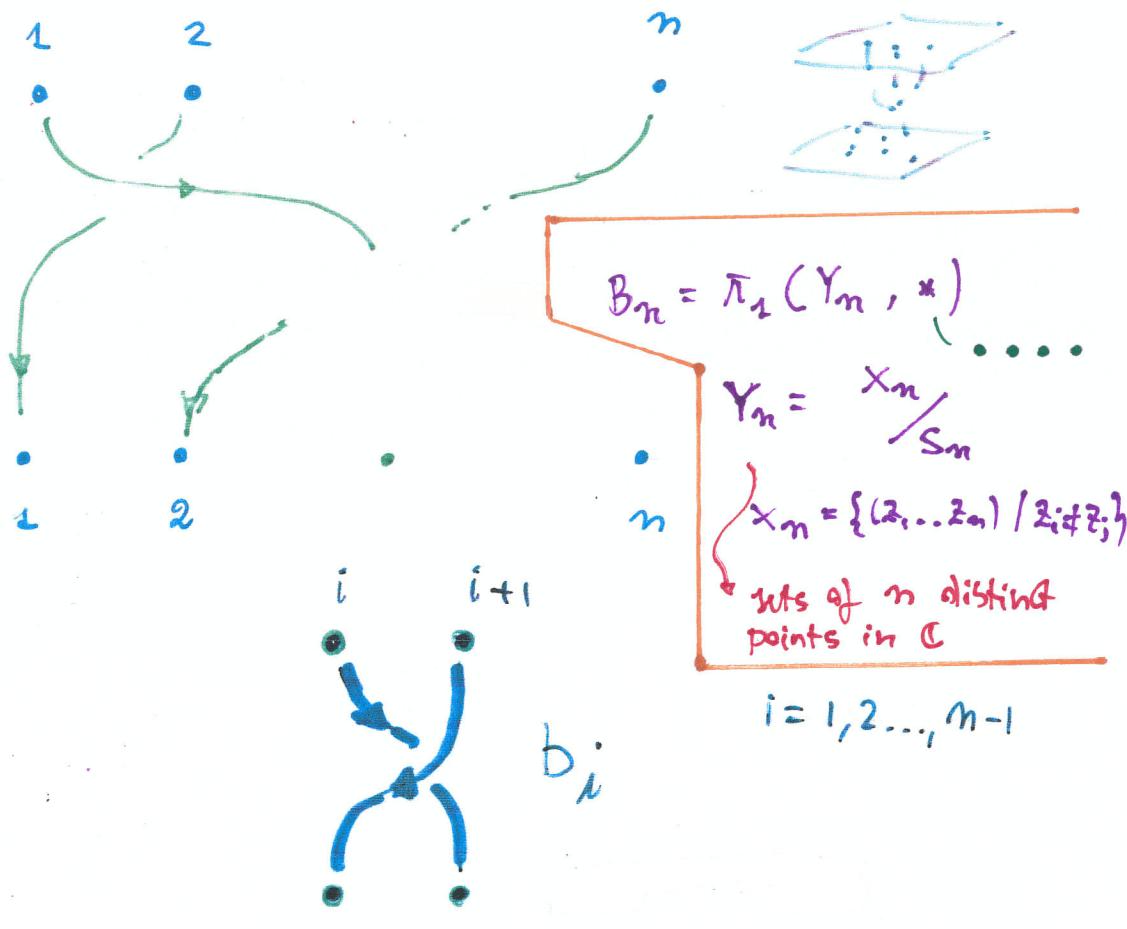
"slope-statistics" formula

get generalised Laughlin wave functions

An early approach: geometric description of
 (standard) Laughlin wave functions (A. Bezanca —
 JKTR 2006)

recall:

B_n Braid group (n -strands)



$$b_i b_j = b_j b_i \quad |i-j| \geq 2$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$$

$$i = 1, 2, \dots, n-2$$

$$G = \text{Sdiff}(R^2)$$

fluid dynamics

$$\tilde{\omega} = \mu \sum_{j=1}^n 2\pi i \delta(z - z_j) dz \wedge d\bar{z}$$

Vorticity 2-form

concentrated on

punctures z_1, \dots, z_n
slow variables

$$Y_n \cong G/G_b$$

Conf(n, C) / S_n

4. Besana & MS '06
area preserving diffeomorphisms
(of R^2 (configuration space of
a perfect 2-d fluid)
or Goldin, Menikoff & Sharp



coadjoint orbit
of G

adiabatic
motion

stabilizer of $b \equiv (z_1, \dots, z_n)$

$$\rightarrow B_n = \pi_*(Y_n, \cdot) \cong G_b^0 / G_b$$

scalar representation
~ character of G

connected component

Feynman-Onsager (FO) quantization

triality: vortices

in general: anyons

$$Y_n:$$

Kähler manifold

inheriting the
(Standard) Kähler form on C^n : $\frac{i}{2} \sum dz_j \wedge d\bar{z}_j$

$$\omega = \mu \sum_{i < j} d \log(z_i - z_j)$$

1-form on $Y_n = \text{Conf}(c, n) / S_n$

stores topological information

$$= i\mu d \sum_{i < j} \arg(z_i - z_j) + dh$$

vortex Hamiltonian



no representation of

$$B_n \cong \pi_1(Y_n, *)$$

$$\text{hol}_b(\omega) = e^{\int_b \omega} = e^{-i\pi \mu \underbrace{w(B)}_{\text{writhe}}} \dots$$

parallel transport
(holonomy)



+ 1 contribution

$L_\mu \rightarrow Y_n$

param
geometric
quantization

line bundles parametrized by $\mu \in \mathbb{R}$)

$$H^1(Y_n, S^1)$$

holonomies of flat connections

$$L_\mu = F_\mu \otimes L$$

flat line bundle \hookrightarrow trivial line bundle on \mathbb{C}^n

$$\mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^n$$

$$\psi_\mu = \prod_{i < j} (z_i - z_j)^\mu$$

holomorphic section
(\in function on \tilde{Y}_n)

universal
covering
space

$$(\psi_\mu, \psi_\mu) := |\psi_\mu|^2 \equiv e^{-f_\mu}$$

metric on F_μ

monodromy encoded in $J_0 = e^{i \arg \psi_\mu}$

$$\Psi_\mu := \psi_\mu \otimes 1$$

standard kähler potential
 $\omega \sim \bar{\partial} f$

$$(\Psi_\mu, \Psi_\mu) := e^{-(f_\mu + f)}$$

$$\equiv e^{-\gamma_\mu}$$

metric on L_μ

plasma
Hamiltonian
vortex + oscillator

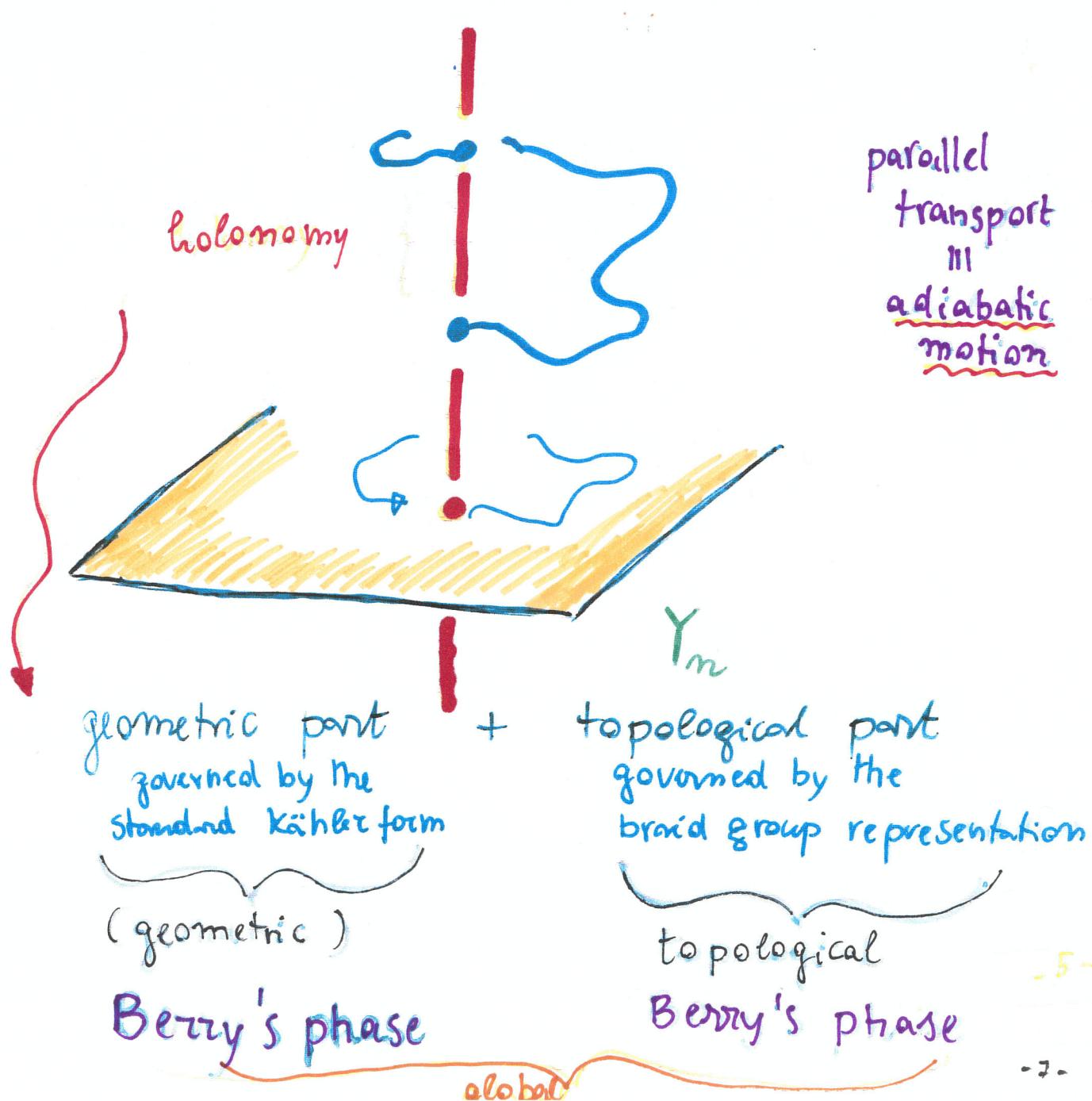
$$\Psi_{\mu} = \prod_{i < j} (z_i - z_j)^{\mu} \cdot e^{-\frac{1}{2} \sum_{j=1}^n |z_j|^2}$$

(Vandermonde)

* get a geometric interpretation of trial anyonic wave function in FQHE
* Laughlin

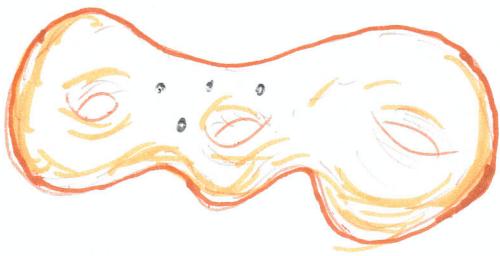
$$\nabla_{\mu}^{(1,0)} = \partial - \partial F_{\mu} \quad \nabla_{\mu}^{(0,1)} = \bar{\partial}$$

canonical Chern-Bott connection



* Bellingen presentation of $B(\mathbb{Z}_g, n)$

Riemann
surface braid
group



closed orientable
surface of genus g

strands

- generators : $\sigma_i = \sigma_{n-i}$ $a_1 = a_g, b_1 = b_g$
 $\underbrace{\phantom{\sigma_i = \sigma_{n-i}}}_{\text{standard generators}}$

* braid relations : $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ $i = 1, 2, \dots, n-2$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| \geq 2$$

"Far commutativity"

* mixed relations

$$(R1) \quad a_r \sigma_i = \sigma_i a_r \quad 1 \leq r \leq g \quad i \neq r$$

$$b_r \sigma_i = \sigma_i b_r$$

$$(R2) \quad \sigma_i^{-1} a_r \sigma_i^{-1} a_r = a_r \sigma_i^{-1} a_r \sigma_i^{-1} \quad 1 \leq r \leq g$$

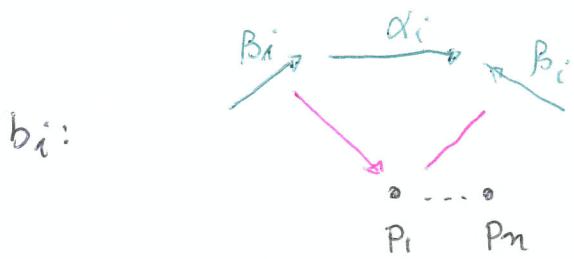
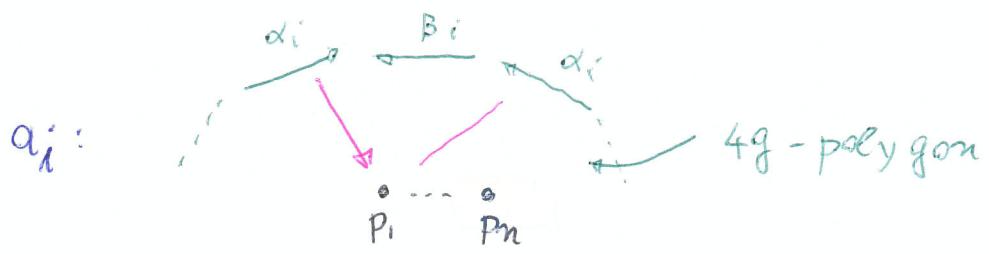
$$(R3) \quad \begin{aligned} \sigma_i^{-1} a_s \sigma_r a_r &= a_r \sigma_i^{-1} a_s \sigma_r \\ \sigma_i^{-1} b_s \sigma_r b_r &= b_r \sigma_i^{-1} b_s \sigma_r \quad s < r \\ \sigma_i^{-1} a_s \sigma_r b_r &= b_r \sigma_i^{-1} a_s \sigma_r \\ \sigma_i^{-1} b_s \sigma_r a_r &= a_r \sigma_i^{-1} b_s \sigma_r \end{aligned}$$

$$(R4) \quad \sigma_i^{-1} a_r \sigma_i^{-1} b_r = b_r \sigma_i^{-1} a_r \sigma_i \quad 1 \leq r \leq g$$

$$(TR) \quad [a_1, b_1] [a_2, b_2] \dots [a_g, b_g] = \sigma_1 \sigma_2 \sigma_{n-1}^2 \dots \sigma_{g-1} \sigma_g$$

$$(\text{here } [a, b] = aba^{-1}b^{-1})$$

* geometric interpretation



Unitary

* Representations of $B(\mathbb{Z}_q, n)$ & Weyl-Heisenberg
commutation relations (CCR)

$$\rho : B(\mathbb{Z}_q, n) \rightarrow U(\mathcal{H})$$

\mathcal{H} : Hilbert space

→ set $\rho(\sigma_j) = \sigma \cdot I$

$$\sigma = (-1)^{\nu} = e^{i\pi\nu} = e^{2\pi i \theta}$$

statistical parameter

$b_1, b_2, b_1^{-1}, b_2^{-1}$ are ok

$$R3: [\rho(a_s), \rho(a_r)] = [\rho(b_s), \rho(b_r)] = I \quad r, s = 1..g$$

$$R4: [\rho(a_r), \rho(b_2^{-1})] = \sigma^{-2} I$$

$$TR: \sigma^{2(n-1+g)} = 1 \quad \nu = \frac{q}{n-1+g} \neq 0$$

→ fractional statistics ("anyons")

$\sigma^2 = 1$: Fermi-Bose

look for $\mathcal{H} := H_1 \otimes H_2$ ← one-dimensional

Weyl-Heisenberg → two types

$$\rho(a_r) = \rho_1(a_r) \otimes I_{H_2}$$

$$\rho(b_2^{-1}) = \rho_1(b_2^{-1}) \otimes I_{H_2}$$

$$\rho(\sigma_j) = I_{H_1} \otimes \sigma I_{H_2}$$

$$\begin{aligned} \rho_1(a_r) &= U(e_r) \\ \rho_1(b_2^{-1}) &= V(e_r) \end{aligned}$$

* Techniques

Finite-dimensional representations stem from the finite CCR.

$$\text{ex: } \beta = 1 \quad \gamma = \frac{q}{r} \quad (q > 0, r > 0, (q, r) = 1)$$

on \mathbb{C}^r : U_1, U_2 satisfying $U_1 U_2 = e^{-2\pi i \gamma} U_2 U_1$

$$\hat{p}(a_i) = U_1 \quad \hat{p}(b_i^{-1}) = U_2 \quad U_f^r = I$$

$$U_1 = \begin{pmatrix} 1 & & & \\ & e(r) & 0 & \\ & 0 & \ddots & \\ & 0 & & e(r-1)r \end{pmatrix} \quad U_2: e_i \mapsto e_{i-1}$$

shift map

$$e(x):= e^{2\pi i x}$$

then complete the definition accordingly

non commutative torus

* Standard Weyl-Neisenberg CCR

$$V(\vec{\beta}) U(\vec{\alpha}) = e^{2\pi i \gamma \vec{\alpha} \cdot \vec{\beta}} U(\vec{\alpha}) V(\vec{\beta})$$

$$\vec{\alpha}, \vec{\beta} \in \mathbb{R}^g$$

both

arising in the representation theory of

Riemann surface braid groups

(Bellingeri's presentation)

no geometric construction *

* Theta functions and canonical commutation relations (after M.S. '86)

$\mathbb{Z} = i\mathbb{H}$ for simplicity

Riemann matrix

$$\mathcal{X} = \mathbb{Z}^2(L, \langle , \rangle)$$

L : h.e.b on an abelian variety X
 $h^0(L) = 1$ via Riemann theta function $\{\theta_i\}$

$L = \text{any } L \text{ via Q.R.}$

▽ Chern-Bott connection

$$\nabla_{\frac{\partial}{\partial q_i}}, \nabla_{\frac{\partial}{\partial p_i}}$$

are skew-hermitian

$$\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i} \in \text{Lie}(X) \cong \mathbb{R}^{2n}$$

$$[\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_j}] = 0$$

$$(0 = \int_X \frac{\partial}{\partial q_i} \langle s', s \rangle = \int_X \langle \nabla_{\frac{\partial}{\partial q_i}} s', s \rangle + \int_X \langle s', \nabla_{\frac{\partial}{\partial q_i}} s \rangle)$$

Theorem (M.S. '86)

$$Q_j = \frac{i}{\sqrt{2\pi}} \frac{\partial}{\partial q_j}, P_j = \frac{i}{\sqrt{2\pi}} \frac{\partial}{\partial p_j}$$

$j=1, 2, \dots, n$ realise a representation of the Heisenberg commutation relations

$$[Q_i, P_j] = \sqrt{-1} \delta_{ij} I$$

$$[Q_i, Q_j] = [P_i, P_j] = 0$$

Consequently, their induced parallel transport operators gives rise to a representation of the Weyl commutation relations

$$V(\beta) U(\alpha) = e^{i \alpha \cdot \beta} U(\alpha) V(\beta)$$

$$U(\alpha) = e^{i \sum_j \alpha_j Q_j}; \quad V(\beta) = e^{i \sum_j \beta_j P_j}$$

The representation is **irreducible**

This is a consequence of the Riemann-Roch theorem in conjunction with the von Neumann uniqueness theorem

from $A_j = \frac{1}{\sqrt{2}} (Q_j + i P_j)$ $\nabla^{\bar{\partial}}$ holomorphic structure
annihilation operator \leftarrow crucial

$H^0(L) = \text{ground state space of the quantum harmonic oscillator defined by}$

$$H = \frac{1}{2} \sum_j (P_j^2 + Q_j^2)$$

also: geometric description of coherent states

$$(| \Psi_{\alpha, \beta} \rangle = e^{\frac{i}{2} \alpha \cdot \beta} U(\alpha) V(-\beta) | \Psi_{0,0} \rangle)$$

$$\sim \text{Pic}^0(X)$$

ground state
of the harmonic
oscillator

→ theory of Landau levels in QH effects

hol. vector bundles over Pic^0

("Brillouin manifold")

→ noncommutative
geometric
generalization
(A. Schwarz '00)

* HE - vector bundles on Jacobians (Matsushima)

Illustration for $g=1$

Start from a r -dimensional representation of the finite Weyl-Heisenberg group corresponding to a 2 -lattice $R \subset \mathbb{C}$ giving rise to the torus \mathbb{C}/R :

$$U(\gamma + \gamma') = U(\gamma)U(\gamma') e^{\frac{i}{2r} A(\gamma', \gamma)}$$

$A = \text{Im } R$ is hermitian form on \mathbb{C} with

$$\left[\frac{1}{2\pi} A(\gamma, \gamma') \in \mathbb{Z} \right]$$

$$A = -2\pi q \cdot w = -2\pi q \cdot dx \wedge dy = -i\pi q dz \wedge d\bar{z}$$

$$J_2 U_2 = U_2 U_1 e^{-\frac{2\pi i}{r} \frac{q}{r}} \quad r = \frac{q}{r}$$

$$R(z, w) = \pi q \cdot z \bar{w}$$

factor of automorphy (theta factor)

$$j(\gamma, z) = U(\gamma) \exp \left[\frac{1}{2r} R(z, \gamma) + \frac{1}{4r} R(\gamma, \gamma) \right]$$

yielding $E_\gamma \rightarrow \mathbb{C}/R$ stable bundle of rank r

$$\left[E_\gamma = (\mathbb{C}/R \times \mathbb{C}^r)/R \right] \quad \text{action of } M:$$

$$\gamma(z, \xi) := (\gamma + z, j(\gamma, z)\xi)$$

equipped with the hermitian metric $h = e^{-\frac{R}{2r}} I_r$

leading to a Chern-Bott connection ∇ with

$$\text{constant curvature } S\nabla = i \frac{A}{r} I_r = -2\pi i D \cdot e_1 \wedge e_2 \cdot I_r$$

One has $h^\circ(E_\gamma) = q$ (Riemann-Roch-Hirzebruch)

$$\dim H^0(E_\gamma) \xrightarrow{\text{!}} \text{holomorphic sections}$$

$r=1$ yields the g -level theta functions

and \star Weyl-Heisenberg again with multiplicity q
via Chern-Bott connection

Construction of H_1 and ρ_1

$$H_1 = L^2(E)$$

$(q_1, p_1, \dots, q_g, p_g) \in J(\bar{\gamma}_g)$
(Darboux)

centre of mass
wave functions

projectively flat

$$\langle \cdot, \cdot \rangle = \int_{J(\bar{\gamma}_g)} h(\cdot, \cdot) \frac{\omega^g}{g!}$$

extra braid group
generators

$$\rho_1(a_r) := \exp\left(\frac{\nabla_z}{\partial q_r}\right)$$

$$\rho_2(b_r^{-1}) := \exp\left(\frac{\nabla_z}{\partial p_r}\right)$$

(parallel transport operators)

∇ : projectively
flat (HE)
connection

* the prime form bundle

towards H_2

↓ prime form theta function, odd characteristic

$$E(x, y) = \frac{\vartheta[\beta](\Delta(y-x))}{\sqrt{\wp(x)} \sqrt{\wp(y)}} \stackrel{\text{Abel map}}{\rightarrow} = -E(y, x)$$

$$\left. \begin{array}{l} \zeta = \sum_{i=1}^g \frac{\partial \wp[\beta]}{\partial z_i}(0) w_i \\ \wp \end{array} \right\} \begin{array}{l} \text{unique 1-form} \\ \text{vanishing on } \beta \\ \text{divisor: } 2\beta \\ \Rightarrow \sqrt{\zeta} \text{ well-defined} \end{array}$$

* needed to cancel the extra zeros of $\wp[\beta]$
 ($\wp[\beta]$, as a function of x , vanishes at y
 + $g-1$ further points)

E : unique holomorphic section of $\mathcal{O}(\Delta)$

Δ = diagonal in $X \times X$

(= form of weight $(-\frac{1}{2}, -\frac{1}{2})$)

on the universal cover $\hat{X} \times \hat{X}$

{ line bundle
pertaining
to the
divisor Δ

the prime form generalizes the genus zero counterpart

$$E_0(x, y) = \frac{x-y}{\sqrt{dx} \sqrt{dy}} \quad \text{anal locally}$$

reduces to it up to a 3rd order term (universal behaviour)

$X^n := \underbrace{X \times X \dots X}_{n \text{ copies}}$ the line bundle

$$\mathcal{L} := \prod_{i < j} \pi_j^* \sigma(\Delta_{ij}) \rightarrow X^n$$

descends to

$$\boxed{\mathcal{L} \rightarrow C_n(X)}$$

(same notation)

where upon $B_n(X)$ acts but we just get ordinary (Yamni-Bose) statistics

* problem: enforce fractional statistics

→ extract roots of a line bundle ⚠

* minimalistic approach: resort to a local description capturing the essence of braiding. Define

$$H_2 := \left\langle \psi_\theta = \prod_{i < j} (\beta_i - \beta_j)^{2\theta} \right\rangle \quad V=2\theta$$

Junking conditions

universal behaviour

* ψ_θ enjoys the correct transformation law under exchanges $x_i \leftrightarrow x_j$

$$\boxed{\psi_\theta \mapsto \frac{(-1)^{2\theta}}{(-1)^V} \psi_\theta = \sigma \cdot \psi_\theta}$$

Upshot:

$$\mathcal{E} \rightarrow J(\mathbb{Z}_g)$$

$$r = q/g$$

Matsuhashima HE-holomorphic vector bundle with slope $\kappa(E) = rg!$

The representation P_1 of the CCR on $H_1 = L^2(\mathcal{E})$, together with the position

$$P_2(\sigma_j) \psi = (-1)^r \psi \quad \psi \in H_2$$

gives rise to a unitary representation

$$\left\{ \rho : B(\mathbb{Z}_g, n) \rightarrow U(X) \quad X = \tilde{H}_1 \otimes \tilde{H}_2 \right\}$$

$$n = r+1 - g$$

P_1 has multiplicity $h^0(\mathcal{E}) = q^g$ (generalised)

$\Psi := \Psi_1 \in \bigoplus_{\mathcal{E}} H^0(\mathcal{E})$:= Laughlin wavefunction
 Matsuhashima theta vector
 "centre of mass"

Remark: compatibility with Sudarshan et al (1988):

They exclude anyons on tori, but their L-wave functions are scalar ones, and in our case, scalar wave functions correspond to m-level theta bundles

★ Noncommutative tori & stable bundles (g=1)

Prologue : Swan's theorem

sections of vector bundles \equiv finitely generated projective modules over the algebra of smooth functions on the base manifold

Noncommutative torus $A_{\varphi} \quad \varphi \in \mathbb{R}/2\pi$

universal unital C^* -algebra generated by unitary operators U_1, U_2 satisfying

$$U_1 U_2 = e^{2\pi i \varphi} U_2 U_1$$

(deformation of the standard commutative algebra of continuous functions on a torus - via Fourier)

\mathbb{T}_{φ}^2 : smooth subalgebra

$$\sum a_{mn} U_1^m U_2^n \quad \{a_{mn}\} \in \mathcal{S}(\mathbb{Z}^2) \\ \text{rapidly decreasing}$$

$$\delta_1(U^\alpha) = 2\pi i n_1 U^\alpha$$

$$U^\alpha = U_1^{n_1} U_2^{n_2}$$

$$\delta_2(U^\alpha) = 2\pi i n_2 U^\alpha$$

Natural (hermitian) \mathbb{T}_{φ}^2 - right modules

$$[E_{p,q} = \mathcal{S}(\mathbb{R}, \mathbb{C}^q)]$$

as vector spaces
they do not depend on p
 $(p>0, q>0, (p,q)=1)$

$$[\mathrm{rr}(E_{p,q}) := \tilde{\tau}_{\mathrm{End}(E_{p,q})}(I) = p - \varphi q \quad (>0)]$$

Murray - von Neumann

$$\tau(a = \{a_{mn} U_1^m U_2^n\}) = a_{00}$$

$$\tau(ab) = \tau(ba)$$

(right module structure:

$$\psi_{\epsilon_{p,q}}$$

$$(\nabla_1 \xi)(s) = e(s) \xi(s)$$

$$(\nabla_2 \xi)(s) = \xi(s - (p/q - \kappa))$$

$$w_1 w_2 = e(p/q) w_2 w_1 \quad w_1^q = w_2^q = 1$$

finite W-H relations for $2\pi/q\mathbb{Z}$

Then set:

$$\sum_i U_i := (\nabla_i \otimes w_i) \sum_{i=1,2}$$

* Connes connection

$$(\nabla_1 \xi)(s) = 2\pi i \frac{q}{p - \kappa q} s \xi(s)$$

$$(\nabla_2 \xi)(s) = \xi'(s) \quad s \in \mathbb{R}$$

$$\mathcal{S} = [\nabla_1, \nabla_2] e_1 \wedge e_2 = -2\pi i \frac{q}{p - \kappa q} e_1 \wedge e_2$$

curvature

$$c_1(\epsilon_{p,q}) = \frac{1}{2\pi i} \text{Tr}(-\mathcal{S}) = q \quad (\text{integer!})$$

1st Chern class

* holomorphic structure (M.S., Polishchuk-Schwarz)

$$\bar{\nabla} := \nabla_1 + i \nabla_2$$

$\text{Ker } \bar{\nabla}$: noncommutative theta vectors (Schwarz, '00)
 (q-dimensional)

$$\xi = \xi(s) = e^{-\pi \frac{q}{p+rq} s^2} v \quad v \in \mathbb{C}^q$$

(quantum harmonic oscillator ground state)

$$\text{Ker } \bar{\nabla}^* = \{0\} \quad (\text{from } [\bar{\nabla}, \bar{\nabla}^*] = 4\pi \frac{q}{p+rq} I)$$

$$\text{ind}(\bar{\nabla}) = \dim(\text{Ker } \bar{\nabla}) - \dim(\text{Ker } \bar{\nabla}^*) = q$$

index formula

Upshot:

Now take $p=r>0$, $q=0$ we get
 a "classical" rank r hermitian holomorphic vector bundle over a complex torus, which is stable,
 equipped with a hermitian connection with constant curvature, having slope

$$\mu(E_r, q) = \frac{q}{r} \quad (= r)$$

and such that

$$h^0(E_r, q) = q$$

(for $r=q=1$ one retrieves the theta line bundle)

(Also: "Schwarz = Laughlin")

centre of mass part

* "statistics" of theta vectors

Let $\nu = \frac{q}{r}$. Consider E_ν

Then set $\nu' = \frac{r}{q} = \frac{1}{\nu}$ & consider

$$w_1 w_2 = e^{-2\pi i \frac{x}{q}} w_2 w_1$$

* realised on $H^0(E_\nu) \cong \mathbb{C}^q$

$$\text{G: } E_\nu \rightarrow \Pi_{\nu'}^2$$

Connes

But the above representation determines an odd-dimensional representation of $B(\mathbb{Z}, q)$ with "dual" statistics parameter $\sigma' = (-1)^{\nu'}$ and can be promoted to a stable bundle $E_{\nu'}$ (Matsusaka)

$$M: \Pi_\nu^2 \rightarrow E_\nu$$

Matsusaka

$$\left[MC: E_\nu \rightarrow E_{\nu'} \quad CM: \Pi_\nu^2 \rightarrow \Pi_{\nu'}^2 \right]$$

Matsusaka - Connes duality

* This can be interpreted via the
Fourier - Mukai - Nahm transform.

★ FMN via noncommutative theta vectors

$J(\mathbb{Z}_2)$ is a self-dual abelian variety

\Downarrow parametrising flat line bundles thereon
 $\alpha \mapsto P_\alpha$

$$E_\alpha = E \otimes P_\alpha$$

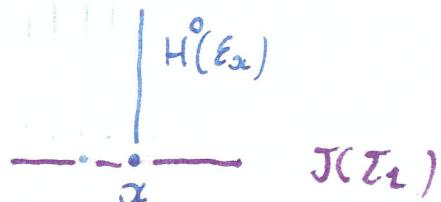
family of Matsushima
bundles

$\xrightarrow{\text{Matsushima}} \ker \tilde{\nabla} \Rightarrow H^0(E_\alpha)$: fibre at α

of a bona fide holomorphic vector bundle on $J(\mathbb{Z}_2)$,

$$\text{FMN}(E) \rightarrow J(\mathbb{Z}_2)$$

equipped with a
natural hamilton
connection $\tilde{\nabla}$



inherited from ∇ on E

we elaborate on this via ncq

$$\tilde{\nabla} = \tilde{\nabla}_1 + i \tilde{\nabla}_2 = \bar{\nabla} - 2\pi i \alpha \mathbb{I}$$

$$\tilde{\nabla}_1 = \nabla_1 - 2\pi i \alpha \mathbb{I}$$

$$\tilde{\nabla}_2 = \nabla_2 - 2\pi i \beta \mathbb{I}$$

$$z = \alpha + i\beta$$

$$\alpha, \beta \in [0, 1]$$

coordinates on $J(\mathbb{Z}_2)$

theta vectors:

$$\left\{ \sum_z = \sum_{\alpha, \beta} = e^{-\pi \frac{r}{q} \left(\frac{q}{r}s - z \right)^2} v, \quad v \in \mathbb{C}^q \right.$$

★ coherent states

L^2 -normalized theta vectors

$$\int e^{-\gamma x^2} = \sqrt{\pi/\gamma}$$

$$\tilde{\xi}_2 = e^{-\pi \frac{r}{q} \beta^2} \left(\frac{2q}{r}\right)^{\frac{1}{4}} \xi_2 \quad (\text{discard } r \text{ temporarily})$$

* Berry-Simon connection (Nahm transform)

$$z \mapsto A_z = \langle \tilde{\xi}_2, d\tilde{\xi}_2 \rangle = \langle \tilde{\xi}_2, \partial_\alpha \tilde{\xi}_2 \rangle dx + \langle \tilde{\xi}_2, \partial_\beta \tilde{\xi}_2 \rangle d\beta$$

$$\text{from } \int e^{-\gamma x^2} x dx = 0$$

* projection
 $P_x: L^2 \rightarrow H^0(E_\omega)$

$$\begin{array}{c|c|c|c} & \parallel & \parallel & \parallel \\ \downarrow & -2\pi i \frac{r}{q} \beta dx \wedge I_q & \downarrow & 0 \\ \hline \end{array}$$

$$* \text{ curvature } \Omega = 2\pi i \frac{r}{q} dx \wedge d\beta I_q = 2\pi i \nu' dx \wedge d\beta I_q$$

$$(Ch_0(\mathcal{E}), Ch_1(\mathcal{E})) = (r, q)$$

$$(Ch_0(FMN^*(\mathcal{E})), Ch_1(FMN^*(\mathcal{E}))) = (q, r) \quad \text{dualization}$$

* Conclusion : $MC = FMN^*$ up to moduli

Also, one finds that if $\mathcal{E} \rightarrow J(E_g)$ is projectively flat, it can be the FMN transform of $\mathcal{E}' \rightarrow E_g$ only for $g=1$

GEOMETRIC QUANTIZATION & LANDAU LEVELS revisited

(A.Galasso & M.S 2016)

Holomorphic geometric quantization of
the harmonic oscillator

$$M = \mathbb{R}^{2n} \cong \mathbb{C}^n \quad n \geq 1 \quad (\hbar = \epsilon) \quad z_j = x_j + iy_j$$

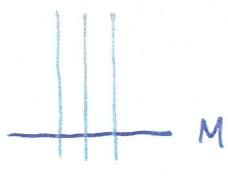
$$\tilde{\omega} = \sum_{j=1}^n dx_j \wedge dy_j = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j = \frac{i}{2} \partial \bar{\partial} \tilde{\gamma}$$

Kähler form $\tilde{\gamma} := \sum_{j=1}^n \bar{z}_j z_j = :2\tilde{h}:$ \tilde{h} oscillator Hamiltonian

Kähler potential

$$L = M \times \mathbb{C} \rightarrow M \quad \text{trivial complex line bundle}$$

$\exists \sigma \in \Omega^1$ trivialising section



$$\text{Hermitian metric } (z, \bar{z}) := e^{-\tilde{\gamma}}$$

$$(s = \tilde{s} \cdot z, s' = \tilde{s}' \cdot z) = \bar{\tilde{s}} \tilde{s}' e^{-\tilde{\gamma}}$$

$$\tilde{H} = L^2(M, \mu) \quad \mu = e^{-\tilde{\gamma}} dx_1 dy_1 \dots dx_n dy_n$$

Chern-Bott connection (in a holomorphic frame)

$$\nabla = d - \partial \tilde{\gamma} = \nabla' + \nabla''$$

$$\nabla' = \partial - \partial \tilde{\gamma}, \quad \nabla'' = \bar{\partial}$$

$$\nabla = d - \partial \tilde{\gamma} = d - \sum_{j=1}^n \bar{z}_j dz_j$$

$$\text{curvature: } \Omega = d(-\partial \tilde{\gamma}) = -2i\tilde{\omega} = -2i d\tilde{\theta}$$

$$\tilde{\theta} = -\frac{i}{2} \sum_{j=1}^n \bar{z}_j dz_j \quad \text{symplectic potential}$$

work in $\hat{\mathcal{H}}$ (actually, suitable domains therein)

$$A_j := \nabla_{\frac{\partial}{\partial z_j}} = \frac{\partial}{\partial \bar{z}_j} \quad A_j^+ := \left(\nabla_{\frac{\partial}{\partial z_j}} \right)^+ = -\frac{\partial}{\partial z_j} + \frac{\partial^2}{\partial \bar{z}_j} = -\frac{\partial}{\partial z_j} + \bar{z}_j$$

\curvearrowleft CCR
annihilation & creation operators

$$[A_j, A_k] = [A_j^+, A_k^+] = 0$$

$$[A_j, A_k^+] = I \cdot \delta_{jk}$$

identification
crucial in theta
function theory
(both classical and
noncommutative)
(M. G. 1986, M.S. 2015
A. Schwarz 2000)

multiplicity = dimension of the
common kernel of the A_j
(von Neumann Uniqueness
theorem 1931)



$$\mathcal{H} = \{ f \in \hat{\mathcal{X}}, f \text{ holomorphic} \}$$

Bargmann-Fock space

already a
Hilbert space

$\bar{\mathcal{H}}$ conjugate space

$$\hat{\mathcal{X}} \cong \mathcal{X} \otimes \bar{\mathcal{X}} \quad \left\{ z_i^{n-k} \right\}_{i,j=1,\dots,n} \quad \begin{array}{l} \text{orthonormal basis} \\ \text{for } \hat{\mathcal{X}} \\ \text{up to constants} \\ i,j=1,\dots,n \\ h>0 \end{array}$$

Prequantum Hamiltonian

$$Q(\hat{h}) := -i \nabla_{X_{\hat{h}}} + \hat{\theta}(X_{\hat{h}})$$

$$= -i X_{\hat{h}} = \sum_{j=1}^n (z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j})$$

\uparrow
Hamiltonian vector field pertaining to \hat{h}

$$\text{contraction} \rightsquigarrow i_{X_{\hat{h}}} \hat{\omega} + d\hat{h} = 0$$

restrict to \mathcal{H} :

$$Q(\hat{u})|_{\mathcal{H}} = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} \quad \begin{array}{l} \text{Euler operator} \\ (\text{number}) \end{array}$$

zero point energy
missing

Set:

$$a_j := A_j|_{\mathcal{H}} = \frac{\partial}{\partial z_j}$$

$$a_j^+ := A_j^+|_{\mathcal{H}} = \bar{z}_j$$

$$\text{CCR: } [a_i, a_j] = [a_i^+, a_j^+] = 0$$

$$[a_i, a_j^+] = \delta_{ij} I$$

Similarly one has CCR on $\tilde{\mathcal{H}}$:

$$b_j = \frac{\partial}{\partial z_j} \quad b_j^+ = \bar{z}_j$$

* Upshot: $\tilde{\mathcal{H}} \in \mathcal{H}_A$ carries a representation π_A of the CCR defined via the Chern-Bott connection. π_A is reducible, with infinite multiplicity given by \mathcal{H} . The latter carries a CCR representation π_b , which is in turn irreducible. Similarly one gets π_a on \mathcal{H}

$$\pi_A = \pi_b \otimes \pi_a, \text{ acting on } \mathcal{H} \otimes \tilde{\mathcal{H}} = \mathcal{H}_b \otimes \mathcal{H}_a = \mathcal{H}_A = \tilde{\mathcal{H}}$$

$$z_i^{m_1} \bar{z}_j^{m_2} \leftrightarrow (b_i^+)^{m_1} (a_j^+)^{m_2} |0\rangle \quad a_j^+ |0\rangle = b_i^+ |0\rangle = 0$$

* Charged particle in a constant magnetic field
(on a plane) classical theory

1st approach $M = T^* \mathbb{R}^2 \cong \mathbb{R}^4$ phase space

$$\omega = dp_x \wedge dx + dp_y \wedge dy \quad \text{symplectic form}$$

$$h = \frac{1}{2} [(p_x - y)^2 + (p_y + x)^2] \quad \text{Hamiltonian}$$

$$l = x p_y - y p_x \quad \begin{matrix} z\text{-component} \\ \text{of angular momentum} \end{matrix}$$

Canonical transformation:

$$\left\{ \begin{array}{l} P_1 = \frac{1}{\sqrt{2}} (x + p_y) \\ Q_1 = \frac{1}{\sqrt{2}} (y - p_x) \\ P_2 = \frac{1}{\sqrt{2}} (y + p_x) \\ Q_2 = \frac{1}{\sqrt{2}} (x - p_y) \end{array} \right. \quad h_i := \frac{1}{2} (P_i^2 + Q_i^2)$$

One gets:

$$h = h_e = \frac{1}{2} [P_e^2 + Q_e^2] \quad l = h_1 - h_2$$

$$\{h, l\} := \omega(x_h, x_e) = 0$$

Also set, for future use

h, l :
complete set of
first integrals

$$z = \frac{1}{\sqrt{2}} [P_1 + i Q_1], \quad \bar{z} = \frac{1}{\sqrt{2}} [Q_2 - i P_2]$$

$$\Rightarrow h_1 = \bar{z}\bar{\xi}, \quad h_2 = z\bar{z}, \quad l = \bar{z}\bar{\xi} - z\bar{z}$$

we have a completely integrable system (no two harmonic oscillators)

2d Liouville tori parametrized by (h_e, h_e) or (h_e, l)

$$\omega = dh_e \wedge d\varphi_e + dh_2 \wedge d\varphi_2 \quad (\vartheta, \varphi) \text{ action-angle variables}$$

2nd approach

equip $M = T^* \mathbb{R}^2$ with a new symplectic form

physical
constants
reinserted

$$\omega' = \omega_{\text{old}} + \frac{eB}{c} dx \wedge dy \quad eB > 0$$

magnetic term

New Hamiltonian: $H' = \frac{1}{2m} (p_x^2 + p_y^2)$
(gauge invariant formulation)

New angular momentum $\ell' = \ell + \frac{eB}{2c} (x^2 + y^2)$

Again $\{H', \ell'\} = 0$

we have $X_H' = \frac{1}{m} (p_x \partial_x + p_y \partial_y - \frac{eB}{c} p_x \partial_{p_y} + \frac{eB}{c} p_y \partial_{p_x})$

$$X_\ell' = -y \partial_x + x \partial_y - p_y \partial_{p_x} + p_x \partial_{p_y}$$

* Translations

Action on (M, ω')

$$(x, y, p_x, p_y) \mapsto (x+a, y+b, p_x, p_y)$$

$\mathbb{R} \times \mathbb{R}$ $t = (\mathbb{R}^2, +)$

$\mathbb{G} = \mathbb{R}^2$ acting on M in a Hamiltonian fashion

$$\partial_x \equiv X_{t_x} \leftrightarrow t_x := p_x - \frac{eB}{c} y + \text{const.}$$

$$\partial_y \equiv X_{t_y} \leftrightarrow t_y := p_y + \frac{eB}{c} x + \text{const.}$$

$$\rightarrow \{t_x, h'\} = \{t_y, h'\} = 0 \quad \leftarrow \text{dynamical symmetry group}$$

* Rotations $S^1 \cong U(1)$ acts on M as

$$\begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}$$

$U(1) \cong \mathbb{R}$ acts via

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mapsto -y \partial_x + x \partial_y - p_y \partial_{p_x} + p_x \partial_{p_y} = X_{e'}$$

$$\{e', h'\} = 0$$

From the general formula $[X_f, X_g] = X_{\{f, g\}}$

one finds that $[X_{t_x}, X_{t_y}] = X_{\{t_x, t_y\}} = X_e = 0$

* "magnetic" central extension

\rightsquigarrow Heisenberg group

Symmetry group of the classical system

the (magnetic central extension of)
the translation group yields
a symmetry of the classical system
but it will break down at the
quantum level

Nevertheless, rotational symmetry
will survive quantization

QUANTIZATION

* Geometric quantization I

Perform holomorphic geometric quantization on the first ("unprimed") system, get BF:

$$\mathcal{H} = \left\{ \mathcal{F} = \mathcal{Y}(\Xi) / \mathcal{Y} \text{ is holomorphic} \quad \Xi = (\xi, z) \right.$$

and $\int |\mathcal{Y}(\Xi)|^2 e^{-2\bar{\Xi}} d\xi d\eta dz dy < \infty \right\}$

$$\Xi = \begin{pmatrix} \xi \\ z \end{pmatrix} = \begin{pmatrix} \xi \\ z+iy \end{pmatrix}$$

$$\downarrow \text{BF} \qquad \qquad \qquad 2\bar{\Xi} := \xi\bar{\xi} + z\bar{z}$$

$$\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2$$

The annals \mathcal{H}_1 and \mathcal{H}_2 are directly (holomorphically) / geometrically quantizable (their flows preserve the holomorphic polarization)

\hat{h}_j of quantum harmonic oscillator

One has an obvious action of $S^1 \times S^1$ on \mathbb{C}^2

* orthonormal basis (up to constants)

$$\sum^{m_1} z^{m_2} \rightarrow (a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2} |0\rangle \quad m_i \geq 0$$

$$a_1 = \frac{d}{d\xi} \quad a_1^\dagger = \xi \quad a_1 a_1^\dagger - a_1^\dagger a_1 = I \quad a_1 |0\rangle = 0$$

acting on \mathcal{H}_1 , and similar expressions for a_2, a_2^\dagger acting on \mathcal{H}_2 . Obviously $[a_1^\dagger, a_2^\dagger] = 0$

creation or annihilation operators

One also has

$$\hat{l} = \hat{h}_1 - \hat{h}_2 = a_1^+ a_1 - a_2^+ a_2$$

quantized angular momentum (\hat{l} -component)

Remark: The above picture is compatible
with Bohr-Sommerfeld

M is foliated by Liouville tori $\Lambda \cong S^1 \times S^1$

cohomological condition: $[\frac{\Theta}{2\pi}] \in H^2(\Lambda, \mathbb{Z})$

$$\Theta = m_1 dq_1 + m_2 dq_2 \quad \Theta: \text{symplectic potential}$$

$$m_i \in \mathbb{N}^*$$

\rightsquigarrow BS-tori (+ circles & a point)

* covariantly constant sections $\nabla \mathbf{f} = 0$

$$\psi_{m_1, m_2} = e^{i(m_1 q_1 + m_2 q_2)}$$

$$\sim \underbrace{\sum}_{\mathcal{Z}}^{m_1, m_2}$$

"WKB-wave functions"

* Geometric quantization II

use the vertical polarization

$$P_m := T_m Q \subset T_m M$$

$$Q = \mathbb{R}^2$$

E-wave functions

$$\mathcal{H}_E := \left\{ s : \langle s, s \rangle = \int_Q (s, s) dx dy < +\infty \right\}$$

(s, s) hermitian structure on $L = M \times \mathbb{C}$

obtained via $\theta = p_x dx + p_y dy + B \left[x dy - y dx + \frac{i}{4\pi} d(x^2 + y^2) \right]_{eB/2\hbar}$

$$d(s, s) = 2\pi i (\theta - \bar{\theta})(s, s) = -B d(x^2 + y^2)(s, s)$$

in the trivialization $s_0 \in \mathbb{C}$ $s \sim \psi$, $s' \sim \phi$, we get

$$(\psi, \phi) = \bar{\psi} \phi e^{-B(x^2 + y^2)}$$

Quantizable classical observables (their flow preserves P)

$$f = \sum_i (q^i p_i + \eta(q))$$

v-field on Q smooth function on Q

* The Hamiltonian is not quantizable

so quantize $h(4) = \text{span} \{ \epsilon, q_x, q_y, p_x, p_y \}$
Heisenberg algebra

and

extend it to the inhomogeneous symplectic algebra

$$\mathfrak{hsp}(4, \mathbb{R}) = \text{span} \{ \epsilon, q_i, p_j, q_i q_j, q_i p_j + p_i q_j \mid i, j = x, y \}$$

via the squaring von Neumann rule

$$Q(p_j^2) = Q^2(p_j) ; Q(q_j^2) = Q^2(q_j)$$

We get

$$\hat{P}_x \psi = -i\hbar \partial_x \psi + \frac{eB}{2c} (y + ix) \psi$$

$$\hat{P}_y \psi = -i\hbar \partial_y \psi - \frac{eB}{2c} (x - iy) \psi$$

and

$$\hat{x} \psi = x \psi$$

$$\hat{y} \psi = y \psi$$

passing to complex coordinates, we get

$$\hat{h} = -\frac{2\hbar^2}{m} \left(\partial_z - \frac{B}{2} \right) \partial_{\bar{z}} + \frac{\hbar eB}{2mc}$$

Schwartz functions

which is essentially self-adjoint on $\mathcal{F}(\mathbb{R}^2, \mathbb{C}) \subset L^2(\mathbb{C}, \mu)$
 via standard arguments (use Nelson's analytic vector theorem) H_p

introduce

$$\hat{a} = -\frac{i}{\sqrt{B}} \partial_{\bar{z}} \quad \hat{a}^\dagger = -\frac{i}{\sqrt{B}} (\partial_z - Bz)$$

$$\boxed{\hat{h} = \frac{eB\hbar}{mc} \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)}$$

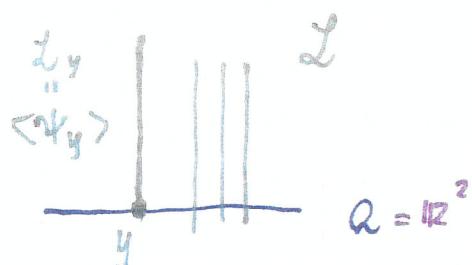
* Translational symmetry breaking

$$\hat{t}_x = -i\hbar (\partial_z + \partial_{\bar{z}}) + i\frac{eB}{4c} (z + \bar{z})$$

$$\hat{t}_y = \hbar (\partial_z - \partial_{\bar{z}}) + \frac{eB}{4c} (z - \bar{z})$$

$$[\hat{t}_x, \hat{t}_y] \neq 0$$

* Geometric interpretation



$\psi_y \in$ ground state space of \hat{h}_y^1
(translated Hamiltonian)

$$a_y \sim \bar{a}_y \quad \text{holomorphic structure}$$

$$\bar{a}_y \psi_y = 0$$

$$\text{and } b_y \psi_y = 0$$

we get a 1-dimensional
space

and an index bundle (as in Yauzer-Mukai-Nahm theory)
carrying a natural connection
(Nahm) with non trivial curvature:

$$\text{Let } \xi = \xi_{\alpha, \beta}(x) = (U(\alpha) V(\beta) \xi_0)(x) = \pi^{-\frac{1}{4}} \exp \left[i\alpha x - \frac{(x-\beta)^2}{2} \right]$$

$$\xi_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}$$

$$[U(\alpha)\phi](x) = e^{i\alpha x}$$

$$[V(\beta)\phi](x) = \phi(x-\beta)$$

$$U(\alpha)V(\beta) = e^{i\alpha\beta} V(\beta)U(\alpha)$$

CCR

$\xi_{\alpha\beta}$: standard coherent states

in Weyl form

Nahm connection form

$$A = \langle \xi, d\xi \rangle$$

curvature

$$\Omega = dA = d\langle \xi, d\xi \rangle =$$

$$[\langle \partial_\alpha \xi, \partial_\beta \xi \rangle - \langle \partial_\beta \xi, \partial_\alpha \xi \rangle] dd \wedge d\beta$$

$$= 2i \operatorname{Im} \langle \partial_\alpha \xi, \partial_\beta \xi \rangle dd \wedge d\beta$$

A routine computation yields

$$\Omega = -i dd \wedge d\beta$$

→ "translational anomaly"

lack of commutativity
with the Hamiltonian
detected via the
curvature of the
coherent state line
bundle

In contrast to translations, rotational
symmetry survives quantization

$$\hat{l} = \pm (\hat{x}\hat{\partial}_y - \hat{y}\hat{\partial}_x)$$
$$+ [\hat{n}, \hat{l}] = 0$$